Physics 235

Notes 4

→ To $k_\parallel > 1$.

→ Recall, have been concerned with transport and diffusion.

Focus:

$$D_M = \int_0^\infty d\ell \sum_k \left|\frac{\delta B_k}{B_0}\right|^2 e^{ik_\parallel \ell}$$

→

$$\sim \left\langle \left(\frac{\delta B}{B_0}\right)^2 \right\rangle \ell_{ac}$$

Scattering: $k_\parallel = 0$ resonances.

$$\rightarrow k_\parallel \sim \frac{\ell_{ac}}{\Delta} \frac{\delta B}{B_0} \sim \frac{1}{\Delta} \delta B / B_0 / [\Delta(k_\parallel)]$$

→ What Happens for $Ku \geq 1$?

— Recall:

~ Stoch Fields

$$Ku \sim \frac{l_{ac}}{\Delta_\perp} \delta B/B_0 \sim \frac{1/\Delta_\perp \; \delta B/B_0}{\Delta |k_\parallel|}$$

$$\sim l_{ac}/l_c$$

$$\sim 1/l_{NL} \,/\, 1/l_{ac} \sim l_{ac}/l_{NL}$$

ratio of autocorrelation to NL mixing length

Large Ku → NL scatt. processes control time/sp. scale.

~ flow

$$Ku \sim \tilde{\tau}_{ac}\, \tilde{V}/\Delta \sim 1/\tilde{\tau}_{NL} \,/\, 1/\tilde{\tau}_{ac} \sim \tilde{\tau}_{ac}/\tilde{\tau}_{NL}$$

$\tilde{V}/\Delta$

ion

i.e. collisional DW's:

$$\tilde{\tau}_{ac} \sim (\Delta |\chi_\parallel k_\parallel^2|)^{-1}$$

$$Ku \sim 1/\tilde{\tau}_L (\Delta |\chi_\parallel k_\parallel^2|)$$

$Ku > 1$

2D GC Plasma — Simple / Compelling Example. B

cf: Taylor + McNamara.

so Point:

$$D_\perp \simeq \int_0^\infty d\tau \, \langle \tilde{V}(0) \tilde{V}(\tau) \rangle$$

Diffusion coeff as integral of correlation (i.e. time history ↔ resp.)

$$\sim \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 \underbrace{R(\tau)}_{\text{Memory fctn}}$$

Diffusion coefficient as integral of correlation function

$$R(\tau)_\omega = e^{+i(\omega - k_\parallel v_\parallel)\tau} \, e^{-\tau/\tau_c}$$

From wpd → scattering

$Ku \gg 1$ limit corresponds to:

- → $k_\parallel \to 0$
- → $\omega \to 0$

⇒ time integral controlled by nonlinear scattering, not wave packet dispersion.

⇒ 2D GC Plasma / Fluid:

i.e.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\rho = D_0 \nabla^2 \rho$$

$$\nabla^2 \phi = -4\pi \rho$$

(Taylor + McNamara)

→ 2D Fluid

→ GC Plasma.

Then generally;

$$D_\perp \cong \int_0^\infty d\tau \sum_{\underline{k}} |\tilde{V}_k|^2 e^{-i\underline{k}\cdot\underline{r}_0} e^{+i\underline{k}\cdot\underline{r}(-\tau)}$$

pupo

$$= r_0(-\tau) + \delta r(-\tau)$$

but $\underline{r}(-\tau) = \underline{r}_0 + \underline{\delta r}(-\tau)$

only evolution off of $\underline{r}_0$ → scattering

δr → stochastic

→ need ensemble average

$$D_\perp \cong \int_0^\infty \sum_k |\tilde{V}_k|^2 \left\langle e^{i\underline{k}\cdot\underline{\delta r}(-\tau)} \right\rangle d\tau$$

$\langle D_\perp \rangle$

$$\cong \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_\perp \tau} d\tau$$

turbulence/scatt. itself controls correl. time.

e.g.,

$$\left\langle e^{i\underline{k}\cdot\underline{\delta r}(-\tau)} \right\rangle \cong \left\langle \left(1 + \cancel{i\underline{k}\cdot\underline{\delta r}(-\tau)} - \frac{(\underline{k}\cdot\underline{\delta r}(\tau))^2}{2} + \dots \right) \right\rangle$$

$$\cong \left\langle \left(1 - \frac{(k_\perp\cdot\delta r)_\perp^2}{2} \right) \right\rangle$$

$$\cong \left\langle \left(1 - k_\perp^2 D_\perp \tau - \right) \right\rangle$$

$$\cong e^{-k_\perp^2 D_\perp \tau}$$

So

$$D_\perp = \int_0^\infty d\tau \sum_k |\tilde{V}_k|^2 e^{-k_\perp^2 D_\perp \tau} d\tau$$

$$= \sum_k |\tilde{V}_k|^2 \frac{1}{k_\perp^2 D_\perp}$$

compare to dispersion

can immediately note:
[illegible]
— $D_\perp$ controls τ_c
— conservation $\partial_t \int \phi \, d^2x \to 0 \Rightarrow$ slow τ_c at large scale

N.B. → note recursive structure of diffusivity!

→ τ_c in integral set by scattering $\sim 1/k_\perp^2 D$, $k_\perp^2$ from conservation (I.C.)

$\Rightarrow$ → $\underline{V} \sim \frac{c}{B} \underline{E} \times \hat{z}$ → $D_\perp \sim 1/B_0 \Rightarrow$ Bohm

$$\boxed{(D_\perp)^2 \cong \sum_k |\tilde{V}_k|^2 / k_\perp^2}$$ → recursive definition

but 2D, assuming symmetric spectrum

$$D_\perp^2 \cong \int dk_\perp \cancel{k_\perp} |\tilde{V}_k|^2 / k_\perp^{\cancel{2}}$$

$\Rightarrow$

$$D_\perp^2 \cong \frac{c^2}{B_0^2}\int dk_\perp \frac{|E_k|^2}{k_\perp}$$

$$\boxed{D_\perp \sim \frac{c}{B_0}\left(\int dk_\perp \frac{|E_k|^2}{k_\perp}\right)^{1/2}}$$ $\Rightarrow$ Diffusivity

$\sim 1/B_0$

Now, can explore different spectra:

~ thermal equilibrium: (what TPM)

$\Rightarrow$ thermal

$$|E_k|^2 = \frac{4\pi}{\ell} k_b T \Big/ (1 + k^2 \lambda_D^2)$$

$e/\ell \to$ charge/length. $\hookrightarrow$ Debye screening

$$= \frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_b T \Big/ (1 + k_\perp^2 \lambda_D^2)$$

$$D_\perp \sim \frac{c}{B_0}\left(\int_{1/L_0}^{1/\lambda_D} dk_\perp \left[\frac{4\pi}{e}\left(\frac{e}{\ell}\right) k_b T \Big/ \left((1 + k_\perp^2 \lambda_D^2)\, k_\perp\right)\right]\right)^{1/2}$$

$$\sim \frac{c k_b T}{eB}\left[(n\lambda^2)^{-1} \ln(L_0/\lambda)\right]^{1/2}$$

so

$$D_\perp \sim D_B \left[(n\lambda_D^2)^{-1} \ln(L_0/\lambda) \right]^{1/2}$$

- recovers basic Bohm scaling, even from thermal flcts
- scales (weakly) with L_0 ⇒ not pure local, or "intensive".

⇒ simple example of "non-locality".

- "non-locality" appears from "slow mode"
  i.e. $1/\tau_c \to 0 \sim k_\perp^2 D_\perp$ as $k_\perp^2 \to 0$

@ is conserved → "conserved order parameter" $\omega \sim k_\perp^2 D_\perp$

- if shear flow:

$$R(t) = \int_0^\infty d\tau \, \exp\left[i(\omega - k_\theta V_\theta)\tau - \tau/\tau_c \right]$$

Interesting to note:

- can consider diffusion due random array charges ("rods")

For spectrum:

$$\nabla \cdot E = 4\pi\rho$$

$$= \frac{4\pi}{\ell} \sum_i q_i \, \delta(x - x_i)$$

$$i k \cdot E_k = \left(\frac{4\pi}{\ell}\right) \sum_i q_i \, e^{-i k \cdot x_i}$$

symmetric distribution $\Rightarrow$ {random array discrete charges

$$|E_k|^2 = \frac{1}{k_\perp^2}\left(\frac{4\pi}{\ell}\right)^2 \left\langle \sum_{i,j} q_i q_j \, e^{i k \cdot (x_j - x_i)} \right\rangle$$

$$= \frac{16\pi^2 n q^2}{k_\perp^2 \ell^2}$$

$$\longrightarrow 1/k_\perp^2$$

$$D_\perp^2 \sim \frac{c^2}{B_0^2} \int dk_\perp \, \frac{16\pi^2 n q^2}{k_\perp^2 \ell^2 k_\perp^2} \, k_\perp$$

$$\sim \frac{c^2}{B_0^2} \int_{k_{min}}^{k_{max}} \frac{16\pi^2 n q^2}{k_\perp^3 \ell^2} \sim \frac{c^2}{B_0^2} \left(\frac{16\pi^2 n q^2}{\ell^2}\right) \frac{1}{k_{\perp\,min}^2}$$

$k_{\perp \min} \sim 1/L_0$

system size

$$\boxed{\Delta_\perp \simeq \frac{c}{B_0}\left(4\pi \frac{n q^2}{f}\right)^{1/2} L_0}$$

strong dependence on system size.

contd

Stochastic Fields — Toward High Ku; Random Conductivity

→ So far:

- reviewed theory of Hamiltonian chaos
- derived QL D_M
- derived $\chi_{\perp e}$ due stochastic fields in $Ku < 1$ regime — diffusion
- focused on interaction of scattering (fluctns), collisions, coarse graining
- discussed transport in $\dot{G}C$ plasma, as example of $\tau_{ac} \to \infty$ regime.

→ Observations

- idea of resonance (small denominator problem) and resonance overlap fundamental to Hamiltonian chaos.
- $Ku \sim \tau_{ac}/\tau_{scatt.}$
- might ask: unified treatment that combines $Ku < 1$, $Ku > 1$ regimes ⇒ renormalized response ?
- in hydro treatment of $\chi_{\perp eff}$ what of nominal 3rd order contribution vs $\chi_{\perp, colln.}$. See (K+D) ↓
- is diffusive treatment of high Ku regime (as in Taylor + McNamara) valid ? See Isichenko Gruzinov papers.
- $\tau_{ac} \to \infty$ can recover strong $D_\perp$ at modest fluctn. level.

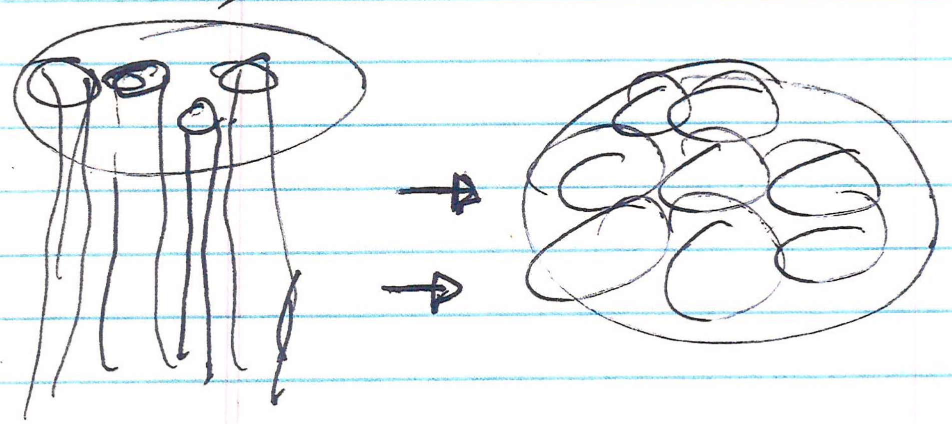
Here:

- general analysis of diffusion
- aspects of percolation, large K_u regime
- Dykhne method → conduction in random media.

⇒ Recall, $K_u \sim l_{ac}\, \delta B / B \Delta_\perp$

- have considered low K_u, with
  - finite l_{ac}
  ⇒ inhomogeneity in z

- now, $\boxed{\text{consider } K_u \to \infty \text{ limit, opposite}}$
  ⇒ random field, in x, y.
  ⇒ homogeneous in z



i.e. akin rods

i.e.
$$\begin{cases} \dfrac{dx}{dz} = b_r = \dfrac{\partial A}{\partial y} \\[2ex] \dfrac{dy}{dz} = b_\theta = -\dfrac{\partial A}{\partial x} \end{cases}$$

from: $\dfrac{dr}{dz} = b_r$

$$r\frac{d\theta}{dz} = \cancel{\frac{\langle B_\theta \rangle}{B_0}} + b_y$$

equivalent, of course, $(L_s \to \infty)$ to G.C. plasma:

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{c}{B}\partial_y \phi \\[2ex] \dfrac{dy}{dt} = \dfrac{c}{B}\partial_x \phi \end{cases}$$

$\Rightarrow$ motivates study of random media transport!

Formally can extend D_M calculation to include resonance broadening

i.e.

$$\frac{\partial \tilde{f}}{\partial z} + b \cdot \nabla \tilde{f} = -\tilde{b}_r \frac{\partial \langle f \rangle}{\partial r}$$

$\Rightarrow$ on beloved model

$$D_M = \sum_k |\tilde{b}_{r,k}|^2 \frac{i}{k_\parallel + i k_\perp^2 D_M}$$

where

$k_\perp^2 D_M / k_\parallel \sim k_u^2$

$$D_M = \sum_k \int_0^\infty e^{i k_\parallel l} e^{-i k \cdot \delta r(l)}$$

$$\left\langle \frac{\delta B}{B_0} \right\rangle_k$$

$$D_M = \int_0^\infty dl \left\langle \frac{\delta B_r(0)}{B_0} \frac{\delta B_r(l)}{B_0} \right\rangle$$

For $K_u < 1$ $\Rightarrow$

$$D_M \simeq \sum_k |\tilde{b}_{r,k}|^2 \delta(k_\parallel) \quad , \text{ à la RSTZ}$$

For $K_u \gg 1$

$$D_M \simeq \sum_k |\tilde{b}_{r,k}|^2 / k_\perp^2 D_M$$

à la Taylor, McNamara.

$\Rightarrow$

$$D_M \cong \left(\sum_{k} |\tilde{A}_k|^2 \right)^{1/2}$$

$$\sim \tilde{b}\,\Delta$$

So

$Ku < 1 \Rightarrow D_M \sim \tilde{b}^2 l_{ac}$

$Ku > 1 \Rightarrow D_M \sim \tilde{b}\,\Delta$

and transport $\sim \langle A^2 \rangle^{1/2}$

$\circledast$ But, is $Ku > 1$ regime ~~really~~ always diffusive ?

$\rightarrow$ recall: $\dfrac{d\underline{x}}{dz} = \underline{\nabla}_\perp A \times \hat{z}$

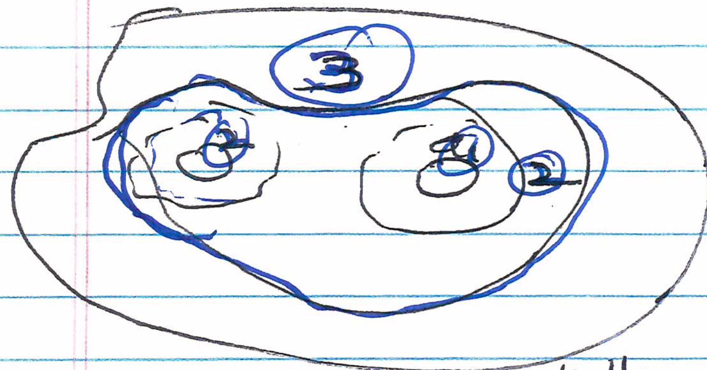
{ akin 2D random media, for
{ A indep z.

simple

$\rightarrow$ Can view physically as:

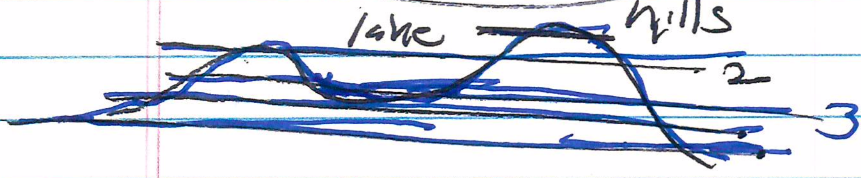
topographical map

i.e.



Map

(what with ambient diffn?)



levels

Now, as $\dfrac{dx}{\partial_y A} = \dfrac{-dy}{-\partial_x A} = \dfrac{dz}{1}$

$\Rightarrow$

$$dy/dx = -\partial_x A / \partial_y A$$

$\Rightarrow$

$$\boxed{\underline{\nabla} A \cdot \underline{dx} = 0}$$

$\Rightarrow$ – Lines traverse const A contours, as on map

– $\langle A \rangle = 0$, $\langle A^2 \rangle = A_0^2$

– avg. depth, height of "lakes", "hills" set by A_0

— most contours closed, isolated
⇒ little contribution to transport

— but contours along "passes"

, i.e 3, can take on long

path lengths.

⇒ transport occurs primarily along
there

⇒ $\boxed{\text{percolation, not diffusion}}$

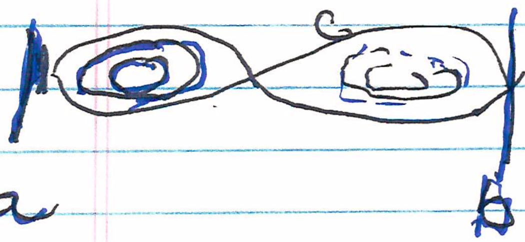
hot

cold

c

a

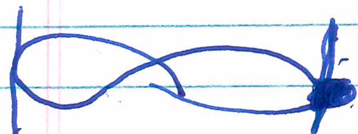
b



a → b transport
isolated along
contour c.

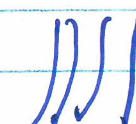
~ more like 'lightning bolt' than
diffusion. Heat "channeled" along C.

~ signature would be sharply
localized strike mark (if b → PFC)
and not periodic or smooth.



vs

spread
load



strike

→ percolation → extension of mean length as $A \to 0$

$$\boxed{l_A \sim A^{-\gamma}}$$

, ~~$\gamma \approx 2.4$~~

→ replaces concept of m.f.p.

Message:

- to understand $k u > 1$ regime, useful to examine:

→ transport in random media –

→ percolation theory –