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Fractional Kinetics

i.) Introduction
ii.) Motivation - Transport on the Standard Map
iii.) Structure of the Theory
iv.) Results ⇒ Physics
v.) Comparison / Contrast with F-P Theory

Refs:
* - Zaslavsky, book & review
- Klafter

→ Introduction

- Fractional Kinetics ⇒ extension of F-P. Eqn. to fractional derivatives

i.e. $\partial_t \to \partial_t^{\alpha}$

$\partial_x \to \partial_x^{\beta}$

α, β not integers.

Exponents α, β ("critical exponents") are the key.

- why?

- anomalous scaling

i.e. $\langle \delta x^2 \rangle \sim t \Rightarrow \langle \delta x^2 \rangle \sim t^{\mu}$

$\mu \to 2\beta/\alpha$, etc.

– phase space "rough" or fractal, so must address roughness in kinetic equation (origin fractional)

i.e. $$\Delta t \frac{\partial \rho}{\partial t} \to (\Delta t)^{\beta} \frac{\partial^{\beta} \rho}{\partial t^{\beta}}$$

n.b.

$$(-i\omega)^{\beta} \rho(\omega) = T\left(\frac{\partial^{\beta} \rho}{\partial t^{\beta}}\right) \qquad \beta \leq 1$$

transform

similar space, etc.

N.B.: Fractional time can obviously represent sticking, flights, etc.

– Relevant if fractal structure in phase space → "sticky" (singular) domains

→ Motivation

Recall the Standard Map (J. B. Taylor, Chirikov),
(see OH, Chapt. 7)

$$I_{n+1} = I_n + K \sin\theta_n$$

K: strength parameter

$$\theta_{n+1} = \theta_n + I_n \quad , \quad \text{mod } 2\pi$$

kicked rotor

obviously represents:

$$\frac{dv}{dt} = \frac{Z}{m} E$$

$K \to |E|$

$$dx/dt = v$$

→ simple dynamical system characteristics of:

$$\partial_t f + v\,\partial_x f + \frac{Z}{M} E\,\partial_v f = 0.$$

$K > K_{crit}$ for chaotic motion ⇒ long story of:
- fixed pt instab.
- h.o. fixed pts.
- overlap
- 2nd ary overlap

⋮

⋮

∪ standard map represents f evolution.
Obviously no analogue of Poisson's
Eqn ⇒ stochastic acceleration.

Now, can define

$$D = \lim_{n\to} \langle (\theta_{n+1} - \theta_n)^2 \rangle$$

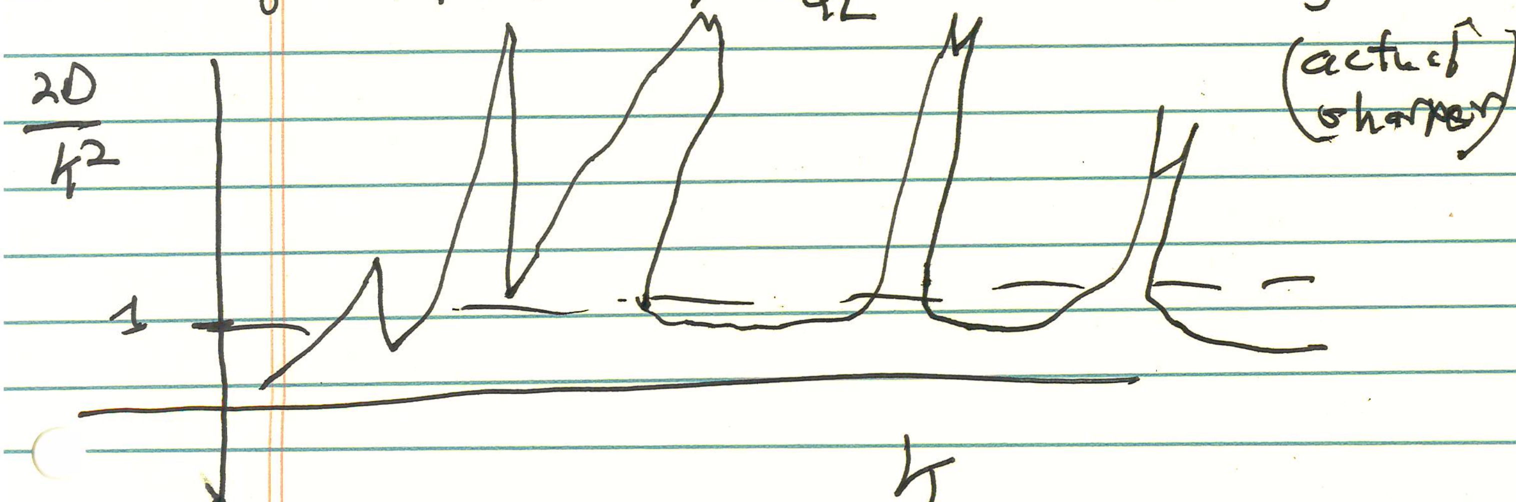
$$D_{QL} = \lim_{n\to\infty} \langle (\theta_{n+1} - \theta_n) \rangle$$

$$= \lim_{n\to\infty} \langle K^2 \sin^2\theta_n \rangle$$

$$= K^2/2.$$

$$D_{QL} = \frac{q^2}{m^2} \sum_{\underline{k}} |E_k|^2 \pi \delta(\omega - kv)$$

So now, if plot D/D_{QL} for large n;



n.b. really spikey

Point:

- deviations beyond QL at certain, fractally distributed values K.

- as n increases, spikes larger grow

i.e. $n \to t$

heuristically,

$$D \sim \frac{\langle \delta x^2 \rangle}{t}$$

if $D \sim n^{\sigma} \sim t^{\sigma}$ $\qquad \sigma > 0$

$$D \sim \frac{\langle \delta x^2 \rangle}{t} \sim t^{\sigma}$$

$$\langle \delta x^2 \rangle \sim t^{1+\sigma} \quad \Rightarrow \text{superdiffusive}$$

- spikes related to accelerator modes of standard map

$$I_{l,\ell} = 2\pi M \qquad M, \ell \text{ integers.}$$

i.e.

$$k \sin \Theta_{l\ell} = 2\pi \ell$$

I at fixed pt increases by $2\pi \ell$ / iteration.

"Always historicize."
— Frederic Jameson

See Lichtenberg and Lieberman
— paper topic!

⇒ accelerator modes produce long time correlations

⇒ dominate transport, drive non-diffusive transport

⇝ intellectual driver for fractional kinetics, (Zaslavsky)

Now:

- many 'accelerator modes', exponents localized in domains
- produce super-diffusion
- special values K underpin flights

Speculative

- unresolved question: fate of flights in self-consistent problem

i.e.

- push particles a la stan. map
- ? distribute initial V according B-O-T
- update K according γ.

would expect K evolution would detune K resonances!?

→ Structure of the Theory

key is critical exponents.

Now, take fractal time:

$$\Delta_t P(x,t) = \left(\frac{\partial P}{\partial t}\right) \Delta t$$

becomes

$$\Delta_t^\beta P(x,t) = \frac{\partial^\beta P(x,t)}{\partial t^\beta} + o\,(\Delta t^{\beta})$$

for corresponding spatial scattering:

$$\Delta_x^\alpha P(x,t) = \int dy\, W(x,y;\Delta t)\, P(y,t) - P(x,t)$$

— formal !.

so, conservation probability:

$$\Delta_t^\beta P(x,t) = \Delta^\alpha P(x,t) + (\Delta t)^\beta$$

so

$$\lim_{\Delta t \to \infty} \frac{1}{(\Delta t)^\beta} \Delta_t^\beta P(x,t) = \lim_{\Delta t \to \infty} \frac{1}{(\Delta t)^\beta} \Delta_x^\alpha P(x,t).$$

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left\{ \int dy\, W(x,y,\Delta t)\, P(y,t) - P(x,t) \right\}$$

Now, expand W as usual, but with fractional exponents:

$$W(x,y;\Delta t) = \delta(x-y) + A(y,\Delta t)\, \delta^{\alpha}(x-y) + \frac{1}{2} B(y,\Delta t)\, \delta^{\alpha_1}(x-y)$$

$$0 < \alpha < \alpha_1 \leq 2$$

$\alpha = 1,\ \alpha_1 = 2 \Rightarrow$ reduces to Fokker-Planck.

N.B.: – A, B indep P (linear).

– W local, P non-local dynamics

equivalent to large time behavior independent local transitions.

Now, seek express A, B in terms W – as moments.

- Strategy:
  - relate B → easy
  - then relate A to B

    (similar to D, V relation).

Now,

$$W(x, y, \Delta t) = \delta(x-y) + A(y,t)\,\delta^{\alpha}(x-y) + \frac{1}{2}\, B(y, \Delta t)\, \delta^{\alpha_1}(x-y)$$

and * $|x-y|^{\alpha_1}$

and $\int dx$

trans prob.

⇒

$$\langle |\Delta x|^{\alpha_1} \rangle = \int dx\, |x-y|^{\alpha_1}\, W(x, y, \Delta t)$$

so

$$\langle |\Delta x|^{\alpha_1} \rangle = \int dx\, |x-y|^{\alpha_1} \Big\{ \delta(x-y) + A(y, \Delta t)\, \delta^{\alpha}(x-y) + B(y, \Delta t)\, \delta^{\alpha_1}(x-y) \Big\}$$

$\alpha_1 > \alpha$ so chop $\Rightarrow$

$$\langle |\Delta x|^{\alpha_1} \rangle = \alpha_1 ! \; B(y, \Delta t)$$

$$\boxed{\langle |\Delta x|^{\alpha_1} \rangle = \Gamma(1+\alpha_1) \; B(y, \Delta t)}$$

Now, to get A:

$$- \int dy \; [\; W \text{ eqn.} \;]$$

$\Rightarrow$

$$1 = 1 + \int dy \, \frac{\partial^{\alpha} A(y, \Delta t)}{\partial y^{\alpha}} \, \delta(x-y)$$

$$+ \int dy \, \frac{\partial^{\alpha_1} B(y, \Delta t)}{\partial y^{\alpha_1}} \, \delta(x-y)$$

i.e. akin:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x} \left\{ + \frac{\langle \Delta x \rangle}{\Delta t} P - \frac{\partial}{\partial x} \frac{\langle \Delta x \Delta x \rangle}{2 \Delta t} P \right\}$$

$$\frac{\partial}{\partial t}\int dx\, x P = \int dx \frac{\langle \Delta x \rangle}{\Delta t} P - \int dx \frac{\partial}{\partial x} \frac{\langle \Delta x \Delta x \rangle P}{2\Delta t}$$

—

$\Rightarrow$

$$\boxed{\frac{\partial^{\alpha} A(x,\Delta t)}{\partial(-x)^{\alpha}} + \frac{\partial^{\alpha_1} B(x,\Delta t)}{\partial(-x)^{\alpha_1}} = 0} \quad *$$

– relates A to B.

$$\boxed{\begin{aligned} A(x) &= \lim_{\Delta t \to 0} \frac{A(x,\Delta t)}{(\Delta t)^{\beta}} \\ B(x) &= \lim_{\Delta t \to 0} \frac{B(x,\Delta t)}{(\Delta t)^{\beta}} \end{aligned}}$$

divide $*$ by $(\Delta t)^{\beta}$ and use A, B defs. $\Rightarrow$

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left[\int dy \left\{ W(x,y,\Delta t) - \delta(x-y) \right\} P \right]$$

so

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^\beta} \left[\int dy \left\{ A(y,\Delta t)\, \partial^\alpha (x-y) + B(y,\Delta t)\, \partial^{\alpha_1}(x-y) \right\} P(y,t) \right]$$

$\Rightarrow$

$$\boxed{\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha} \left\{ A(x) P(x,t) \right\} + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} \left\{ B(x)\, P(x,t) \right\}}$$

- FKE
- $\alpha, \alpha_1, \beta \to$ critical exponents
- A, B expressed as moments W.

~ not easy to calculate with.

Some observations:

- $\alpha_1 = \alpha + 1$ and use A, B relation.

$$\boxed{\frac{\partial^\beta P(x,t)}{\partial t^\beta} = -\frac{\partial^\alpha}{\partial(-x)^\alpha}\left(B(x)\frac{\partial P}{\partial x}(x,t)\right)}$$

- if $\alpha = \beta = 1$, $B = 1/2\,D$
regular F-P.

- if $B(x)$ neglible

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha}\left(A(x) P(x,t)\right)$$

$\rightsquigarrow$ $\beta = 1$, $\alpha = 2$ $\Rightarrow$ diffusion

$\rightsquigarrow$ $0 < \beta < 1$, $\alpha = 2$ $\Rightarrow$ fractal B.M.

$$\frac{\partial^\beta}{\partial t^\beta} P(x,t) = \frac{\partial^2}{\partial |x|^2}\left(A(x) P(x,t)\right)$$

~ $\beta = 1$ $\quad 1 < \alpha < 2$

⇝ Lévy Process

$$\frac{\partial P}{\partial t} = \frac{\partial^\alpha}{\partial |x|^\alpha} \left(A(x) P(x,t) \right)$$

⇝ Physics of FKE

- primarily interested in moments of $P(x,t)$

⇒ observables

$$\langle |x|^\delta \rangle = \int dx \, |x|^\delta P(x,t)$$

if, as before, $A \sim$ slow, B negligible

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha} \left(A \, P(x,t) \right)$$

$* \, |x|^\alpha$

$$\frac{\partial^\beta}{\partial t^\beta} \left(|x|^\alpha P(x,t) \right) = A |x|^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}$$

Integrating, using moments:

$$\frac{\partial^\beta}{\partial t^\beta}\langle |x|^\alpha \rangle = A\int dx\, |x|^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}$$

$$= A\int dx \left(\frac{\partial^\alpha}{\partial |x|^\alpha}|x|^\alpha\right) P(x,t)$$

$$= \alpha! \, A \int dx\, P(x,t)$$

$$= \Gamma(1+\alpha)\, A$$

integrating ⇒

$$\langle |x|^\alpha \rangle = \frac{A\,\Gamma(1+\alpha)}{\Gamma(1+\beta)}\, t^\beta$$

For self-similar soln., expect

$$\langle |x| \rangle \sim t^{\beta/\alpha} = t^{\mu/2}$$

$$\Rightarrow \quad \boxed{\mu = 2\beta/\alpha}$$

so for finite variance;

$\langle x^2 \rangle \sim t^{\mu}$

Point: $A \sim$ const, B negligible FKE produces (self-similar) result that allows for anomalous diffusion

i.e. $\boxed{\text{Fractional kinetics realizes anomalous diffusion scaling.}}$

(but needs input $\alpha, \beta \Leftrightarrow W$).

of course:

$\mu > 1 \rightarrow$ super-diffusion (flights)

$\mu < 1 \rightarrow$ sub-diffn. (sticky walks, long waits)

$\mu \longleftrightarrow \alpha, \beta \Leftrightarrow$ moments of transition probability (+ their short time evolution).

where from $\rightarrow$ structure of phase space.

N.B.: Critical exponents depend on windows.

→ Relation to F-P.

	F-P	F-K
stochastic variable	Δx	$\Delta x, \Delta t$
time	fixed clock	variable pdf (as CTRW)
variance	$\langle \lvert x \rvert^2 \rangle \sim t$	$\langle \lvert x \rvert^2 \rangle \sim t^{\mu}$, $\mu < 2$
input	T	W, α, β
[redacted] Reln. bet A, B (V, D)	$A = \frac{1}{2} \frac{\partial B}{\partial y}$	$\frac{d^{\alpha} A(x)}{\partial (-x)^{\alpha}} + \frac{d^{\alpha_1} B(x)}{\partial (-x)^{\alpha_1}} = 0$

cont'd

	F-P	F-K
C-K eqn	usual	fractal
$A(y, \Delta t)$ (drift)	$\langle \Delta y \rangle$	no simple form
$B(y, \Delta)$ (diffn)	$\langle (\Delta y)^2 \rangle$	$\dfrac{\langle \lvert \Delta x \rvert^{\alpha_1} \rangle}{\Gamma(1+\alpha_1)}$

Comments

N.B.: F-K expands kinetics to anomalous diffusion, but no panacea
- general formalism
→ need W; α, β (as need T)
→ can have many critical exponents

Non-trivial

Comments

→ Real question in F-K:

Link to self-consistent nonlinear dynamics.

i.e. can understand resonance broadening as:

- stochastic orbits, modelled as diffn.

- limit of renormalized perturbation theory

and can calculate:

$\epsilon(k, \omega \mid D)$.

recall MNR advocated:

$\epsilon_{IM}(k, \omega \mid D) = 0 \Rightarrow D$

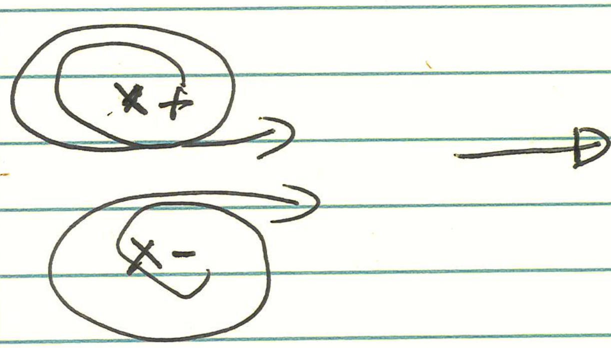
for M-instabilities.

How complement for F-K ??

→ Is there B-O-T problem for F-K? How do accelerator (correlated scattering) modes appear in B-O-T.

→ N.B. Zaslavsky on flights

2D H-M:



dipole pairs.