To Self-Organized Criticality and Avalanches I

- An Intellectual History of SOC → how we come to it.

- 1/f Noise re-visited

recall $\begin{cases} \text{Zipf's Law} & P \sim 1/\Delta X \\ 1/f & \end{cases}$

Lognormal (intermittent/multiplicative) approximated by power law $P \sim 1/X$ for finite range.

Recall pdf (T_c) → 1/f

- SOC → BTW
  → Kadanoff model

- Profiles

- Continuum Models

A Brief Intellectual History of 'SOC'

- Storylines

I)

Hydrology
Characterizing Time Series

H, Hurst and Holder

II)

(50's)

'Concentrated' pdf,
Intermittency
Multiplicative Processes

Lognormality,
Pareto-Levy Distributions

fractals are unifying theme

Intermittency
Fractals, Self-similarity

MW '68

(70's)

(80's)

1/f Noise

1/f noise "universal"
→ associated with $H \to 1$

SOC

BTW '87

(physical system realizing 1/f noise)

- Lognormal ←→ Zipf ←→ 1/f related

i.e.

- $P\left(\frac{x}{\bar{x}}\right) = P(\log x)\frac{d\log x}{dx} = g\left(\frac{x}{\bar{x}}\right) d\left(\frac{x}{\bar{x}}\right)$

Probability

$x/\bar{x}$ lies in $d(x/\bar{x})$ at $x/\bar{x}$

$\log(g) = -\log f$ + variance corrections

$$\boxed{f = 1/(x/\bar{x})}$$

- Lognormal well approximated by power law $P \sim \frac{1}{x}$ (Zipf's law), over finite range! (Montroll '82)
- Multiplicative processes related to Zipf's law trend
- Link to 1/f noise?

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- 1/f Noise?

  A few observations:

  – Zipf and 1/f related but <u>different</u>

    Zipf → $P(\Delta B) \sim 1/|\Delta B|$

    $1/f$ → $\langle(\Delta B)^2\rangle_\omega \sim 1\,/\,\omega$

    Both embody:

    - Self-similarity
    - Large events rare, small events frequent → intermittency phenomena
    - 1/f linked to H→1

  – 1/f noise (flickers, shot…)

    - Ubiquitous, suggests 'universality' → why ? ?
    - Poorly understood, circa 80's

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- N.B.: <u>Not</u> easy to get 1/f …
- In usual approach to ω spectrum; $\leftarrow\rightarrow$ (DIA, EDQNM, Dupree, Kadomtsev, Kraichnan, Krommes):

  $$\langle \phi(t_1)\phi(t_2)\rangle = \left|\hat{\phi}\right|^2 e^{-|\tau|/\tau_c}$$

  $$\rightarrow S(\omega) = \frac{1/\tau_c}{\omega^2 + 1/\tau_c^2} \sim \frac{1}{\omega^2}$$

  i.e. τ_c imposes scale, but 1/f scale free !?
- N.B.: Conserved order parameter may restore scale invariance
- But, consider ensemble of random processes each with own τ_c (Montroll, BTW)

$$S(\omega)_{eff} = \int_{\tau_{c1}}^{\tau_{c2}} P(\tau_c) S_{\tau_c}(\omega)\, d\tau_c$$

Probability of τ_c

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- And... demand $P(\tau_c)$ scale invariant, i.e.

$$P(\tau_c) = d\tau_c/\tau_c$$

$$S(\omega) = \left.\frac{\tan^{-1}(\omega\tau_c)}{\omega}\right|_{\tau_{c1}}^{\tau_{c2}} \sim 1/\omega, \qquad \text{recovers } 1/f\,!$$

→ but what does it mean? ...

- So, circa mid 80's, need a simple, intuitive model which:
  - Captures 'Noah', 'Joseph' effects in non-Brownian random process (H→1)
  - Display 1/f noise

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SOC at last !

- Enter BTW '87:

Self-Organized Criticality: An Explanation of $1/f$ Noise (7000+ cites)

Per Bak, Chao Tang, and Kurt Wiesenfeld
Physics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 13 March 1987)

We show that dynamical systems with spatial degrees of freedom naturally evolve into a self-organized critical point. Flicker noise, or $1/f$ noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

- Key elements:

Very well written paper.

  - Motivated by ubiquity and challenge of 1/f noise (scale invariant)
  - Spatially extended excitations (avalanches) *

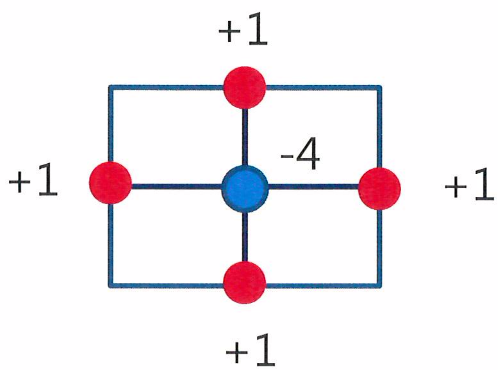
  Comment: statistical ensemble of collective excitations/avalanches is intrinsic

  - Evolve to 'self-organized critical structures of states which are barely stable'

  Comment: SOC state ≠ linearly marginal state!

  SOC state is <u>dynamic</u>

- Avalanches and Clusters:
  - BTW – 2D CA model



$Z \equiv \text{occupation}$

$Z > Z_{crt} = K$

$Z(x, y) \rightarrow Z(x, y) - 4$

$Z(x \pm 1, y) \rightarrow Z(x \pm 1, y) + 1$

$Z(x, y \pm 1) \rightarrow Z(x, y \pm 1) + 1$

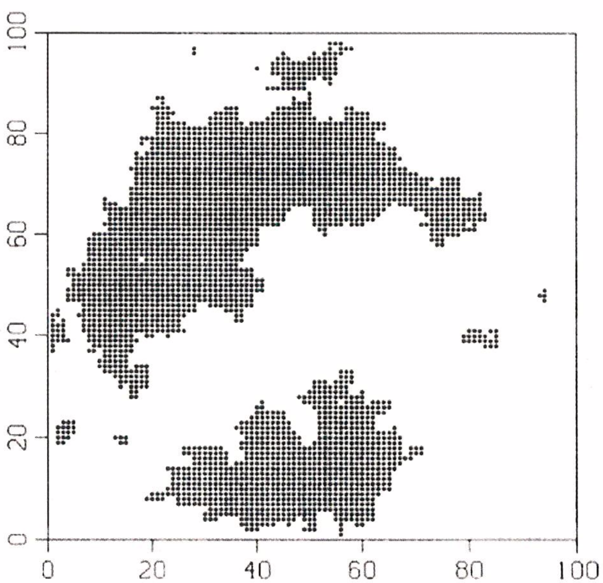


FIG. 1. Self-organized critical state of minimally stable clusters, for a 100×100 array.

- SOC state with minimally stable clusters
- 'Cluster' ≡ set of points reached from toppling of single site (akin percolation) !
- Cluster size distribution $D(s) \sim s^{-\alpha}$, $\alpha \sim 0.98$

➔ Zipf, again !

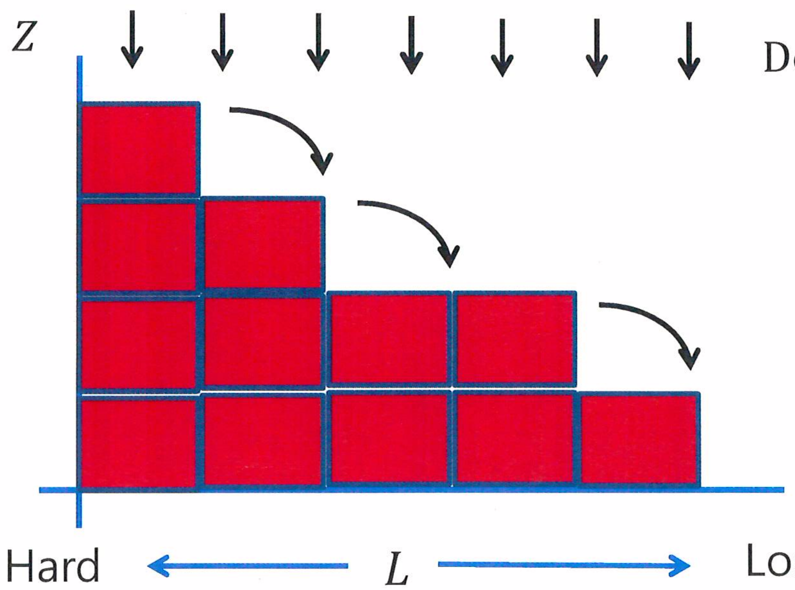
- Key elements, cont'd:
  - "The combination of <u>dynamical</u> minimal stability and spatial scaling leads to a power law for temporal fluctuations"
  - "Noise propagates through the scaling clusters by means of a "domino" effect upsetting the minimally stable states" *

    Comment: space-time propagation of avalanching events *
  - "The critical point in the dynamical systems studied here is an attractor reached by starting far from equilibrium: ~~the scaling properties of the model"~~

    Comment: Noise essential to <u>probe</u> dynamic state *

  N.B.: BTW is example of well-written PRL

- The Classic – Kadanoff et al '89 1D driven lossy CA



Deposition → random, can profile

Lossy bndry

If
$$\begin{cases} Z_i - Z_{i+1} > \Delta Z_{crt} \\ Z_{i+1} \to Z_{i+1} + N \\ Z_i \to Z_i - N \end{cases}$$
Etc.

Grains ejected at boundary

Why of interest for MFE?

- Interesting dynamics only if $L/\Delta \sim N \gg 1$ ↔ equivalent to $\rho_* \ll 1$ condition – analogy with turbulent transport obvious

TABLE I. Analogies between the sandpile transport model and a turbulent transport model.

Turbulent transport in toroidal plasmas	Sandpile model
Localized fluctuation (eddy)	Grid site (cell)
Local turbulence mechanism:	*Automata rules*:
Critical gradient for local instability	Critical sandpile slope (Z_{crit})
Local eddy-induced transport	Number of grains moved if unstable (N_f)
Total energy/particle content	Total number of grains (total mass)
Heating noise/background fluctuations	Random rain of grains — location
Energy/particle flux	Sand flux
Mean temperature/density profiles	Average slope of sandpile
Transport event	Avalanche
Sheared electric field	Sheared flow (sheared wind)

What is SOC?

General Thoughts

(cf: Jensen)

- (Constructive)

  Slowly driven, interaction dominated threshold system

  Classic example: sandpile

- (Phenomenological)

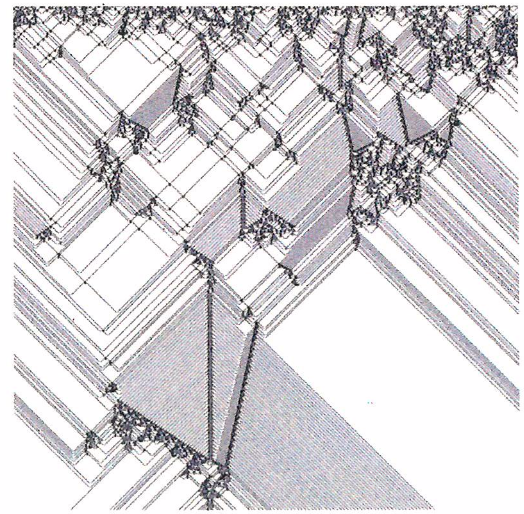
  System exhibiting power law scaling without tuning.

  Special note: 1/f noise; flicker shot noise of special interest

  See also: sandpile

N.B.: $1/f$ means $1/f^{\beta}$, $\beta \leq 1$

What is SOC?, cont'd



- Elements:

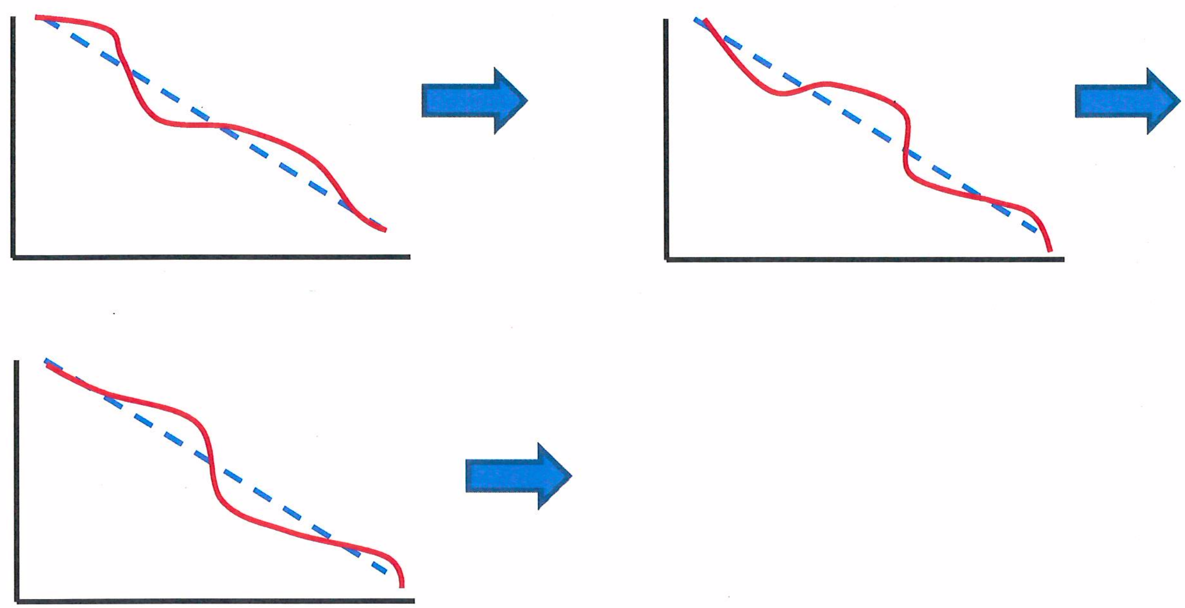
→ Interaction dominated

- Many d-o-fs $\left\{ \begin{array}{l} \text{Cells} \\ \text{Modes} \end{array} \right\}$
- Dynamics dominated by d-o-f interaction i.e. couplings

→ Threshold and slow drive

- Local criterion for excitation
- Large number of accessible meta-stable, quasi-static configuration
- 'Local rigidity' ←→ "stiffness" !?

- Multiple, metastable states



- Proximity to a 'SOC' state → local rigidity
- Unresolved: precise relation of 'SOC' state to marginal state

- Threshold and slow drive, cont'd
  - Slow drive <u>uncovers</u> threshold, metastability
  - Strong drive <u>buries</u> threshold – does not allow relaxation between metastable configurations
  - How strong is 'strong'? – set by toppling/mixing rules, box size, b.c. etc.
- Power law ←→ self-similarity
  - 'SOC' intimately related to:
    - Zipf's law: P(event) ~ 1/(size) (1949)
    - 1/f noise: S(f) ~ 1/f

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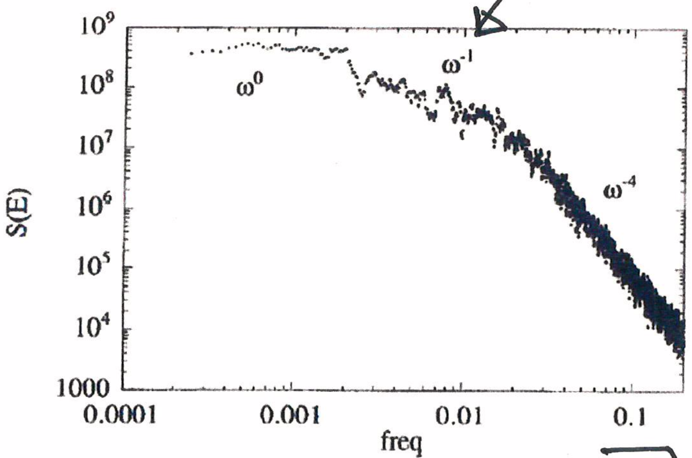
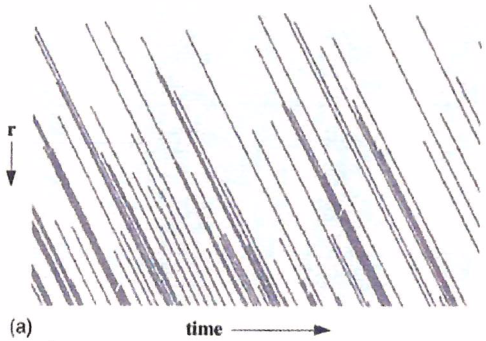
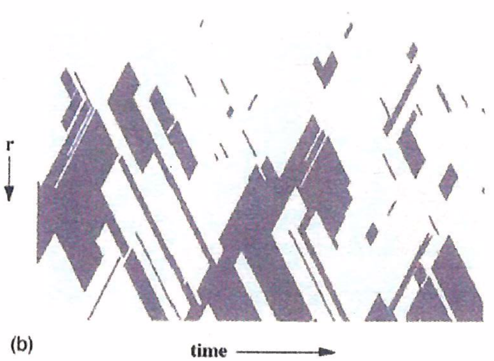
1/f Generic structure – spectra.



Power spectrum of over-turnings $\langle(\Delta Z)^2\rangle_\omega$





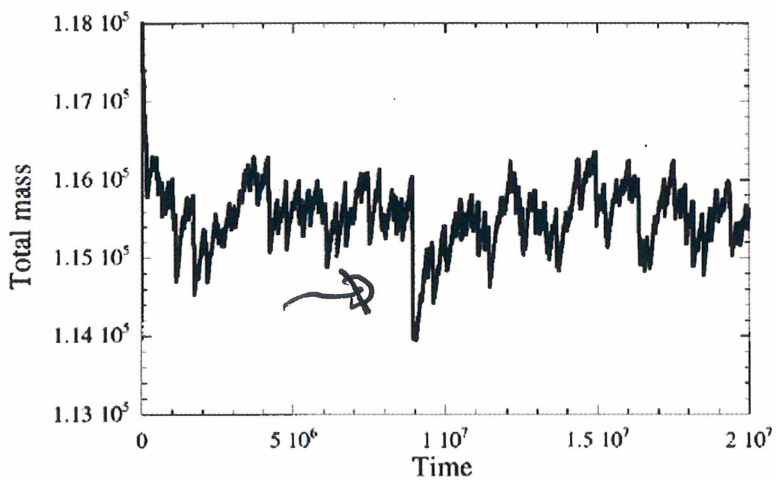
Avalanching

dark → over-turning
light → stable

- Some generic results
  - 1/f range manifest
  - Large power in slowest, lowest frequencies
  - Loosely, 3 ranges:
    - ω^0 → 'Noah'
    - 1/f → self-similar, interaction dominated (Joseph)
    - $1/f^4$ → self correlation dominated
    - Space-time → distribution of avalanche sizes evident

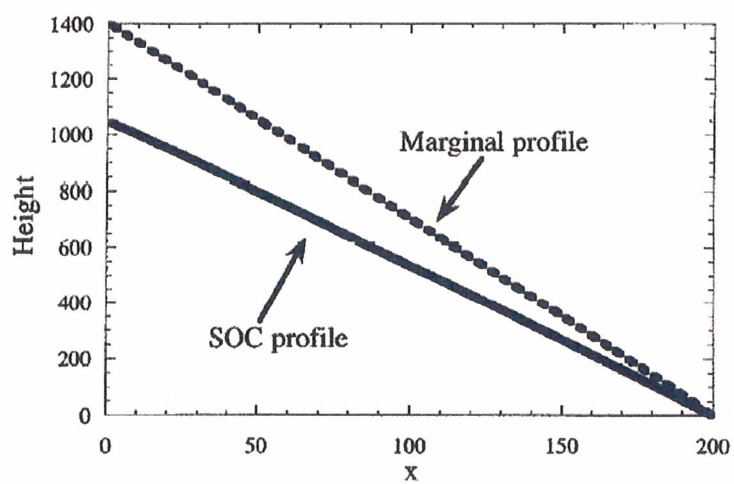
→ Outward, inward avalanching …

- ## Global Structure



- Time history of total grain content
- Infrequent, large discharge events evident *

SOC vs Marginal?



- SOC $\neq$ Marginal
- SOC → marginal at boundary
- Increasing N_{dep} ➔ SOC exceeds marginal at boundary *
- Transport bifurcation if bi-stable rule
- Simple argument for L-H at edge *

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- An Important Connection Hwa, Kardar '92; P.D., T.S.H. '95; et seq.
  - 'SOC' intimately connected to self-similarity, 'cascade' etc ultimately rooted in fluid turbulence – relate?

And:

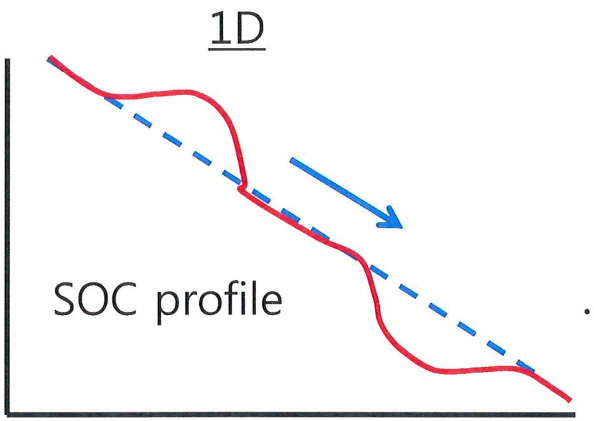
  - C in 'SOC' → criticality
  - Textbook paradigm of criticality (tunable) is ferromagnetic ala' Ginzburg, Landau → symmetry principle!?

$$\frac{d\eta}{dt} - D\nabla^2\eta = -(T-T_c)a\eta - b\eta^3$$

And:

  - Seek hydro model for MFE connections

$$H = D(\nabla\eta)^2 + (T-T_c)a\frac{\eta^3}{2} + b\frac{\eta^4}{4}.$$



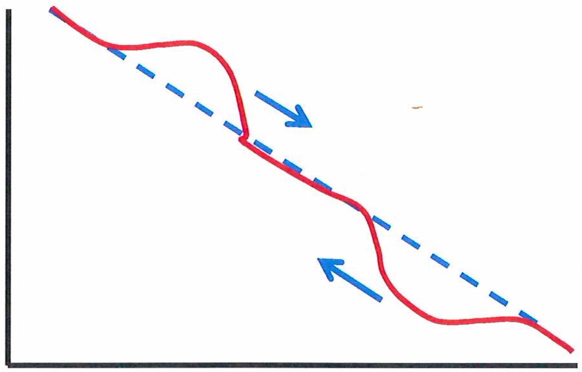
$\delta P \equiv P - P_{SOC}$ → order parameter

→ Local excess, deficit

How does it evolve?

If dynamics conservative;

- $\partial_t \delta P + \partial_x \Gamma(\delta P) - D_0 \partial_x^2 \delta P = \tilde{S}$
- Simple hydro equation
- δP conserved to $\tilde{S}$ boundary
- How constrain δP? → symmetry !
- Higher dimension, $\partial_x \to \partial_{\parallel}$, and $D_{\perp,0}, \nabla_{\perp}^2$ enter



$\delta P > 0$ → bump, excess

→ Tends move down gradient, to right

$\delta P < 0$ → void, deficit

→ Tends move up gradient, to left

- Joint reflection symmetry principle

$x \to -x$

$\delta P \to -\delta P$

➔ $\Gamma(\delta P)$ unchanged

i.e. flip pile, blob

→ void structure → rt.

- Allows significant simplification of general form of flux:

$$\Gamma(\delta P) = \sum_{m,n,q,r,\alpha} \{A_n(\delta P)^n + B_m(\partial_x \delta P)^m + D_\alpha(\partial_x^2 \delta P)^\alpha + C_{q,r}(\delta P)^q(\partial_x P)^r + \cdots\}$$

similar spirit to Ginzburg–Landau.

- So, lowest order, smoothest model:

  $\Gamma(\delta P) \approx \alpha\, \delta P^2 - D\partial_x \delta P$; α, D coeffs as in G.-L.

  N.B.: Heuristic correspondence

$$\alpha \delta P^2 \leftrightarrow -\chi \left(\frac{1}{P} \nabla P|_{threshold} - \frac{1}{L_{P_{crit}}} \right) \nabla P$$

  And have:

$$\partial_t \delta P + \partial_x (\alpha \delta P^2 - D \partial_x \delta P) = \tilde{s}$$

  - Noisy Burgers equation
  - Solution absent noise → shock
  - Shock ←→ Avalanche
  - Manifests shock turbulence → widely studied //

- More on Burgers/hydro model (mesoscale)
  - Akin threshold scattering
  - $V \sim \alpha\, \delta P$ relation → bigger perturbations, faster, over-take
  - Extendable to higher dimensions
  - Cannot predict SOC state, only describe dynamics about it. And α, D to be specified
  - $\langle \delta P \rangle$? → corrugation (!?)
  - Introducing delay time → traffic jams, flood waves, etc (c.f. Whitham; Kosuga et al '12)

- Avalanche Turbulence
  - Statistical understanding of nonlinear dynamics → renormalization
  - Conserved order parameter

$$\partial_x\,(\alpha\delta P^2) \;\rightarrow\; \nu_T k^2\,\delta P_k$$

$$\nu_T \approx \left(\alpha^2 S_0^2 \int_{k_{min}}^{0} dk/k^4\right)^{1/3} \rightarrow (\alpha^2 S_0^2)^{1/3} k_{min}^{-1}$$

Infrared divergence
due slow relaxation

$$\sim (\alpha^2 S_0^2)(\delta l)$$

  - $(\delta l)^2 \sim \nu_T \delta t$ ➔ $\delta l \sim \delta t$
    - H → 1
    - 'Ballistic' scaling

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- Infrared trends ←→ non-diffusive scaling, recover self-similarity
- Amenable to more general analyses using scaling, RG theory
- Pivotal element of 'SOC' theory as connects 'SOC' world to turbulence world, and enables continuum analysis