Closures and 'Noisey Burgulence'

- Issues in Turbulence
- OV of Closures
- Noisey Burgers
- Response Function
- Time Scales
- Spectral Equation

A Quick Look at Closures

⇒ Turbulence, so far:

- satisfied, from "physicists' perspective"?

- ① scalings — rooted in phenomenology?!

  ② mixing length models — also rooted in phenomenology!?

$\nu_T = u_T l$

$\langle V_1 V_2 \rangle = -\nu_T \frac{\partial u}{\partial x}$

⇒ where have Navier-Stokes Equations gone?

⇒ Might one:

- derive eddy viscosity
- derive K41 spectrum

from some systematic procedure starting from NSE?

⇒ Apply to more complex problems → MHD, stratified turbulence etc.

⇒ Framework.

~~59a.~~ 3.

References on Closures: See Physics 216

- Kraichnan 59 → Basics of DIA
- Kraichnan 61 → Random Coupling Model
- Kraichnan 76 → Test Field Model

→ Forster, Nelson, Stephen '77 → Forced Burgers Turbulence

→ Hunt '90 → Rapid Distortion Theory.

~~3.~~

Spectral Equation

so

viscous damping

turbulent viscous damping

$$\left(\frac{\partial}{\partial t} + \nu k^2\right)\langle \tilde{V}^2\rangle_k + 2\sum_{k'}(k+k')^2\,\Theta_{k,k',k+k'}\,\langle \tilde{V}^2\rangle_{k'}\,\langle \tilde{V}^2\rangle_k$$

$$= S_k + 2\sum_{p,q}(p+q)^2\,\Theta_{k,p,q}\,\langle \tilde{V}^2\rangle_p\,\langle \tilde{V}^2\rangle_q$$

random stirring

mode-coupling induced stirring — nonlinear noise. ($>> S_k$ in inertial range)

- structure is that of Langevin equation, with noise <u>and</u> drag renormalized, of course same origin.

i.e. $$\frac{\partial \tilde{V}}{\partial t} + \underbrace{\mu}_{\text{Stokes drag}} \tilde{V} = \underbrace{\tilde{f}}_{\text{thermal noise}}$$

$$\mu = \frac{6\pi\eta}{\rho}a$$

$$\Rightarrow$$

NL noise

$$\frac{\partial E_k}{\partial t} + \underbrace{\nu^T}_{\text{turbulent viscosity}}\langle \tilde{V}^2\rangle_k = S^T_{p+q=k}$$

- energetics — does renormalized theory respect primitive equations?

- $\sum_k T_k \overset{?}{=} 0$

$$\sum_k T_k = \sum_k \sum_{k'} 2(k+k')^2 \theta_{\substack{k,k' \\ k+k'}} \langle \tilde{V}^2 \rangle_{k'} \langle \tilde{V}^2 \rangle_{k}$$

$$- \sum_k \sum_{\substack{p,q \\ p+q=k}} 2(p+q)^2 \theta_{\substack{p,q \\ p+q=k}} \langle \tilde{V}^2 \rangle_p \langle \tilde{V}^2 \rangle_q$$

$= 0$
(re-label)

RPA, $\theta_{k,p,q}$ → Molec. Chaos

Comment: Equilibrium closure as $\underline{\underline{C(\theta)}}$ → H thm ⟺ ⇒ relaxation to equipartition spectra (stat. mech).

N.B.: Upon summation, coherent damping conserves energy vs. incoherent emission.

$\underline{\underline{\text{i.e.}}}$ cascade as sequence of coherent damping → incoherent emission → coherent damping → ...., a la' band models.

Closure zoology: based upon use of coupled response fctn, spectral eqns

① i.e. $\frac{\delta V}{\delta f}$ response fctn ⟷ depends on spectra $\langle \tilde{V}^2 \rangle_{k'}$

② $\frac{\partial}{\partial t} \langle \tilde{V}^2 \rangle_k$ depends on C_k, L_k, etc.

→

DIA: solve $\underline{\text{coupled}}$ equations for $\delta V / \delta f$ and $\langle \tilde{V}^2 \rangle_k$

EDQNM : parametrize C_k in terms $\langle \tilde{v}^2 \rangle_k$, yielding spectral equation

Eddy viscosity models / R! B.T. : dV/dt equation

Weak Turbulence : neglect $C_{k+k'}$ in $L_{k+k'}$.

⋮

Comments on Closures:

- consistent with conservation laws, albeit trivially;

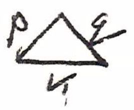
- based upon assumed weak coupling / RPA hypothesis (The Swindle Occurs Here!)

$$N \sim C_{k+k'} V_{k+k'} + (\;) V_k V_{k'}, \text{ etc.}$$

- $\omega_{\substack{\text{triad}\\ \text{decorreln.}}} = \sum (\omega_{decorr})_{triad}$

- no restriction on shape of interacting triad, i.e. → confusion of $\left\{ \begin{array}{l} \text{sweeping} \\ \text{strainings} \end{array} \right.$?

$p + q = k$



, etc. ⟺ sweeping ?

→ Foundations of the D.I.A. and Issues in Turbulence Closure (R.H. Kraichnan, J. Math Phys. 2, 124 (1961)).

① – reprise of the D.I.A. and the D.I.A. propagator for N.-S. T.

② – stochastic oscillator models ↔ general structure

③ – random coupling model and the problem of realizability

① <u>Reprise</u>

Recall D.I.A. → coupled equations for
- propagator
- spectrum.

Interesting to note:

→ essential physics is nonlinear scrambling in triad coherence (i.e. sets coherence time)

→ spectrum } sets scrambling time ⇒ cascade dynamics

sets response → propagator

Useful to note for later that for $N=S$, Eq 1;
D.I.A. for propagator evaluation gives;

$$\underbrace{\frac{\partial g(k,\tau)}{\partial \tau}}_{\text{propagator fctn.}} + \overset{\text{molecular viscosity}}{\nu} k^2 g(k,\tau) \qquad \underline{k}+\underline{p}+\underline{q}=0$$

$$= -\frac{k}{2}\int d\underline{p}\int d\underline{q}\, \frac{p}{q}\, \underbrace{b(k,p,q)}_{\text{coupling coeff}}\, \underbrace{E(q)}_{\text{background energy}} \int_0^{\overset{\text{non-Markovian structure}}{\tau}} ds\, g(k,\tau-s)\, \underbrace{g(p,s)}_{\text{test-wave response propagator (from closure)}}\, \underbrace{N(q,s)}_{\text{self-correlation of backgnd. spectrum}}$$

Can simplify using:

a) $E(q)$ largest at small q → energy containing range.

b) and $\underline{p}+\underline{q}=\underline{k}$ ⇒ $|\underline{p}| \sim |\underline{k}| \gg q$ (selection rule)

c) $N(q,s) \cong N(q,0)$ → i.e. large eddys have long lifetimes, treat as slow, relative to high k response.

so ...

$$\frac{\partial g(k,\tau)}{\partial \tau} + \nu k^2 g(k,\tau)$$

$$= -\underbrace{k^2 V_0^2}_{} \int_0^\tau g(k,\tau-s)\, g(k,s)\, ds = 0$$

↳ non-Markovian - convolution

$$\underbrace{k^2 V_0^2}_{\text{effective straining/sweeping (?) time}} = +\frac{k}{2}\int dp \int dq \; \frac{p}{q}\, b(k,p,q)\, E(q)$$

Can solve via Laplace transform (n.b. convolution!) so:

$$\boxed{g(k,\tau) = e^{-\nu k^2 \tau} J_1(2kV_0\tau)/kV_0\tau}$$

"trademark" D.I.A. propagator

Some observations:

i) sweeping vs straining → physics of eddy lifetime !? → V_0^2 ?!?

ii) $g(k,\tau)$ oscillates ?! - physical meaning ?

iii) ultimately gives $E(k) \sim k^{-3/2}$, not $E(k) \sim k^{-5/3}$.

see ·59a for Refs.

Closures and Renormalization - Overview

Refs. → W.D. McComb: "The Physics of Fluid Turbulence"
"Renormalization - A Guide for Beginners."

and see postings

Closures

⇒ object of closure to derive equations for observables of turbulence from Navier-Stokes Eqn. - dynamics, not just geometry ....
(contrast fractals)

→ observables typically:
- response function
- spectrum

{ (effective eddy viscosity), time scale

i.e. (low order moments) → not full pdf...

→ procedure is perturbative / RPT (a la' QLT, mean field theory)

⇒ closure methodology usually involves:

a) RPA / weak coupling approximation
Test field model

i.e. $$\frac{\partial}{\partial t} a_{\underline{k}} + \gamma_{\underline{k}} a_{\underline{k}} + \sum_{\underline{k}'} C_{\underline{k},\underline{k}'}\, a_{\underline{k}+\underline{k}'}\, a_{-\underline{k}'} = f_{\underline{k}}$$

generic NL model eqn.

$$|a_{\underline{k}}|^2 = E(\underline{k})$$

$$\frac{\partial}{\partial t} E(\underline{k}) + \gamma_{\underline{k}} E(\underline{k}) + \sum_{\underline{k}'} C_{\underline{k},\underline{k}'}\, a_{-\underline{k}}\, a_{-\underline{k}'}\, a_{\underline{k}+\underline{k}'}$$

i.e. $$\frac{\partial}{\partial t} \langle a^2 \rangle_{\underline{k}} \sim \langle a a a \rangle$$

Key issue → i.e. coupled moment hierarchy; how treat.

and moment hierarchy $\Rightarrow$

$$\frac{\partial}{\partial t}\langle a^3\rangle \sim \langle aaaa\rangle$$
$$\sim \langle a^2\rangle\langle a^2\rangle$$
$$\uparrow$$

- application of RPA to $\langle a^4\rangle$
- on $\langle a^4\rangle \sim \langle c\,c\,a\,a\,a\,a\rangle$

quasi-Gaussian $\sim |c|^2\langle a^2\rangle^2$

(random coupling)

b) to renormalization

$$\langle a^3\rangle \sim \tau_c\langle a^2\rangle\langle a^2\rangle$$
$$\downarrow$$

what controls this?

- if simple perturbation theory,
  (→ is this physical?)

$1/\tau_c \sim \nu k^2$, necessarily

$\Rightarrow \tau_c \sim (\nu k^2)^{-1} \to \infty$, relative to inertial range time scales $O(\tau_c)$

so

$$\frac{\partial}{\partial t}\langle a^2\rangle \sim \tau_c\langle a^2\rangle\langle a^2\rangle$$
$$\Downarrow$$

Too much energy transfer due to spectral depletion

transfer unphysically large, due to long correlation times (also unphysical) response time $\rightarrow$ eddy visc

12, 6/

- mindless perturbation theory yields unphysically long correlations $\Rightarrow$ $\frac{\partial}{\partial t}\langle a^2\rangle$ large $\Rightarrow$ $E < 0$ results!
  ∴ unphysical! $\rightarrow$ "realizability problem" $\rightarrow$ model

must "renormalize" $\tau_c \rightarrow (\nu_T k^2)^{-1}$ (i.e. treat
$\rightarrow$ time-scale self-consistently), so that modal coherence consistent with inertial range scrambling rate!

Example: Burgerlence (Driven Burgers/KPZ Equation)

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \nu \partial_x^2 v = f$$

$\uparrow$ stochastic forcing

n.b.: perturbative closure will completely miss shock formation physics.
[Pdf(v') asymmetry!]

$\boxed{\frac{d}{dt} v = -v^2}$ c.f. Saffman '68

a.) Response function

NL Langevin Equation

$$\frac{\partial v_k}{\partial t} + \frac{ik}{2}\sum_{k'} v_{-k'} v_{k+k'} + \nu k^2 v_k = f_k(t)$$

Eddy viscosity!

now, seek $\delta v_k/\delta f_k$ $\Rightarrow$ response function for mode k.

key physics: space/time scales.

For Re << 1,

$$\frac{\partial}{\partial t} V_k + \nu k^2 V_k + i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'} = f_k(t)$$

$$(-i\omega + \nu k^2) V_k = f_{k,\omega}$$

$$R_{k,\omega} = dV_{k,\omega}/df_{k,\omega} = 1/-i\omega + \nu k^2$$

⇒ time scale set by viscosity !?

for Re >> 1 ⇒ idiotic! → need faster time scale

∴
- need extract effective time-scale from nonlinearity
- physics is time scale of nonlinear scrambling/coupling — NL response → how calculate?

i.e. $$\frac{\partial}{\partial t} V_k + \nu k^2 V_k + C_k V_k = f_k(t)$$

$$\overset{\Updownarrow}{}$$

- reflects $i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'}$ physics
- phase coherent with f_k

$$C_k V_k \Leftrightarrow i\frac{k}{2}\sum_{k'} V_{-k'} V_{k+k'}$$

i.e. seek response of test mode wave, interacting with rest of turbulent spectrum....

so, in lowest order

$$C_k \sim |V|^2$$ (C phase independent)

Now, to calculate C_k;

$$(-i\omega+\nu k^2)V_{\substack{k\\ \omega}} + i\frac{k}{2}\sum_{k'} V_{\substack{-k'\\ -\omega'}} V_{\substack{k+k'\\ \omega+\omega'}} = f_{\substack{k\\ \omega}}$$

$V_{\substack{k+k'\\ \omega+\omega'}} \to V^{(2)}_{\substack{k+k'\\ \omega+\omega'}} \Leftrightarrow$ V driven by direct beat interaction of V_k, $V_{k'}$ (hence DIA)

$$(-i\omega+\nu k^2)V_{\substack{k\\ \omega}} + ik\sum_{k'} V_{\substack{-k'\\ -\omega'}} V^{(2)}_{\substack{k+k'\\ \omega+\omega'}} = f_{\substack{k\\ \omega}}$$

where: $ik\sum_{k'} V_{\substack{-k'\\ -\omega'}} V^{(2)}_{\substack{k+k'\\ \omega+\omega'}} \equiv C_{\substack{k\\ \omega}} V_{\substack{k\\ \omega}}$ (5.+)

so, when calculated:

$$(\partial f_{k,\omega}/\partial V_{k,\omega})^{-1} = 1/\Big(-i\omega + \overbrace{\underset{\substack{\uparrow\\ \text{bare}}}{\nu} k^2 + \underset{\substack{\uparrow}}{C_{k,\omega}}}^{\text{dressed viscosity}}\Big)$$

limits reso $\longrightarrow$ reflects inertial range scrambling

[fast-field hypothesis]

Now, to calculate: NL scrambling rate (self-consistent) → [all other interaction then those selected]

$$\Big(-i(\omega+\omega') + \nu(k+k')^2 + C_{\substack{k+k'\\ \omega+\omega'}}\Big) V^{(2)}_{\substack{k+k'\\ \omega+\omega'}}$$

$$= -i\frac{(k+k')}{2}\left(V_{k'}V_k + V_k V_{k'}\right) = -i(k+k')(V_{k'}V_k)$$

~~6.~~ 7.

Now, define

NL interaction

$$L^{-1}_{\substack{k+k' \\ \omega+\omega'}} = -i(\omega+\omega') + \nu(k+k')^2 + \underset{\uparrow}{C_{\substack{k+k' \\ \omega+\omega'}}}$$

(renormalized propagator)

$$V^{(2)}_{\substack{k+k' \\ \omega+\omega'}} = L_{\substack{k+k' \\ \omega+\omega'}}\left(-i(k+k')\right)V_{\substack{k' \\ \omega'}}V_{\substack{k \\ \omega}}$$

so, self consistently,

$$C_{\substack{k \\ \omega}}V_{\substack{k \\ \omega}} = ik\sum_{\substack{k' \\ \omega'}} V_{\substack{-k' \\ -\omega'}} L_{\substack{k+k' \\ \omega+\omega'}}(-i)(k+k')V_{\substack{k' \\ \omega'}}V_{\substack{k \\ \omega}}$$

$$= \left(k^2\sum_{k',\omega'}|V_{\substack{k' \\ \omega'}}|^2 L_{\substack{k+k' \\ \omega+\omega'}}\left(1+\frac{k'}{k}\right)\right)V_{\substack{k \\ \omega}}$$

$$\text{so}\begin{cases} dV_{k,\omega}/df_{k,\omega} = 1\Big/-i\omega+\nu k^2 + C_{\substack{k \\ \omega}} \\ C_{k,\omega} = \nu_{\substack{k \\ \omega}}k^2 \equiv k^2\sum_{k',\omega'}|V_{\substack{k' \\ \omega'}}|^2 L_{\substack{k+k' \\ \omega+\omega'}}\left(1+\frac{k'}{k}\right)\end{cases}$$

→ {renormalized response function}

renormalized turbulent viscosity

$\nu \to \nu + \nu_{k,\omega}$.

→ nonlinear scrambling rate

→ recursively defined.

About $\nu_{k,\omega}$:

- at long wavelength / low frequency } $k < k'$, $\omega < \omega'$ $\Rightarrow$ quasilinear limit
Markovian
(parity)

$$\nu_{k,\omega} \to \nu^T \cong \sum_{k',\omega'} |V_{k',\omega'}|^2 L_{k',\omega'}$$

$\downarrow$ effective transport coefficient (diffusion) $\Leftrightarrow$ sets NL/turbulent time scale

$$\nu^T \sim \langle \tilde{V}^2 \rangle \tau_c \sim \tilde{V}_{rms} l_c \qquad l_c \sim \tilde{V} \tau_c$$

$\nu \to$ F.-P. E.

$\nu_{k,\omega} \to$ need Zwanzig-Mori theory.

- important to note:

$$\nu_{k,\omega} \to \nu^T = \sum_{k',\omega'} |V_{k',\omega'}|^2 (k'^2 \nu_{k',\omega'}) \Big/ \left\{ \omega'^2 + (k'^2 \nu_{k',\omega'})^2 \right\}$$

→ response function in ν^T also renormalized $\Leftrightarrow$ (self-consistency) $\Leftrightarrow$ $\nu^T \Leftrightarrow$ random Doppler shift

input
$\downarrow$
(RPA)

→ irreversibility / dissipation from inertial range mixing / transfer

i.e. contrast QLT, with resonance, i.e.

→ to estimate;

$$D = \frac{q^2}{m^2} \sum_k |E_k|^2 \pi \delta(\omega - kv)$$

~~6~~6a

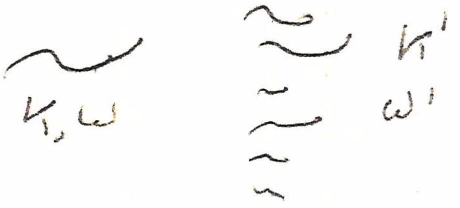
$$\begin{cases} \nu^2 \sim \frac{1}{k^2} V_{rms}^2 \\ \nu \sim \frac{1}{k} V_{rms} \end{cases}$$

– $\nu^T_{k,\omega}$ vs. $\nu^T_{\overset{?}{0}}$

$k, \omega \to 0$ if $k < k'$, $\omega < \omega'$

$\Rightarrow$ Markovian limit $\Leftrightarrow$ no memory (a la' F.P.E)
Fokker-Planck Eqn.

i.e. consider interaction of 'test wave' k, ω with background k', ω'.



$\Leftrightarrow$ for $\gamma', \lambda' \ll \gamma, \lambda$

$\Rightarrow$ interaction appears as random, memory-less kick, as in walk.

for $\gamma', \lambda' \sim \gamma, \lambda$

$\Rightarrow$ interaction is one of mutual sloshing, etc.
i.e. test wave "feels" space time history of turbulence background.

- also,

$$\nu^{T} k^2 V_{\substack{k\\ \omega}} \rightarrow -\nu \partial^2 V$$

eddy viscosity

$$\nu^{T}_{\substack{k\\ \omega}} k^2 V_{\substack{k\\ \omega}} \rightarrow + \int dx' \int d\tau \, \underset{\substack{\downarrow \\ \text{memory}}}{C(x-x', t-\tau)} \, V(x', \tau)$$

memory convolution (space/time)

- why "renormalization":

i.e. QED

$$\frac{1}{\not{p} - m_0} \rightarrow \text{electron Fermion propagator (bare)}$$

↳ bare mass, electron

"renorm" ⇒

$$\frac{1}{\not{p} - m_0 + \Sigma} \rightarrow \frac{1}{\not{p} - m} \quad \text{(renormalized)}$$

↳ self-energy; due electron interaction with vacuum polarization cloud (ambient fluctuations)

turbulence:

$$\frac{1}{-i\omega + \nu_0 k^2} \rightarrow V \text{ propagator}$$

↳ bare (collisional) viscosity

renorm.

$$\Rightarrow \frac{1}{\left[-i\omega+(\nu_0+\nu_{T_{\substack{k\\ \omega}}})k^2\right]} \rightarrow \text{v propagator}$$

↙ renormalized viscosity (dressing)

↳ interaction of mode/eddy with turbulence (dressing)

$\Sigma \Leftrightarrow \nu_{T_{\substack{k\\ \omega}}}$

D.I.A. is procedure for calculation of self-energy.

→ Aside: Candidate Time Scales for Modal Interaction

① νk^2 → viscous damping rate

② γ_{NL} → nonlinear energy transfer rate

③ $\left|\left(\frac{\omega}{k} - \frac{\partial\omega}{\partial k}\right)\Delta k\right|$ → wave-(resonant particle) autocorrelation rate

④ $|\Delta\omega_{MM}|$ → wave-wave autocorrelation rate, set by mis-match dispersion

⑤ $\Delta\omega_{\underline{k}}$ → nonlinear scrambling rate (NL acts on self)

Examples:

i.) Weak Turbulence Theory → Wave-Wave (includes weak wave turbulence)

④ < ②, ⑤

Wave-Particle → ③ < ②, ⑤, $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t}$

(ii.) N.S.T. → no resonances, $Re \gg 1$

②, ③, ④ → 0

① < ⑤ ⇒ renormalization needed.

→ Spectral Equation — <u>spectrum</u> is goal.

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \nu\frac{\partial^2 v}{\partial x^2} = \tilde{f}$$

$$\frac{\partial\langle v^2\rangle}{\partial t} + \left\langle v^2\frac{\partial v}{\partial x}\right\rangle + \langle\nu(\partial_x v)^2\rangle = \langle\tilde{f}\tilde{v}\rangle$$

$$\downarrow$$

$$\cancel{\left\langle\frac{\partial}{\partial x}\left(\frac{v^3}{3}\right)\right\rangle} \xrightarrow{\ \text{end points}\ } 0$$

→ [NL conserves energy (to boundary terms)]

∴ have energy balance:

$$\underbrace{\frac{\partial}{\partial t}\langle v^2\rangle}_{\text{net K.E.}} = \underbrace{\langle\tilde{v}\tilde{f}\rangle}_{\substack{\text{source}\\ \text{(forcing)}\\ S}} - \nu\underbrace{\langle(\partial_x v)^2\rangle}_{\substack{\text{viscous}\\ \text{dissipation}}}$$

in k:

$$\frac{\partial}{\partial t}\langle\tilde{v}^2\rangle_k = S_k - \nu k^2\langle\tilde{v}^2\rangle_k + \overset{\text{Nonlinear transfer}}{\underset{\substack{\uparrow\\ \text{inertial range}\\ \text{interaction}}}{T_k}}$$

where $\sum_k T_k = 0 \iff$ <u>NL transfer</u> <u>conserves energy</u>

i.e. expect T_k is sum of two cancelling terms (upon summation) $\Rightarrow$ is anti-symmetric in k.

$\hookrightarrow$ renormalized theory must respect symmetry, conservation laws of original, primitive eqn.

Now!

$$T_k = \frac{1}{3}\left\langle \frac{\partial}{\partial x}\,\frac{V^3}{3}\right\rangle_k$$

coherent mode coupling

$\hookrightarrow \sim \nu \langle v^2\rangle$

$$= i\,2\sum_{k'} \tilde{V}_{-k}\left(\tilde{V}_{-k'}\,\tilde{V}^{(2)}_{k+k'}\,(k+k')\right)$$

$$-2i\sum_{\substack{p,q\\ p+q=k}} \tilde{V}_{-p}\,\tilde{V}_{-q}\,\tilde{V}^{(2)}_{p+q}\,(p+q)$$

incoherent mode coupling (nonlinear noise $\leftrightarrow$ I.R. cascade)

i.e. coherent:

$$\cong \tilde{V}_{-k}\left(C_k\,\tilde{V}_k\right)$$

$\rightsquigarrow$ $\textcircled{v}$ same as renormalized response function

$$\cong C_k\,\langle \tilde{V}_k^2\rangle$$

$\rightarrow$ dissipation of $\langle\tilde{V}^2\rangle_k$ due turbulent viscosity (death)

incoherent:

$$\cong -\langle\tilde{V}^2\rangle_p\,\langle\tilde{V}^2\rangle_q \qquad p+q=k$$

$\rightarrow$ (birth) nonlinear noise emission into k, via mode coupling

Now,

must treat beat / virtual mode self-consistently ⟺ include NL mixing in time response history ⟺ self-consistent field

$$\ast\ \frac{\partial}{\partial t}\tilde{V}^{(2)}_{k+k'} + \left[\nu(k+k')^2 + C_{k+k'}\right]\tilde{V}^{(2)}_{k+k'} = -(i)(k+k')\left[\tilde{V}_{k'}\tilde{V}_{k}\right]$$

$\Rightarrow$

$$\tilde{V}^{(2)}_{k+k'} = -i(k+k')\int L^{+1}_{k+k'}\,\tilde{V}_{k'}\tilde{V}_{k}\,d\tau$$

$$\tilde{V}^{(2)}_{p+q} = -i(p+q)\int L^{+1}_{p+q}\,\tilde{V}_{p}\tilde{V}_{q}\,d\tau$$

$$T^{C}_{k} = 2i\sum_{k'}\tilde{V}_{-k}(t)\tilde{V}_{-k'}(t)(k+k')\Big(-i(k+k')\Big)\Big) \ast \int_0^{\infty} L^{+1}_{k+k'}(\tau)\,\tilde{V}_{k}(t-\tau)\tilde{V}_{k}(t-\tau)\,d\tau$$

need model of temporal self-coherence!

$$= 2\sum_{k'}(k+k')^2\left\langle\tilde{V}_{-k}(t)\tilde{V}_{-k'}(t)\int_0^{\infty}L^{+1}_{k+k'}(\tau)\,\tilde{V}_{k}(t-\tau)\tilde{V}_{k}(t-\tau)\right\rangle d\tau$$

Now, take:

$$\langle\tilde{V}(t)\tilde{V}(t+\tau)\rangle_k = |\tilde{V}_k|^2 e^{-C_k\tau}$$

→ self-correlation decays at rate given by response time

(neglect νk^2 for convenience)

so

$$T_k^{c} = 2\sum_{k'}(k+k')^2\int_0^\infty d\tau \exp\left[-(c_{k+k'}+c_k+c_{k'})\tau\right] *$$

↓
coherent

$$\underbrace{\langle \tilde{v}^2\rangle_{k'}\,\langle \tilde{v}^2\rangle_{k}}_{\text{slow time residual}}$$

$$\Theta_{k,k',k+k'} = \int_0^\infty d\tau\ \exp\left[-(c_{k+k'}+c_k+c_{k'})\tau\right]$$

↓
triad coherence time → set by modal decorrelation rates!

Similarly;

$$T_k^{I} = 2\sum_{\substack{p,q\\ p+q\\ =k}}(p+q)^2\,\Theta_{p,q,k}\langle\tilde{v}^2\rangle_p\langle\tilde{v}^2\rangle_q$$

↓
incoherent

⇒ energy equation becomes:

$$\left.\begin{aligned}
&\frac{\partial}{\partial t}\langle\tilde{v}^2\rangle_k + \nu k^2\langle\tilde{v}^2\rangle_k + T_k = S_k\\
&T_k = 2\sum_{k'}(k+k')^2\,\Theta_{k,k',k+k'}\langle\tilde{v}^2\rangle_{k'}\langle\tilde{v}^2\rangle_k\\
&\qquad - 2\sum_{p,q}(p+q)^2\,\Theta_{p,q,k}\langle\tilde{v}^2\rangle_p\langle\tilde{v}^2\rangle_q
\end{aligned}\right\}$$