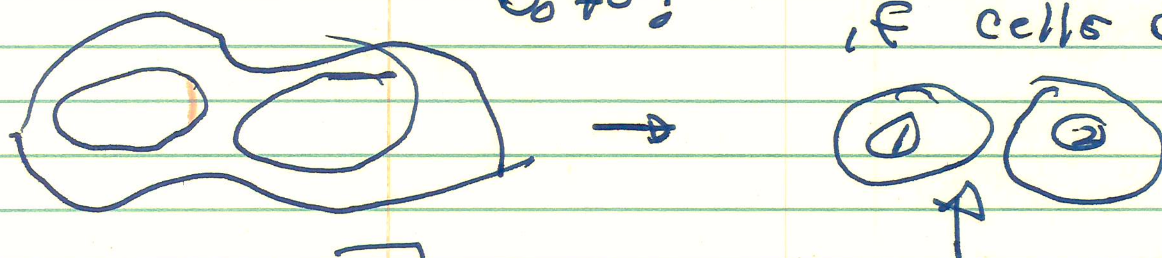
Notes 5 — Collisional Diffusion + Scattering
→ Taylor Cells
→ Taylor Shear Dispersion

→ Recall 2D transport in Stochastic Field

$D_0 \neq 0$!

if cells close



theme: diffusive kick off line.

collisional diffusion can kick 1 → 2

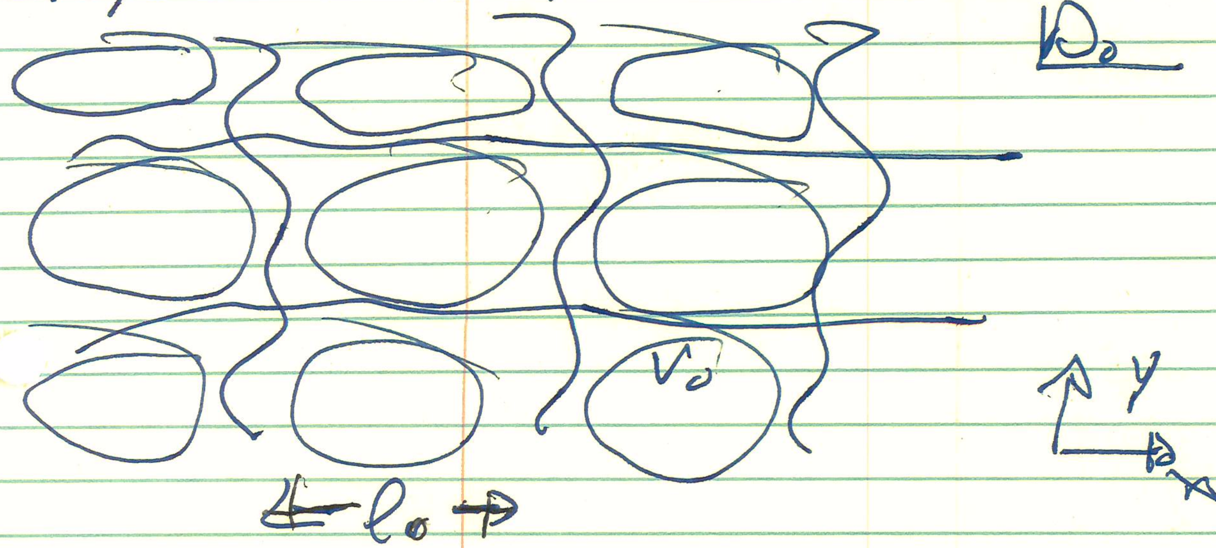
$$D_{\perp\,\mathrm{eff}} = D_\perp(\tilde{b}, \Delta_\perp, D_0)$$

corresponds to D_M here

→ Many problems involve synergy between turbulent scattering and collisional diffusion

Geometry matters

→ Taylor Problem — the Classic



For passive scalar problem:

$$\frac{\partial n}{\partial t} + \underline{v} \cdot \underline{\nabla} n - D_0 \nabla^2 n = 0$$

can define: $Pe = V_0 l_0 / D_0$

→ Peclet number

→ Pe >> 1 of interest

Interest: – Effect transport coefficient, i.e. diffusivity, for scales $L \gg l_0$

(*) – effective medium problem with 2 transport process:

fast → convection → operating on cells

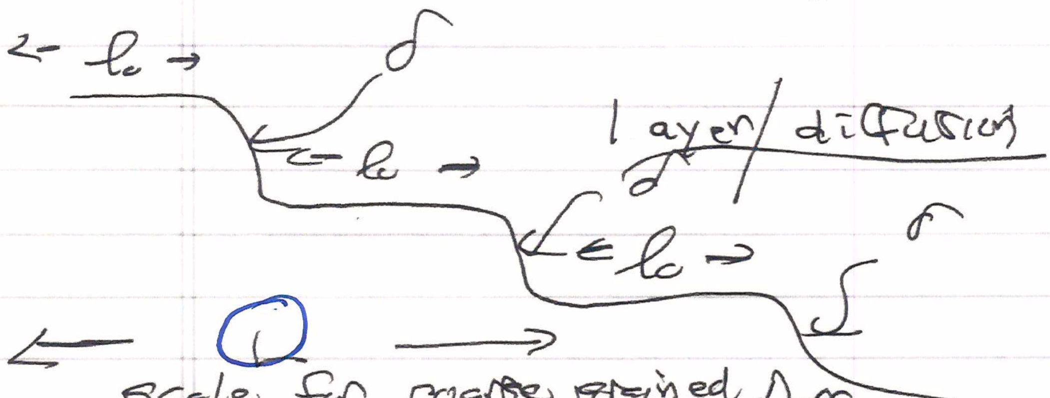
slow → diffusion → operating on boundary layers

What is D_{eff}?

Point: – transport is hybrid of
- fast kicks thru cell
- slow diffn thur BL

– diffusion is ultimate origin of irreversibility for static cells. Only BL particles transported.

Can envision concentration profile:



scale for coarse grained D_{eff} (finite)

"staircase"?!

Heuristic argument:

→ for random walk:

↱ fraction of active region → fraction where diffusion occurs.

$$D_{eff} \cong \text{factor} \, \frac{(\Delta x)^2}{\Delta t}$$

factor: active fraction for diffusion

$\sim \delta / l_0$ small in BL thickness

Δt: cell circulation time

$\sim l_0 / V_0$

then $\boxed{\delta^2 \sim D_0 \Delta t \sim D_0 \frac{l_0}{V_0}}$

Diffusion in transit time thru layer

~~8.~~

and $\Delta x \sim l_0$ (cell scale).

so

$$D_{eff} \cong \frac{\delta}{l_0} \frac{l_0^2}{l_0/V_0}$$

$$\cong \left(D_0 \frac{l_0}{V_0}\right)^{1/2} \frac{\cancel{l_0} V_0}{\cancel{l_0}}$$

$$\cong (D_0 V_0 l_0)^{1/2}$$

$$D_{eff} \cong (D_0 D_{cell})^{1/2} \cong \boxed{D_0 (Pe)^{1/2}}$$

→ D_{eff} is geometric mean of D_0 (slow) and D_{cell} (fast), $\sim Pe^{1/2}$, large Pe

→ resembles Dykhne result, but
⇒ Dykhne → equal areas σ_1, σ_2
cells → $\delta_{active}/l_0 \ll 1$

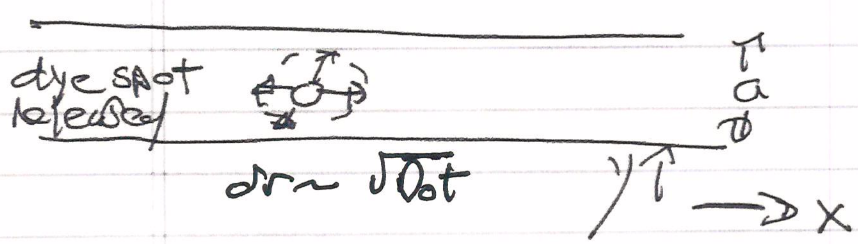
→ see Rosenbluth, et. al. 1987 for details of calculation (tedious).

→ Result is <u>not</u> simple addition

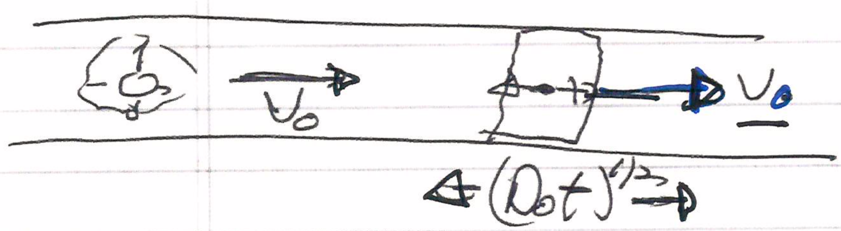
Related problem: Shear dispersion

see Taylor 1953 et seq. (many postings)
Young & S. Jones especially good

Problem stated by comparison of three laminar flows, into which dye with molecular diffusion D_0 is injected.

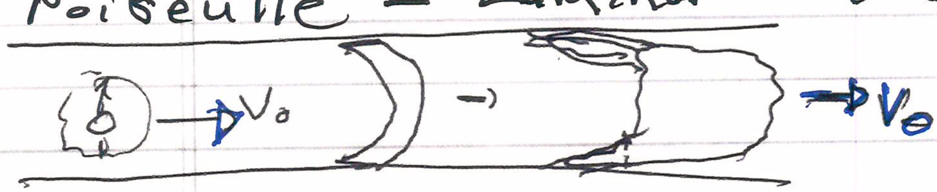


Flow ① stationary water



Pipe ② Laminar Flow at V_0.
slip BC

necessarily a shear flow,
Poiseulle – Laminar (Parabolic)



Pipe ③ No-slip BC

Pipe ②: – slug CM advects at V_0

– slug expands axially at $(D_0 t)^{1/2}$ → molecular diffusion

Shear Dispersion:

→ what is effective along stream diffusivity of passive scalar in a laminar shear flow.

Simplest - Poiseulle Flow
- BC → shear.

" the fundamental character of the result that differential unidirectional convection and transverse diffusion yield a longitudinal diffusion process far downstream "

Laminar Flow

→ but, Pipe ③:

- experiments (cf. 1953 paper) indicate more rapid dispersal of dye in sheared flow, i.e. effective axial diffusion enhanced

$$D_{eff\ axial} > D_0$$

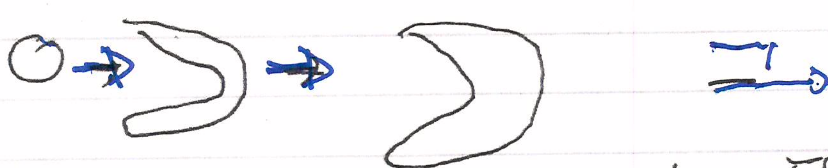
$\Rightarrow D_{eff} = D_0 + D_{shear}$ why?

$\langle\ \rangle \to \perp$ avg

$$\frac{\partial c}{\partial t} + \underline{v}\cdot\nabla c = D_0 \nabla^2 c \to \partial_t \langle c\rangle + \partial_x \langle \tilde{v}\tilde{c}\rangle = D_0 \partial_x^2 \langle c\rangle$$

- though CM velocity same, V_0?!

- velocity shear stretches cloud,



spreading it more rapidly. Effect is shear + D_0

How calculate?!

$\perp$ collisional scattering sets

Consider dye concentration field $c(x,y,t)$

Calculation – simple

Now, consider scalar equation:

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla}\, c = D_0 \nabla^2 C$$

$$\left.\begin{aligned} \langle c \rangle &= \frac{1}{a}\int_{-a/2}^{a/2} dy \; c(x, y, t) \\ \langle v \rangle &= \frac{1}{a}\int_{-a/2}^{a/2} dy \; v(y, x) \end{aligned}\right\} \text{section avg (2D, 3D)}$$

$$\frac{\partial c}{\partial t} + \underline{v} \cdot \underline{\nabla}\, c = D_0 \nabla^2 C \qquad (1)$$
$$= D_0 (\partial_x^2 + \partial_y^2)\, C$$

$$\partial_t \langle c \rangle + \langle v \rangle \partial_x \langle c \rangle + \partial_x \langle \tilde{v} \tilde{c} \rangle \qquad (2)$$

$$= D_0 \partial_x^2 \langle c \rangle$$

$$\tilde{v} = v - \langle v \rangle$$

$$\tilde{c} = c - \langle c \rangle$$

Now, subtract (2) from (1):

$$\partial_t \langle C \rangle + \partial_t \tilde{C} + \langle V \rangle \partial_x \langle C \rangle + \tilde{V} \partial_x \langle C \rangle + \langle V \rangle \cdot \nabla \tilde{C} + \partial_x \langle \tilde{V} \tilde{C} \rangle + \nabla \cdot \widetilde{\tilde{V} \tilde{C}} = D_0 \left(\partial_x^2 \langle C \rangle \right) + D_0 \partial_x^2 \tilde{C} + D_0 \partial_y^2 \langle C \rangle + D_0 \partial_y^2 \tilde{C} \qquad (1')$$

$$\partial_t \langle C \rangle + \langle V \rangle \partial_x \langle C \rangle + \partial_x \langle \tilde{V} \tilde{C} \rangle = D_0 \partial_x^2 \langle C \rangle \qquad (2')$$

subtracting:

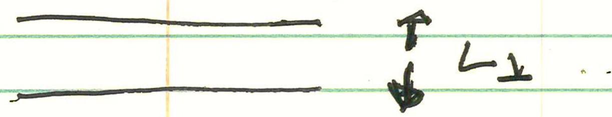
$$\partial_t \tilde{C} + \langle V \rangle \cdot \nabla \tilde{C} - D_0 \nabla^2 \tilde{C} + \tilde{V} \partial_x \langle C \rangle = D_0 \partial_y^2 \tilde{C}$$

Define:

$$\left(\frac{d}{dt} \tilde{C} \right) = \partial_t \tilde{C} + \langle V \rangle \cdot \nabla \tilde{C} - D_0 \nabla^2 \tilde{C}$$

$$\left(\frac{d}{dt}\tilde{c}\right) + \tilde{v}\,\partial_x \langle C \rangle = D_0\, \partial_y^2 \tilde{C}$$

Now,



→ pipe has finite ⊥ scale

→ this defines natural time scale — in frame co-moving with mean flow — of

$$\tau_{\perp \mathrm{diff}} \sim L_\perp^2 / D_0$$

$\left[L_\perp \ll L_\parallel \rightarrow \text{time scale separation} \right]$

For $t \gg \tau_{\mathrm{diff}} \cong L_\perp^2 / D_0$, $\tilde{c}$ diffusively damped.

So, time asymptotically, i.e.

$$t \gg \tau_{\perp\, \mathrm{diff}}$$

have dominant balance:

$$\tilde{V}_x\, \partial_x \langle C \rangle \cong D_0\, \partial_y^2 \tilde{C}$$

Recall: $\tilde{V}_x = V - \langle V \rangle$

Fourier expand:

$$\tilde{c} = \sum_{k_y} e^{ik_y} \tilde{c}_{k_y}$$

$\tilde{v}$ similar.

$k_{y\,min} \cong 2\pi/L_\perp$

$\Rightarrow$

$$\tilde{v}_{x\,k_y} \partial_x \langle C \rangle = -k_y^2 D_0 \tilde{c}_{k_y}$$

$$\tilde{c}_{k_y} = \frac{-1}{k_y^2 D_0} \tilde{v}_{x\,k_y} \partial_x \langle C \rangle$$

so, what is sought:

$$\langle \tilde{v}_x \tilde{c} \rangle = \sum_{k_y} - \frac{|\tilde{v}_{x\,k_y}|^2}{k_y^2 D_0} \frac{\partial \langle C \rangle}{\partial x}$$

so, recalling:

$$\partial_t \langle C \rangle + \partial_x \langle \tilde{v}_x \tilde{c} \rangle = D_0 \partial_x^2 \langle C \rangle$$

so

$$D_{eff} = D_0 + D_{shear}$$

$$D_{shear} = \sum_{k_y} |\tilde{V}_{k_y}|^2 / k_y^2 D_0$$

$$\rightarrow \# \frac{V_0^2 L_\perp^2}{D_0}$$

$$\boxed{\begin{aligned} D_{eff} &= D_0 + D_{shear} \\ &= D_0 + \# \frac{V_0^2 L_\perp^2}{D_0} \end{aligned}}$$

Note that for Laminar flow

$D_{shear} > D_0$ quite possible

$\Rightarrow$ enhanced along stream / in frame diffusivity.

$\Rightarrow$ enhanced dispersion / "shear dispersion"

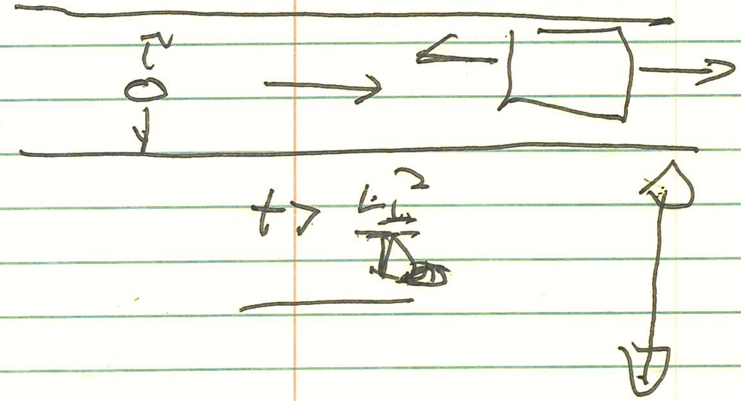
Several Points:

→ What is happening?

"the fundamental character of the result | differential unidirectional convection and transverse diffusion yield a longitudinal diffusion process far downstream"

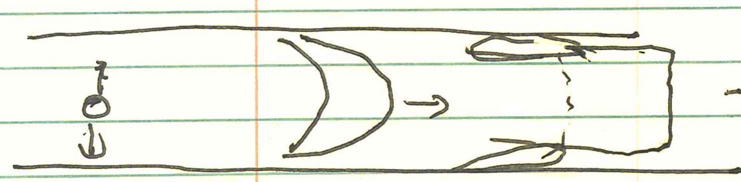
i.e.

$(D_0 t)^{1/2}$



slip

$\underline{t > L_\perp^2 / D_0}$



$\to u_0$

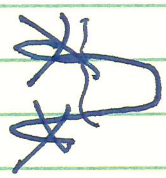
no slip

$\underline{t \gg L_\perp^2 / D_0}$

→ asymptotic result:

$t \gg L_\perp^2 / D_0$

→ uni-directional velocity essential



→ Details:

$$v(y,z) = 2(U/a^2)(a^2 - y^2 - z^2)$$

$$\# = 1/48$$

$$D_{eff} = D_0 + \frac{1}{48}\frac{V_0^2 a^2}{D_0}$$

$D_{sh} > D_{eff}$ for $P_e > \sqrt{48}$

→ For turbulent flow,

$$D_{shear} \sim \# \frac{u_*^2 a^2}{D_{shear}}$$

$$D \sim u_* a \, .$$

More generally:

→ Origin of irreversibility is D_0 (laminar flow). $\tilde{\tau}_c^{-1} \sim D_0/L_\perp^2$

→ Akin cell, laminar flow + molecular diffusion yields transport

→ Taylor proposed shear dispersion as mechanism for distributing nutrients in blood flow.

→ time scale ordering is crucial.

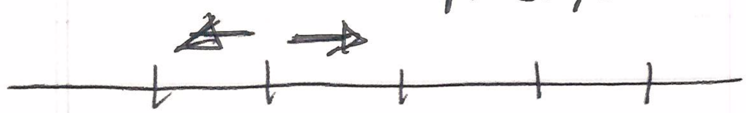
→ Excellent topic for further study.

To percolation →

Percolation processes 1

Percolation vs. Diffusion → Comparison/Contrast

a) Diffusion ⇒ 1D random walk
prob. 1/2 for each way, each step



- Medium fixed
- particle motion stochastic

$\delta x^2 \sim D t$

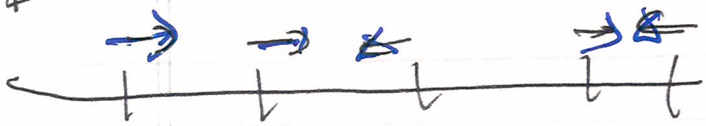
$\sim N$

↳ # steps

and particle can return.

b.) Percolation – assign left/right orientation to each site, with probability 1/2

particle ⇒



etc.

- Medium stochastic
- particle motion deterministic

i.e. random conductivity

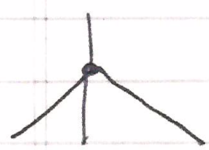
Simply: → diffusion: medium deterministic, motion stochastic — $Ku < 1$

→ percolation: motion deterministic, medium stochastic — $Ku > 1$

Examples: ~ "random" media at low, high Ku.

a.) Cascade process

$(p+q) = 1$

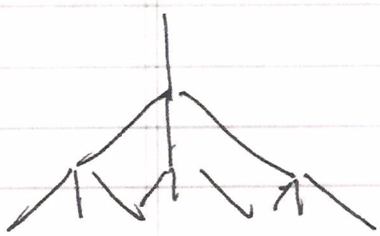


0 desc. 1 desc. 2 desc

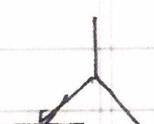
q^2 $2pq$ p^2.

descendants

can think of as diffusion

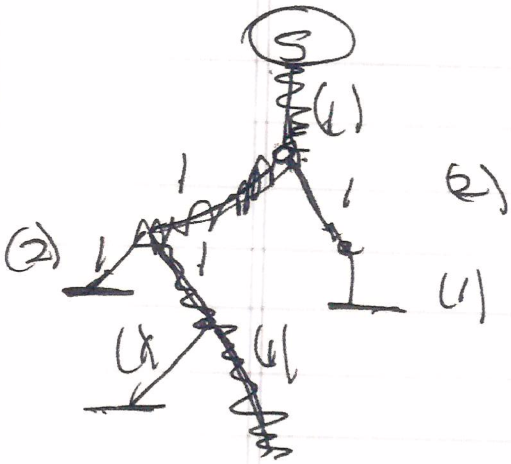


or, does one generation "reach" N_1 into future generations.



0 1 or 2

$\underline{\underline{\infty}}$



Σ probability of blocked

p of 1, 2 cont.

percolation

In percolation:

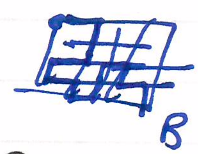
→ intrinsic and random properties of medium determine motion. i.e. cell arrangement

→ in diffn / stoch. motion indeterminate. Prop. particle fundamental (i.e. stochastic orbits) ..

Origin of random characteristics of medium:

→ randomly dam / cut connections
→ dimensionality matters ☺

→ result is random maze



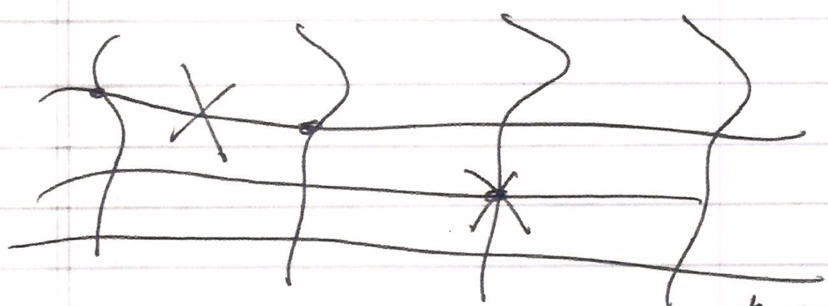
→ flow can traverse A→B only if there is an un-dammed, un-cut self-avoiding random walk connecting A, B

SAW visits intermediates at most once.

→ General aspects (mostly topological)

– can have "bond" or "site" percolation

i.e



bond → some fraction of bonds/connections missing

site → some fraction of sites missing

⇒ game played either way.

– Consider:

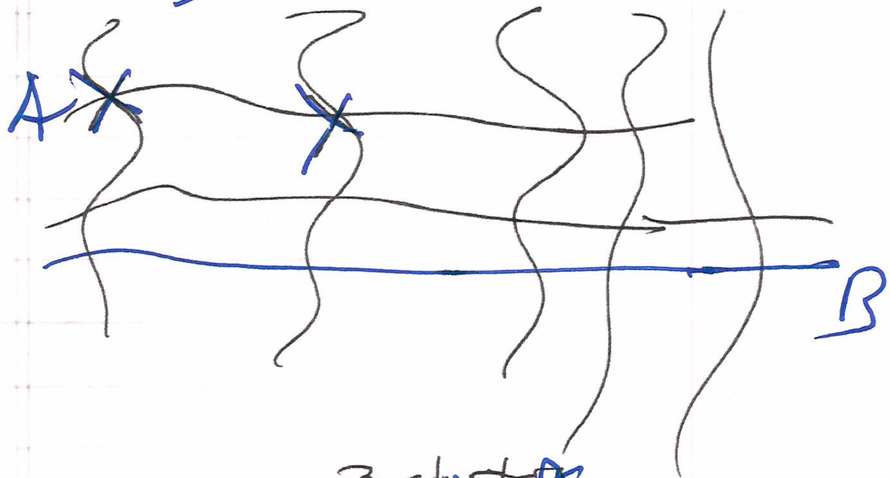
→ lattice of N sites, $N \gg 1$

→ concentration of allowed sites X

→ can envision increasing X from below, i.e.

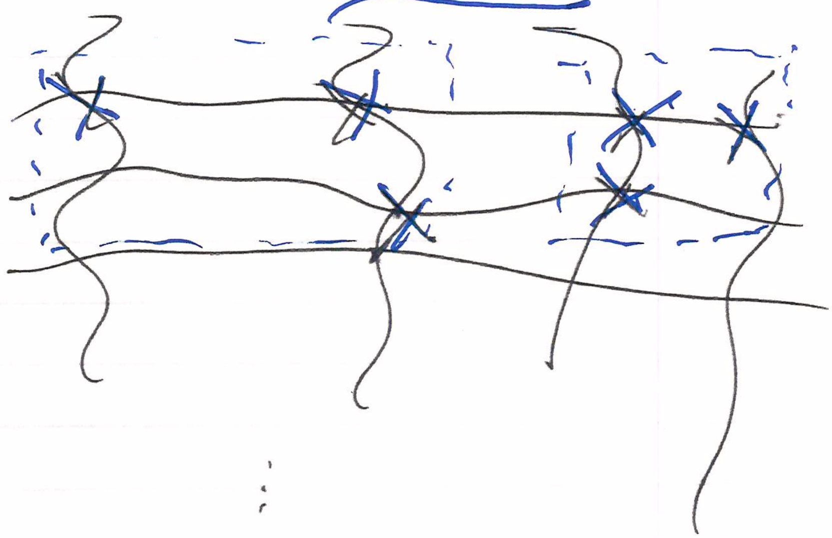
i.e.

2 cluster



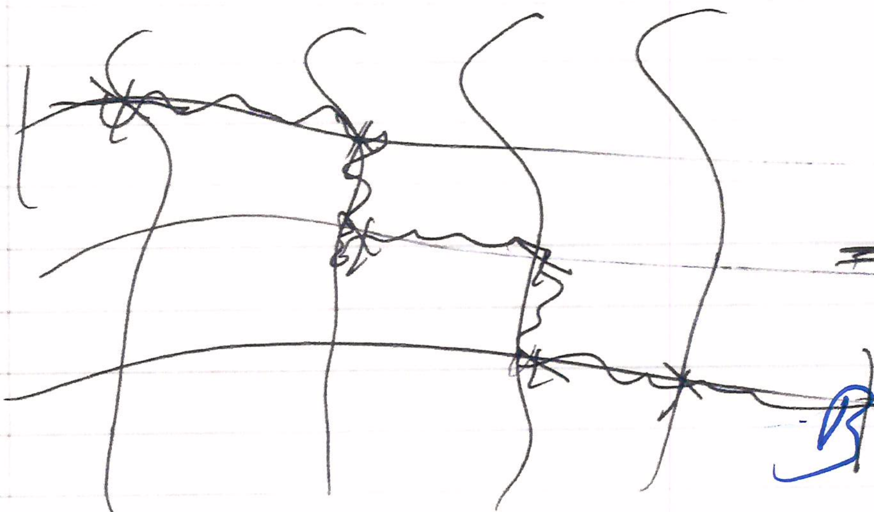
X_r

3 clusters



$X_2 > X_1$

$N >> 1$



connecting
⇒ cluster
spans
network

can traverse.

∞ $N \to \infty$.

- $x \ll x_c \to$ isolated small clusters
- $x \uparrow \to$ larger clusters form.
- $\ell(x) \uparrow$ with x
  size
- as $x \uparrow$, few large clusters form
- $\ell(x_c) \to \infty$ as $N \to \infty$
- one cluster for $x > x_c$

$\boxed{\text{phase transition}}$

$P^s(x) \equiv$ site percolation probability

$\equiv$ ratio of # sites in big (infinite) cluster to # sites in lattice

$\equiv$ fraction of system in which DC conduction is possible.

And:

- there is a threshold concentration of active sites, x_c
- near threshold:

  $\rho^s(x) \sim (x - x_c)^s$ → perc. exponent

  percolation exponent

  fraction of conduction

  geometry

  universality
- 3D: $s \sim .3 \to .4$

  corresponding lattice structure

⇒

- percolation is a type of phase transition

  i.e., $s \leftrightarrow$ critical exponent

  $x_c \leftrightarrow$ critical temp / pt.

  $\ell_A \sim \ell_c$.
- key question is effective medium model near x_c.
- connection to turbulence: intermittency → reduced packing fraction.