Notes 14: More on CTRW and FK

→ Some Results.

Saw:

- FK as general, systematic formal structure
- FK recovers various limits such as F-P, BM, Fractal BM, Levy Walks etc.
- Formalism ponderous, difficult to extract α, α_1, β etc.
- Physics content somewhat opaque

Recall CTRW

- less general, somewhat ad-hoc

but

- physically more transparent and <u>more</u> <u>tractable</u>

- CTRW intensively analyzed. See especially key works by E. Montroll

ref.: Hughes.

Now showed:

in CTRW:

joint pdf — 2 random variables

$$Q(x,t) = \int d(\Delta x) \int_0^t d(\Delta t) \; P(\Delta x, \Delta t) \, Q(x-\Delta x, t-\Delta t)$$

i.e. now Δt becomes distributed random variable.

Approaches:

- assume separable (n.b. literature on non-separable CTRWs exists)

$$P(\Delta x, \Delta t) = P_1(\Delta x) \, P_2(\Delta t)$$

$P_2(\Delta t) \to 1$, recover F.P Eqn, for well behaved $P_1(\Delta x)$

– branch to 2 approaches:

i) → waiting time

ii) → velocity model

if ignore spatial scattering, for tractability
⇒
i) waiting time model

$$Q(x,t) = \int_0^t d(\Delta t)\, P(\Delta t)\, Q(x, t-\Delta t)$$

if must need wait at least Δt

$$\phi_w(\Delta t) = \int_{\Delta t}^{\infty} dt'\, P(t')$$

and

$$Q(x,t) = \int_0^t d(\Delta t)\, \phi_w(\Delta t)\, Q(t-\Delta t)$$

need specify $P(t')$ for progress
⇒ waiting time distribution

ii.) Velocity Model → based on

(presumed) insight into characteristic velocity, separate P via:

$$P(\Delta x, \Delta t) = \delta\left(\Delta t - \frac{|\Delta x|}{v}\right) P(\Delta x)$$

~ converts $P(\Delta t)$ to $P(\Delta x)$

etc. (avalanche, $V \to V_*$)

→ Waiting Time Models (heavily studied).

- usual random walk - regular inputs in time $t = N\tau$ (τ = step)

- τ now random, sampled from 1 sided waiting time pdf $\psi(t)$

$\psi(t) \equiv$ waiting time distribution

- before, F.T. for walks over ∞ spatial domains

– Laplace Transform for time
(1 sided).

$$\psi(s) = \int_0^\infty e^{-st}\psi(t)\,dt$$

$$\psi(t) = \int_{c-i\infty}^{c+i\infty} e^{st}\psi(s)\,\frac{ds}{2\pi i}$$

c for convergence.

Can play same game with waiting times using moments

i.e. $p(k)$ encodes $\langle x^n p(x)\rangle$ moments

$\psi(s)$ encodes $\langle t^\alpha \psi(t)\rangle$

$$\boxed{\langle T^n\rangle = \int_0^\infty t^n\psi(t)\,dt = (-1)^n \left.\frac{d^n\psi}{ds^n}\right|_{s=0}}$$

and if power law tail

$$\psi(t) \sim \frac{A}{t^{1+\alpha}} \quad (t\to\infty)$$

analogous $p(x)$).

then

$$\psi(s) \sim 1 - Bs^{\alpha} \quad (s \to 0)$$

and related:

$$\psi(t) \sim \frac{A t^{\beta}}{\Gamma(1+\beta)} \quad t \to \infty, \qquad \psi(s) \sim \frac{A}{s^{1+\beta}}$$

$s \to 0.$

$$[f * g](t) = \int_0^t f(t')\, g(t-t')\, dt'$$

so

$$[\widetilde{f * g}](s) = \tilde{f}(s)\, \tilde{g}(s),$$

– Random Waiting Times

(Distribution between arrival in lines, traffic, etc.).

Now, assume

– no correlation between steps.

– define $P(N,t)$ as
probability of N events in t

$\psi(t)$ is waiting time distribution.

so

$P(0,t) \equiv$ probability of no events to time t.

$$P(0,t) = \int_t^\infty \psi(t')\,dt' \equiv \Psi(t).$$

$\Updownarrow$

ϕ_W

For $P(1,t)$ → calculate prob. 1st at t', no subsequent

$$P(1,t) = \int_0^t \psi(t')\,\Psi(t-t')\,dt' = \psi * \Psi$$

N events:

$$P(N,t) \equiv \psi * \psi * \psi * \dots \Psi$$

$$= \underbrace{\psi^N}_{\text{conv.}} * \Psi$$

Then,

$$P(N,s) = \psi^N \Psi = \frac{\psi(s)^N [1 - \psi(s)]}{s}$$

Now seek moments → # steps over time t.

example: ① Consider Poisson Process:

$$\psi(t) = \lambda e^{-\lambda t} = \frac{1}{\bar{\tau}} e^{-t/\bar{\tau}} \quad \text{(char. time)}$$

$$\psi(s) = \lambda / \lambda + s = 1 / 1 + \bar{\tau} s$$

$$\simeq 1 - \bar{\tau} s \qquad (s \to 0)$$

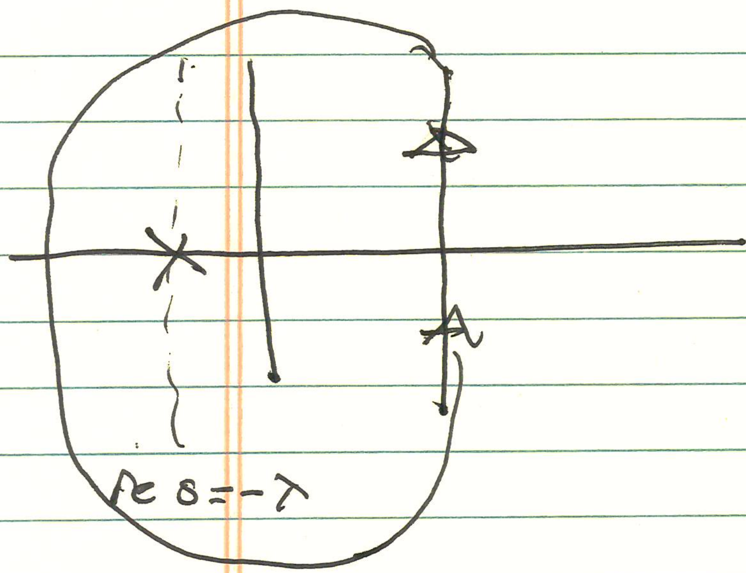
$\langle \tau \rangle = \bar{\tau}$ → waiting time.

②

$$P(N,s) = \left(\frac{\lambda}{\lambda + s}\right)^N \frac{1}{s} \left[1 - \frac{\lambda}{\lambda + s}\right]$$

$$= \lambda^N / (\lambda + s)^{N+1}$$

$$P(N,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{ds\, e^{st}\lambda^N}{(\lambda+s)^{N+1}}$$



integrate by residues

$$f^N(z_0) = \frac{n!}{2\pi i}\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

so

$$\text{res}\,(s=-\lambda) = \frac{\lambda^N}{N!}\frac{d^N}{ds^N}\left[e^{st}\right]\Big|_{s=-\lambda}$$

$$= \frac{(\lambda t)^N e^{-\lambda t}}{N!}$$

$N(t)$ is also Poisson, if $\nu(t)$ is Poisson.

(2) More generally, if consider N as # steps in a walk

$$\langle N(t) \rangle = \sum_{N=0}^{\infty} N \, P(N,t)$$

$$\langle N(s) \rangle = \sum_{N=0}^{\infty} N \, P(N,s)$$

$$= \frac{1}{s} \left[1 - \psi(s) \right] \sum_{N=0}^{\infty} N \, \psi(s)^N$$

$$= \frac{1}{s} \left[1 - \psi(s) \right] \psi(s) \frac{d}{d\psi} \left[\frac{1}{1-\psi(s)} \right]$$

$$N(s) = \psi(s) \Big/ s \left[1 - \psi(s) \right]$$

Need invert. Now $\rightarrow$ 2 cases:

- $\overline{T} = \langle T \rangle < \infty$ well defined

- $\langle T \rangle = \infty$

Case 1: $\bar{T} = \langle T \rangle < \infty$ (well defined)

$$\psi(s) \simeq 1 - \bar{T}s \quad \text{so}$$

$$\langle N(s) \rangle \simeq \frac{1 - \bar{T}s}{s[1 - 1 + \bar{T}s]} \sim 1/\bar{T}s^2$$

so

$$N(t) = t/\bar{T}$$

i.e. Mean # steps increases linearly with $t/\bar{T}$
Random waiting times not import.

Case 2: $\langle T \rangle = \infty$

$$\langle N(s) \rangle \sim \frac{1 - Bs^{\alpha}}{s[1 - 1 + Bs^{\alpha}]} \sim \frac{1}{Bs^{1+\alpha}}$$

so

$$\langle N(t) \rangle \sim t^{\alpha}/B\,\Gamma(1+\alpha)$$

as $\alpha < 1$ → $\langle x^2 \rangle$ sublinear in time.

can push further.

→ CTRW, again (Wrap).

- in CTRW, each step has waiting time τ, Δx displ.

- can be separable

- position:

$$X(t) = \sum_{n=0}^{\infty} \Delta x_n$$

- formulate: 2 approaches

a) - Montroll, Weiss - work with $\psi(t)$

b) - $P(N,t)$

a) → anomalous diffn / walks divergent mean wait times

b) $P(N,t)$ better when τ_w well defined.

Seek $\text{pdf}[x(t)] = P(x,t)$

$$P(x,t) = \sum_{N=0}^{\infty} \underbrace{P(N,t)}_{P_N \text{ steps in } t} \underbrace{P_N(x)}_{P \text{ at } x \text{ after } N}$$ (separability)

so

$$P(k,s) = \sum_{N=0}^{\infty} P(N,s)\, P_N(k)$$

$$= \frac{1-\psi(s)}{s} \sum_{N=0}^{\infty} \left[\psi(s)\, p(k)\right]^N$$

$$= \left(\frac{1-\psi(s)}{s}\right)\left[\frac{1}{(1-\psi(s)\, p(k))}\right]$$

separability

$$\boxed{P(k,s) = \left(\frac{1-\psi(s)}{s}\right)\left[\frac{1}{1-\psi(s)\, p(k)}\right]}$$

→ Montroll–Weiss Eqn.

→ Gen Fctn. fo $P(x,t)$

so, for position:

$$\langle x(t) \rangle = -i \frac{dP}{dk}\Big|_{k=0} = \frac{\langle \Delta x \rangle \psi(s)}{s(1-\psi(s))}$$

$\langle |\Delta x| \rangle < \infty$.

so

$$\langle x(s) \rangle = \langle \Delta x \rangle \langle N(s) \rangle$$

so

$$\boxed{\langle x(t) \rangle = \langle \Delta x \rangle \langle N(t) \rangle}$$

and of course can get $P(k,t)$.

- Results hold for anomalous cases
- Also get second cumulant / std dev

$$\langle x^2(t) \rangle - \langle x(t) \rangle^2 = \sigma_x^2$$

variance step #

$$= \sigma_{\Delta x}^2 \langle N(t) \rangle + \sigma_N^2(t) \langle \Delta x^2 \rangle$$

variance step size

- can't decouple by looking at diffn

N.B. Terminology

- Leapers → jump at Δt
- Creepers → move at V (velocity model)

Much more → insufficient time.

Next { - Avalanches
- Self-Organized Criticality.