Avalanches and Self-Organized Criticality II → Intro to Hydrodynamic Models

(Avalanche Turbulence) (aka "Flipping Burgers")

see: – FNS
– Hwa and Kardar
– Gil + Sornette

Recall

→ SOC idea
→ Sandpile Model (CA)

Now, natural to ask: Analogy SOC – Turbulence.

– is there a continuum model, as $l_{avalanche} \gg \Delta$? → akin granular flow

Can one think in terms of avalanche turbulence

– can one exploit ~~symmetry~~ in deriving SOC model, much as symmetry exploited in Ginzburg–Landau model
i.e. $M \to -M$ symmetry – even terms.

These bring us to the hydrodynamic theory / model of SOC

→ continuum model

→ Valid for large scales, long time scales.

$L \gg l \gg \Delta$

$\tau_{conf} \gg \tau \gg \tau_{step}$

→ Basics

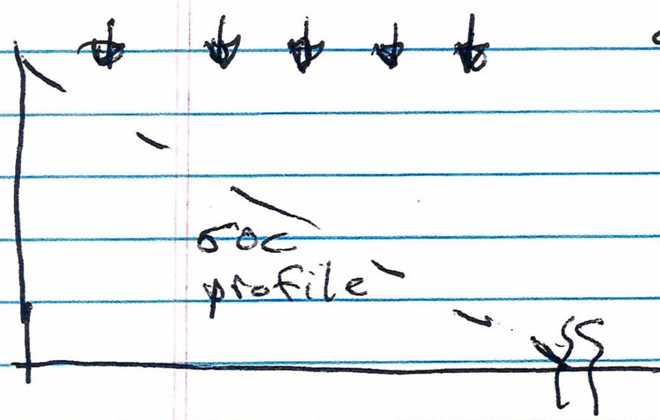
Consider:

→ box with ejecting bndry on RHS, accumulating bndry on LHS

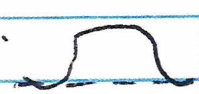
→ SOC profile, TBD

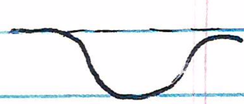
n.b. what is SOC profile?

→ noise → random rain (uncorrelated. correlated drops irrelevant.)



→ Now, consider deviations from SOC profile, i.e.

 → bumps, "blobs"

 → voids, "holes"

(no self-binding mechanism).

→ Also assume conservation of "stuff" in the profile up to boundary losses and noise source. Call stuff ρ.
(could represent pressure)

→ Idea is to describe dynamics of deviation from SOC state

i.e. $\rho = \rho_{SOC} + \delta\rho$

→ but evolve only deviation

→ only small deviation theory

formally posited, not calculated.

$\Rightarrow$, have

$$\boxed{\partial_t \delta\rho + \partial_x \left[\Gamma(\delta\rho) - D_0 \partial_x \delta\rho \right] = \tilde{S}}$$

- $\Gamma(\delta\rho)$ is flux induced by deviation from SOC state

- obviously, ρ conserved so $\delta\rho$ evolves via $\nabla \cdot \Gamma$ only

- background diffusion posited.
- can generalize to higher dimensions. See Hwa + Kardar.

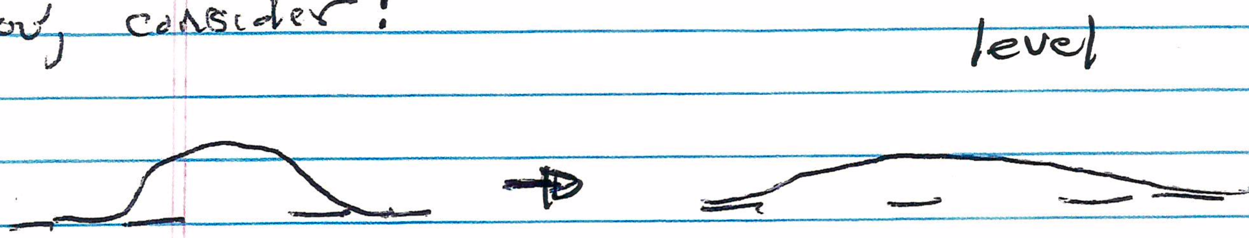
— $\Gamma(\delta p) \to 0$ as $\delta p \to 0$

$\delta p \to 0$ as $\tilde{S} \to 0$.

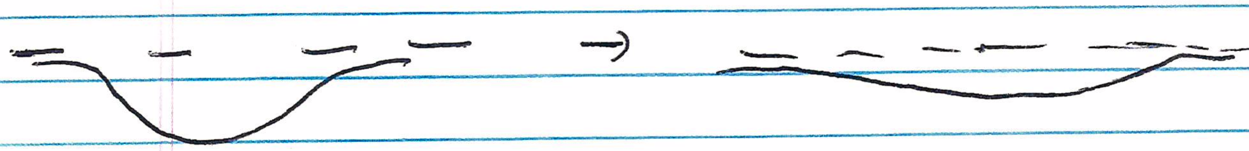
— How constrain $\Gamma(\delta p)$? → $\boxed{\text{Symmetry}}$!

in spirit of Ginzburg / Landau prescription.

Now, consider:

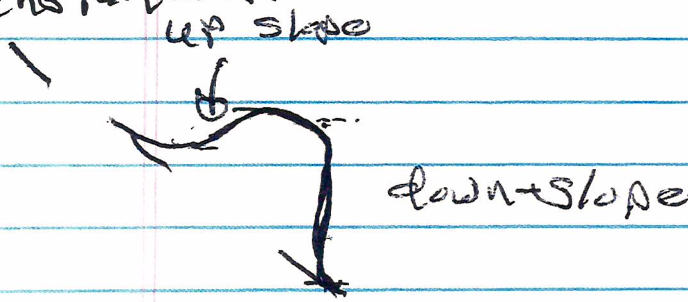


blob spreads out, conserving area.



likewise void.

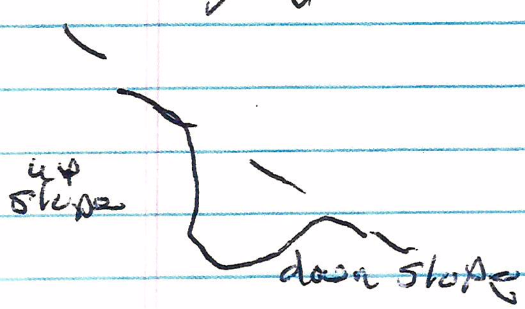
Now, if left-right symmetry broken by
$\nabla p_{sou} \neq 0$
instantaneous



bump $\Rightarrow$
greater (extent / steepness) on down-slope

$\Rightarrow$ bumps / local excesses propagate down gradient, to right

Necessarily;



void $\Rightarrow$
greater (extent / steepness) on up-slope than down slope

$\Rightarrow$ voids / local deficits propagate up gradient, to left.

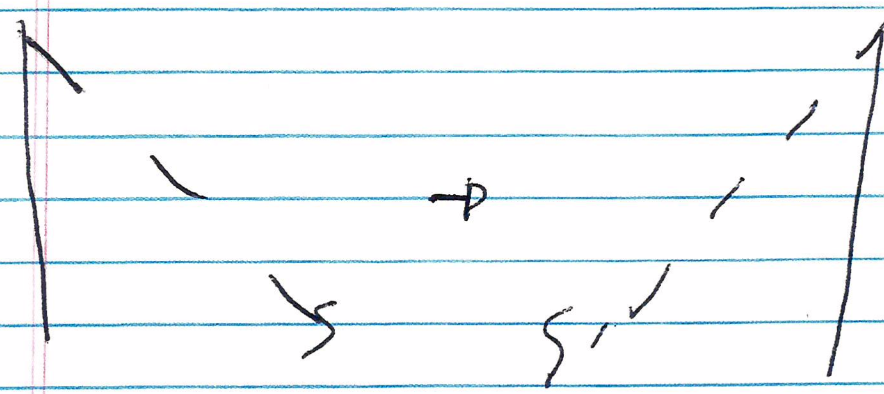
→ Both criteria local,

→ Both criteria common sense.

Now, observe:

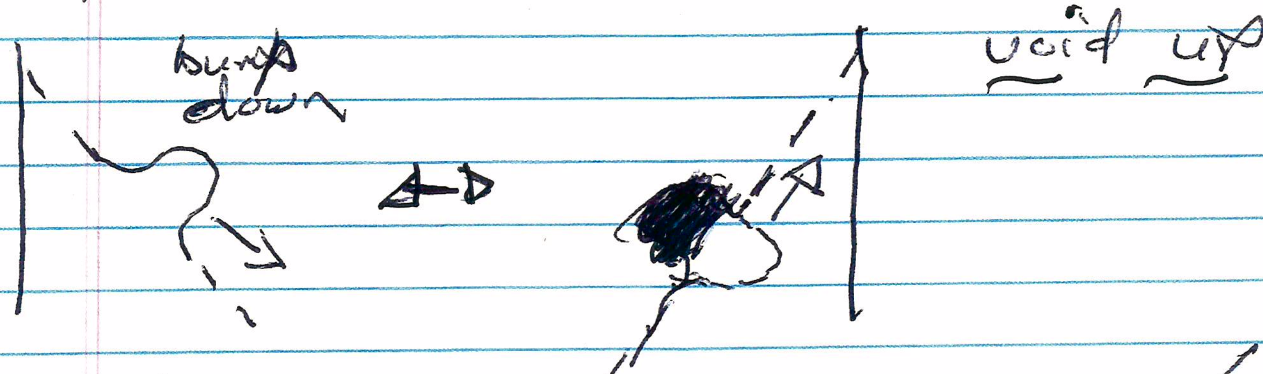
① reflection $x \to -x$

i.e.



and

② bump ↔ hole interchange



→ right ✓

right → ✓

same flux direction!

→ $\boxed{\text{This brings us to the principle of joint reflection symmetry!}}$

$$\Gamma\Big|_{\substack{x\to -x \\ \delta p \to -\delta p}} = \Gamma$$

akin $\vec{\Gamma}$ invariant under direction flip

This constrains the form of $\Gamma(\delta P)$!

How?

N.B.: - Full flux is complicated.

- seek flux in large scale, long time limit ⇒ smoothest form

interested in long time, large scale > biggest non-trivial

so have

$$\partial_t \delta P + \partial_x \left[\Gamma(\delta P) - D_0 \partial_x \delta P \right] = \tilde{S}$$

$\Gamma(\delta p)$ must satisfy joint reflection symmetry.

Then formally:

$$\Gamma(\delta P) = \sum_{\substack{m,n,\\ \xi,r,\alpha}} \Big\{ A_n (\delta P)^n + B_m (\partial_x \delta P)^m + D_\alpha (\partial_x^2 \delta P)^\alpha + C_{\xi,r} (\delta P)^\xi (\partial_x \delta P)^r + \Big\}$$

JRS ≡ joint reflection symmetry.

① $n=1$ violates JRS

① $\cong \alpha\, \delta P^2 + h.o.t.$

$\alpha > 0$

② $m=1$ ok

$m=2$ ok

② $\cong -D\, \partial_x \delta P + h.o.t.$

$D > 0$ (well behaved)

③ $\alpha = 1$ violates JRS

$\alpha = 2$ too fine scaled, ignore..

④ $q=1$, $r=1$ violates JRS

so, to lowest order in roughness:

$$\boxed{\partial_t \delta P + \partial_x\left[\alpha\, \delta P^2 - D\, \partial_x \delta P\right] - D_0\, \partial_x \delta P = \tilde{S}}$$

α, D are constants to be specified, as a, b in G-L theory are.

Re-absorb D_0 into D:

$$\boxed{\partial_t \delta P + \partial_x\left[\alpha\, \delta P^2 - D\, \partial_x \delta P\right] = \tilde{S}}$$

- hydro model limit is noisey Burgers $\partial_t V + V \partial_x V - \nu \partial_x^2 V = \tilde{S}$
- exactly solvable for $\tilde{S} = 0$
- basic solution structure is shocks! (shocks produce entropy).

- shocks akin avalanches, ...

  shock trains →

Now, seek long wavelength approximation to nonlinear flux

i.e. $\left[\partial_x(\alpha\,\delta P^2)\right]_k \rightarrow d_k\,\delta P_k$

$$\cong \underbrace{\nu}_{\text{turbulent viscosity}} k^2\,\delta P_k$$

N.B. $\alpha\,\delta P^2 \longleftrightarrow D_T(\delta P)\,\delta P$

$$\longrightarrow -D(\nabla P - \nabla P_{crit})\,\nabla\delta P$$

"critical gradient"

δP [illegible] $\quad I \sim \delta P$

[clear correspondence to expected QL expression for flux, with threshold.]

Now,

$$N_{k,\omega} = \left[\partial_x(\alpha\, \delta P^2)\right]_{k,\omega} \xrightarrow{\text{renormalized NL}} \nu k^2 \delta P_{k,\omega}$$

$$= ik\alpha \sum_{k',\omega'} \delta P_{-k',-\omega'}\, \delta P_{k+k',\, \omega+\omega'}$$

in spirit QL.

$$\cong ik\alpha \sum_{k',\omega'} \delta P_{-k',-\omega'}\, \delta P^{(2)}_{k+k',\, \omega+\omega'}$$

where:

nonlinear scrambling of coupling time

$$\left[-i(\omega+\omega') + (k+k')^2 D_0 + (k+k')^2 \nu_k\right]$$

$$= -i\alpha(k+k')\, \delta P_{k',\omega'}\, \delta P_{k,\omega}$$

and substituting gives:

$$N_{k,\omega} = \nu_T k^2 \delta P_{k,\omega}$$

where:

for $k, \omega \to 0$; $\left\{ \begin{array}{l} \text{long, smooth} \\ \text{slow limit} \end{array} \right.$

$$\nu_T \cong \sum_{k',\omega'} \left| \delta P_{k',\omega'} \right|^2 \frac{k'^2 \nu_T}{\left[\omega'^2 + (k'^2 \nu_T)^2 \right]}$$

where neglected ν_0 relative to ν_T.
Note recursive defn ν.

Now, need related $\delta P_{k', \omega'}$ to noise (i.e. $k', \omega' \to$ high freq, short wavelength modes excited). This must also include nonlinear response, self-consistently

so drop sub T

$$\left[(-i\omega' + k'^2 \nu) \delta P_{k', \omega'} = \tilde{S}_{k', \omega'} \right]$$

so

$$\nu = \alpha^2 \sum_{k',\omega'} \frac{|\tilde{s}_{k',\omega'}|^2}{(k'^2\nu)^3} \frac{1}{\left[1+(\omega'/\nu k'^2)^2\right]^2}$$

$$\sum_{k',\omega'} = \int_{k_{min}}^{\infty} dk' \int d\omega'$$

noise 'color' in space time significant.

and

$$|\tilde{s}_{k,\omega}|^2 = s_0^2 \quad \Rightarrow \text{white noise}$$

$\Rightarrow$

$$\nu = \frac{\overset{\#\,\text{const}}{c_1}\, \alpha^2 s_0^2}{\nu^2} \int_{k_{min}}^{\infty} \frac{dk'}{k'^4}$$

$\rightarrow$ infared divergence !

$\rightarrow$ why ? $\Rightarrow$
- conserved order parameter (flux form) $\partial_x \Gamma$
- slow modes $1/\tau_{c_k} \sim k^2 \nu$

slow modes → damping drops

$$\gamma \sim -k^2 \nu$$
$$\Downarrow$$
$$\rightarrow 0 \text{ as } k \to 0$$

weak noise + tiny decay ⇒ strong intensity

⇒ general point: weakly damped modes dangerous if any excitation available.

So

$$\nu_T = \left(c_1 \alpha^2 S_0^2 \int_{k_{min}}^{\infty} \frac{dk'}{k'^4} \right)^{1/3}$$

$$\simeq \left(c_1' \alpha^2 S_0^2 \right)^{1/3} k_{min}^{-1}$$

⇒ $\boxed{\nu_T \text{ depends explicitly on cut-off scale.}}$

Now, meaning? → what is physics message of inferred divergence?

$$\boxed{k_{min}^{-1} \equiv \delta l}$$ → scale being observed

$\underline{\underline{\delta l' < \delta l}}$ ⇒ scatterers

so

$$\nu_T \sim \nu_{T,0}\, \delta l$$

$\left\{ \nu_T \text{ grows with scale of interest} \right.$

but ν_T is diffusion ⇒

$$\frac{d}{dt}\langle \delta l^2 \rangle \sim \nu_T \quad , \text{ but}$$

$$\delta l^2 \sim \nu_{T_0}\, \delta l\, t$$

$$\Rightarrow \boxed{\delta l \sim \nu_{T,0}\, t} \Rightarrow$$

δl pulse propagates ballistically, not diffusively.

pulse.

→ infered divergence ultimately identifies uncovers ballistic propagation

⟺ conserved order param. is key.

→ supported by scaling analysis

see { H+K
FNS

→ if 2D, anisotropic pile:

$$\partial_t \delta P + \partial_{\parallel}\left\{ \alpha\, \delta P^2 - D\, \partial_{\parallel} \delta P \right\} = r_0 \nabla_{\perp} \delta P$$

$$= \tilde{S}$$

$$\partial_{\parallel} = \frac{\nabla P}{|\nabla P|} \cdot \nabla$$

→ derivative parallel to pile gradient of surface.

see refs for more.

See also: Gil + Sornette, I and P eqn.