Notes 6 → Percolation Basics, I

→ Why Percolation?

- The problem of large Ku ...

recall: $Ku \sim \frac{l_{ac}}{\Delta_\perp} \frac{\delta B}{B}$

$\sim \tilde{V} \tau_{ac} / \Delta_\perp$ etc.

⇒

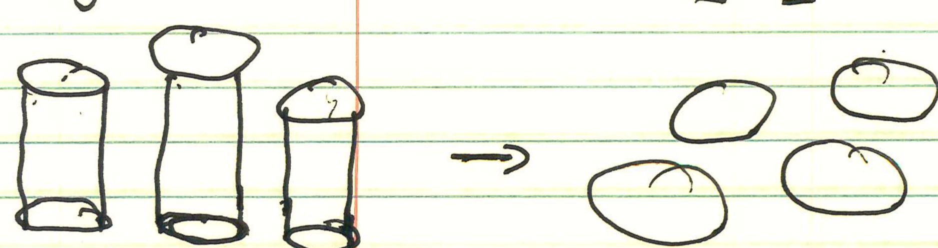
Limiting case, well defined paradigms:
(Toy models!?)

$Ku \ll 1$ → diffusion, extensively studied
$l_{ac} \to 0$

$Ku \gg 1$ → $Ku \to \infty$. ⇒ $l_{ac} \to \infty$

i.e. no variation of $\delta B_\perp$ in z (or time).

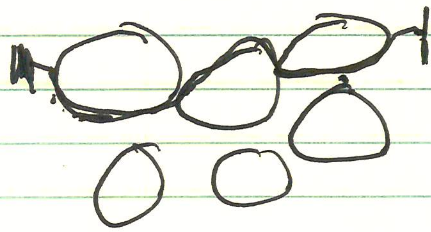
→ random (by assumption) array of magnetic cells ($\nabla \cdot B = 0$!)



static, disordered 2D medium.
random

→ Akin array of randomly placed static vortices, as in Taylor–McNamara problem (see notes)

→ ignoring D_0, Θ_0 etc. need macroscopic connection — a percolation — to transport (hills and lakes, as before)



n.b. need not have connection if $D_0 \neq 0$, though expect

$$D_{eff} \sim D_0^{\alpha} D_{cell}^{\beta} \qquad (\alpha + \beta = 1)$$

explicit dependence on collisional diffusivity

→ Problem is percolation of static random function of <u>2 variables</u>

akin to problem of effective conductivity of medium with random mixture of conducting and insulating elements, ~ identical.—

Some properties of system / problem, sticking to magnetic example:

→ spectrum of A, $B = \nabla A \times \hat{z}$

- random phases
- $\langle A \rangle = 0$
- $\langle A^2 \rangle = 1$

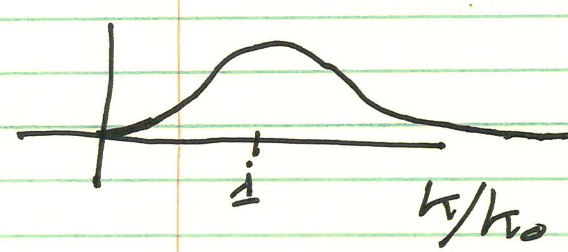
and small auto-correlation at large distances,

$$l_{\perp ac} \ll \underset{\substack{\uparrow \\ \text{system size}}}{L}$$

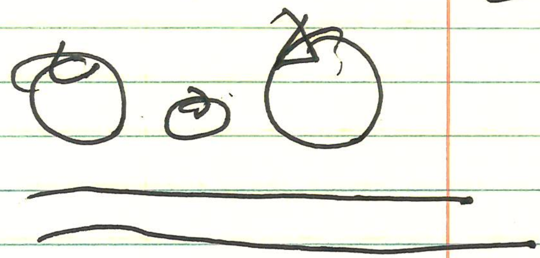
$$\langle A^2 \rangle_k \cong e^{-k^2/k_0^2}, \quad k > k_0$$

$$\langle A^2 \rangle_k \cong \left(\frac{k}{k_0}\right)^{2n} \quad \begin{array}{l} n > 0 \\ k < k_0 \end{array}$$

so $\langle A^2 \rangle_k$



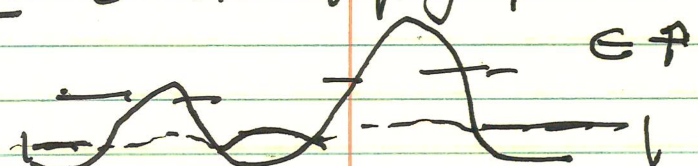
$A > \epsilon \rightarrow \begin{cases} \text{island} \\ \text{Magnetic vortex} \end{cases}$



$\rightarrow$ $\underset{\text{reg conf.}}{A < \epsilon}$, $\epsilon > 0$ $\Rightarrow$ percolating

$\epsilon = 0$, critical phase

i.e. recall topographic map



percolation/connection for $A < \epsilon$.

$l(\epsilon) \equiv$ length island or isoline
surrounding island.

$l \uparrow$ as $\epsilon \downarrow$; $\underset{\text{percolation}}{l \rightarrow L \rightarrow \infty}$

→ in such 2D topography with $D_0 \to 0$ (i.e. nothing kicking particle off field line)

$$D_{eff} = 0$$

i.e. turbulent diffusion approximation <u>not</u> applicable (i.e. all particles trapped in/near closed cells)

→ either no transport or 'burst' along macroscopic connection.

→ Interesting to consider:

2D system $\Longleftrightarrow$ $Ku \gg 1$.

$\langle B \rangle \neq 0$

$\tilde{B} \gg \langle B \rangle$ $\qquad$ $\boxed{B_{rms} \gg \langle B \rangle}$

akin: solar tachocline

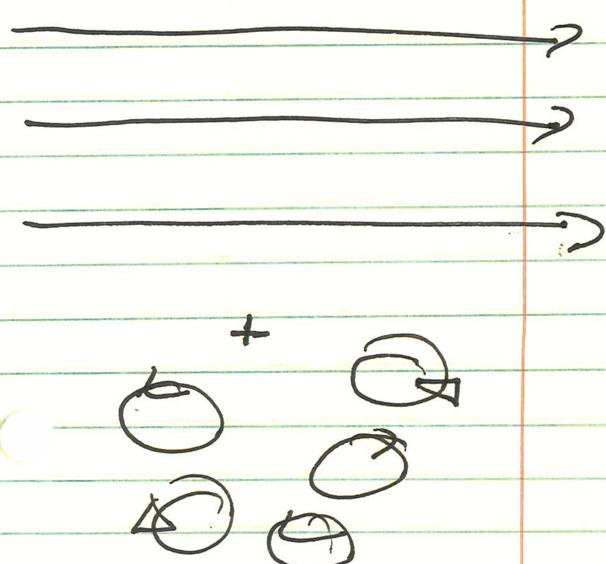
Aside: References

- "The Almighty Chance"
  - Zeldovich, Ruzmaikin, Sokolov
  - serious OV of statistical dynamics.
  - dated, highly recommended
  - percolation chapter posted.

see also:

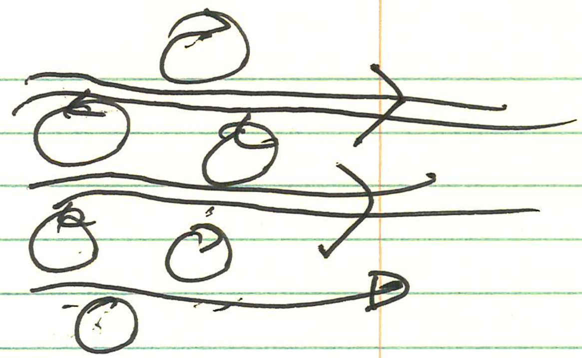
- Ya. B. Zeldovich, 1983 (posted)
  (magnetic problem)
- Stauffer 1973 (review) (posted)
  (review of percolation theory).
- Stauffer and Aharony, monograph.

Now, view as



$\langle B \rangle \to B_0$

unperturbed large scale field

= 

B_L confined to narrow channels.

i.e. real field pattern is ensemble of cells + sinews of deformed B_L

Question, does B_L percolate $\rightarrow$ i.e. extend to a, system size as $a \rightarrow \infty$.

Physics : - Will Alfvenic excitation propagate?

- More generally, what is response of such a system to external excitation?

Now, seek $\langle B \rangle \rightarrow$ mean field. what is it?

- define system as strip of width a

$$k_{min} \simeq 2\pi/a \rightarrow 1/a.$$

$\overleftrightarrow{a}$.

- take $A_k \sim k^m$ (see before)

so $B_k \sim k^{m+1}$

- $$\langle B \rangle = \left(\langle b^2 \rangle_{k < 1/a} \right)^{1/2}$$

long wavelength components define mean field

so

$$\langle B \rangle = \left(\int_0^{1/a} \underset{2D}{dk\, k}\; k^{2m}\, k^2 \right)^{1/2}$$

$$\cong (1/a)^{-m-2}$$

then, for $\langle B \rangle \neq 0$ as $a \to \infty$
(i.e. not decrease with increasing distance, a)

for $a \to \infty$, $\quad \underline{\underline{m = -2}}$

$\langle B \rangle \neq 0$

$\langle B \rangle \to$ const.

$m \sim -2$ white noise! currents

$j_{z_k} \simeq k^2 A_k \sim k^0 \sim$ const.

random currents

i.e. Random currents will result in percolating mean large field structure set by currents.

N.B. More generally:

- need $\langle j(\underline{x})\, j(\underline{x}+\underline{r})\rangle \geq 0$ (correlated currents) for percolating $\langle B\rangle$.

- no percolation for anti-correlated currents.

→ Some General Features of Percolation Problems

- explore formation of $\ell \sim L$ linked paths.

$L \sim (N)^{1/D} \Delta x$. $N \to \infty$ thermodynamic limit

— Basic game is some D-lattice:

x x x x

x x x x

x x x x etc.

With probability p of occupation of a site,

— connections $\uparrow$ with p.

— bond x—x

site x

percolation

— Universality is key concept

for $\frac{\Delta}{L} \ll 1$, $L \to \infty$,

details of specific site element irrelevant,

i.e. classic is details of, say, conductor or insulator shape element.

- Caveat :→ Fractal structure elements can complicate universality

→ example: pancakes in 3D.

Universality →

- define as simple problem
- concept as for phase transition
  - → $l_c \to \infty$ as $T \to T_c$.
  - → $\langle \tilde{T}^2 \rangle \sim (T-T_c)^2$ in regime interest
  - → scaling!

c.f. N. Goldenfeld, "Intro to Phase Transitions"
Landau & Lifshitz, "Stat. Mech".

- contrast with diffusion:

→ diffusion (i.e. on lattice): randomness arises from particle / trajectory dynamics

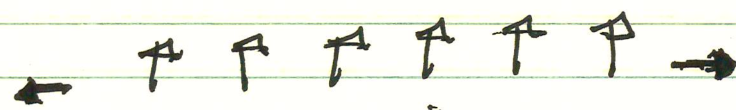
i.e. → island overlap

→ percolation: randomness encoded in site probabilities.

→ D matters

if $p = p_c$ → critical occupation dens. threshold for percolation

1D; $p_c = 1$



all sites occupied.

2D; $p_c = .59$ triangular lattice

$p_c = .50$ square lattice

etc.

How describe percolation?

→ "Scaling Theory of Percolation Clusters" - c.f. Stauffer (also relevent to Avalanches)

percolatin - lattice

random occupation p yes

$1-p$ no.

$p > p_c$ → 1 cluster spanning lattice.

$p < p_c$, no spanners

$p_c \to L_{syst}$

→ s-cluster

a cluster of s-sites, nearest neighbor coupled, ends empty.

→ seek describe scaling of cluster properties as $p \to p_c$

i.e. → characterize distribution of of cluster sizes

→ how behave as $p \to p_c$

n_s = # s-clusters

$$\boxed{\begin{array}{l} n_s = \text{avg \# / site of } s \text{ clusters} \\ \quad = n_s(p) \\ \text{population density for } p \text{ of } s \end{array}}$$

→ Of interest is:

— n_s, $\sum_s n_s(p)$ → cluster population

— $\sum_s s\, n_s(p)$ → cluster probability

$\xi(p) \equiv$ correlation length

etc.

- Aim of scaling theory is n_s.
  This yields more info.

i.e. $P_\infty \equiv$ percolation probability

$=$ fraction of sites belonging a percolation network

$\sim$ "strength" of infinite network.

Now, can relate different quantities;

i.e. any lattice site can be:

→ empty → Prob $= 1-p$

→ part of a percolation cluster

Prob $= p P_\infty$

→ part of a finite cluster

Prob $= p(1-P_\infty)$

of course, being part of a finite cluster

$$\text{Prob} = p(1-P_\infty) = \sum_s n_s s$$

so

$$1-p + P_\infty + \sum_s s n_s = 1$$

so computing n_s gives P_∞!

→ what we are interested in
→ scaling exponents, near p_c.

As for phase transitions, aim will be to (understand / predict) the relationship between scaling exponents. So, determining 1 from simulation / experiment / RG calculation (!, ?) etc. gives all.

→ Scaling Exponents

$$\sum_s n_s(p) \sim |p - p_0|^{2-\alpha}$$

$$\sum_s s\, n_s(p) \sim |p - p_0|^{\beta}$$

$$\sum_s s^2 n_s(p) \sim |p - p_0|^{-\gamma}$$

$$\xi(p) \sim (|p - p_0|)^{-\nu}$$

etc.

For 2D,

$\alpha \sim -0.7$

$\beta \sim .14$

$\gamma \sim 2.4$

$\delta \sim 18$

$\nu \sim 1.35$

Aim is to get relation.

→ 1D Percolation → Trivial but Illustrative

of course $p_c = 1$.

L 

– prob occ ~ p

– cluster → s sites + 2 ends (empty!)

i.e. $p(s \text{ cluster}) \sim \underbrace{p^s}_{s \text{ sites}}\underbrace{(1-p)^2}_{2 \text{ ends}}$

$L \gg s \Rightarrow$

$$\boxed{\underbrace{n_s}_{\#\, s \text{ clusters/site}} = p^s(1-p^2)}$$

{can generate distribution

~ thermodynamic limit ($N \to \infty$)

~> no worry re: over-hang

$n_s \to 0$ for $s \to \infty$ (exponentially)

Prob. of site in cluster of size s = $s n_s$

Prob any cluster $= \sum_s n_s s = p$

how?

$$\sum_s \underbrace{p^s (1-p)^2}_{n_s} s = (1-p)^2 \sum_s p \frac{d}{dp} p^s$$

$$= (1-p)^2 p \frac{d}{dp} \sum_s p^s$$

$$= (1-p)^2 p \frac{d}{dp} (p/1-p)$$

$$= p \, !$$

$\downarrow$

Probability occupied.

N.B. Standard trick in Statistical Mechanics $\rightarrow$ relate index to derivative.

Now: define

W_s = Probability of cluster to which arbitrary site belongs, contains s sites

$$W_s = \frac{n_s s}{\sum_s n_s s}$$

so average cluster size:

$$\bar{s} = \sum_s s\, W_s = \frac{\sum_s n_s s^2}{\sum_s n_s s}$$

(why moments scaling of some concern).

so

$$\bar{s} = \frac{\sum_s (1-p)^2 p^s s^2}{\sum_s (1-p)^2 p^s s} = \frac{1+p}{1-p}$$

after some trick.

obviously, $\bar{s} \to \infty$ $\quad p \to p_0 = 1$.

if $p = 1 - \delta$

$\bar{s} \sim \dfrac{2}{\delta}$!

Coming $\to$ further consideration of scaling theory.