Notes 12 — Physics of Levy Flights, CTRW and Fractional Kinetics.

Recently: – introduced L-stable distributions
– ex: Gaussians ⟺ C.L.T.
– extended range of L-stable, not require finite variance.
– used self-similarity

$$\Rightarrow L_\alpha(a,k) = \exp\left[-a|k|^\alpha\right]$$

$\Downarrow$

Levy Distrib $\qquad 0 < \alpha \leq 2$

$\alpha \rightarrow$ power law

$a \rightarrow$ akin extended strength parameter (i.e. D)

of course: $P_\alpha(x,a) = \int \frac{dk}{2\pi} e^{ikx} L_\alpha(a,k)$

Properties: $L_\alpha > 0 \qquad \left(\psi(k\mu(\lambda)) = \lambda\psi\right)$

$\int dx\, P_\alpha(a,x) < \infty$.

obviously: $P_\alpha(x,a) \xrightarrow[x\to\infty]{} 1/|x|^{\alpha+1}$

$\alpha \rightarrow \mu$

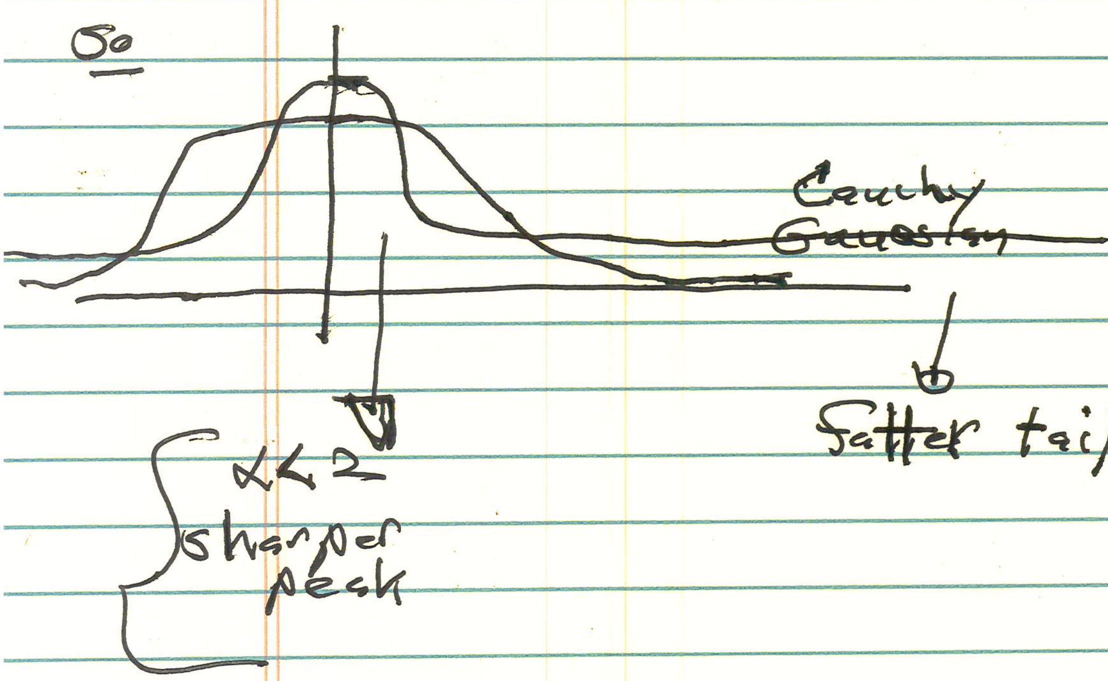
$\alpha = 2 \rightarrow$ Gaussian.

$\frac{d\psi}{dk} = \psi/\mu'(1)k$

$\alpha = 1 \rightarrow$ Cauchy (Lorentzian)

$$p_1(x) = a/\pi(x^2+a^2)$$

So



And can consider time-dependent evolution, analogous to diffusion

- first, if discretize:

$$p_N(x) = \frac{1}{N^{1/\alpha}} L_\alpha(a, x/N^{1/\alpha})$$

width $\sim N^{1/\alpha}$

$\alpha < 2$, $N^{1/\alpha} \gg N^{1/2} \rightarrow$ super-diffusive

– making time explicit :

$$P_\alpha(k, ct) = \exp\left[-ct|k|^\alpha\right]$$

$\alpha \to$ index

$c \leftrightarrow a \to$ strength

characteristic Fctn.

$$P_\alpha(x,t) \xrightarrow[x\to\infty]{} t\,/\,|x|^{\alpha+1}$$

N.B. Can see description of Levy process involves fractional calculus.

i.e.

$$P_\alpha(x,t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} P_\alpha(k,t)$$

so $\partial_t P_\alpha(x,t) =$

$$\int \frac{e^{ikx}}{\sqrt{2\pi}} \left(-c|k|^\alpha\right) \exp\left[-ct|k|^\alpha\right]$$

$\Downarrow$

Fractional derivative for $\alpha \neq 2$.

if $\alpha = 2$

$c \Rightarrow D$

$$\partial_t P_{\alpha=2}(x,t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} (-Dk^2) \exp\left[-Dtk^2\right]$$

$$= D \frac{\partial^2}{\partial x^2} \int \frac{e^{ikx}}{\sqrt{2\pi}} \exp\left[-Dtk^2\right]$$

$$= D \frac{\partial^2}{\partial x^2} P_2(x,t)$$

i.e. $\frac{\partial^2}{\partial x^2} \Leftrightarrow -k^2$

$$\partial/\partial x \Leftrightarrow ik \Rightarrow (\partial/\partial x)^\alpha \Leftrightarrow (ik)^\alpha$$

so $(ik)^\delta \Leftrightarrow (\partial/\partial x)^\delta$

meaning of fractional calculus
(i.e. integral transform defines
fractional derivative

→ What does it all mean ??

– Levy Walks self-similar

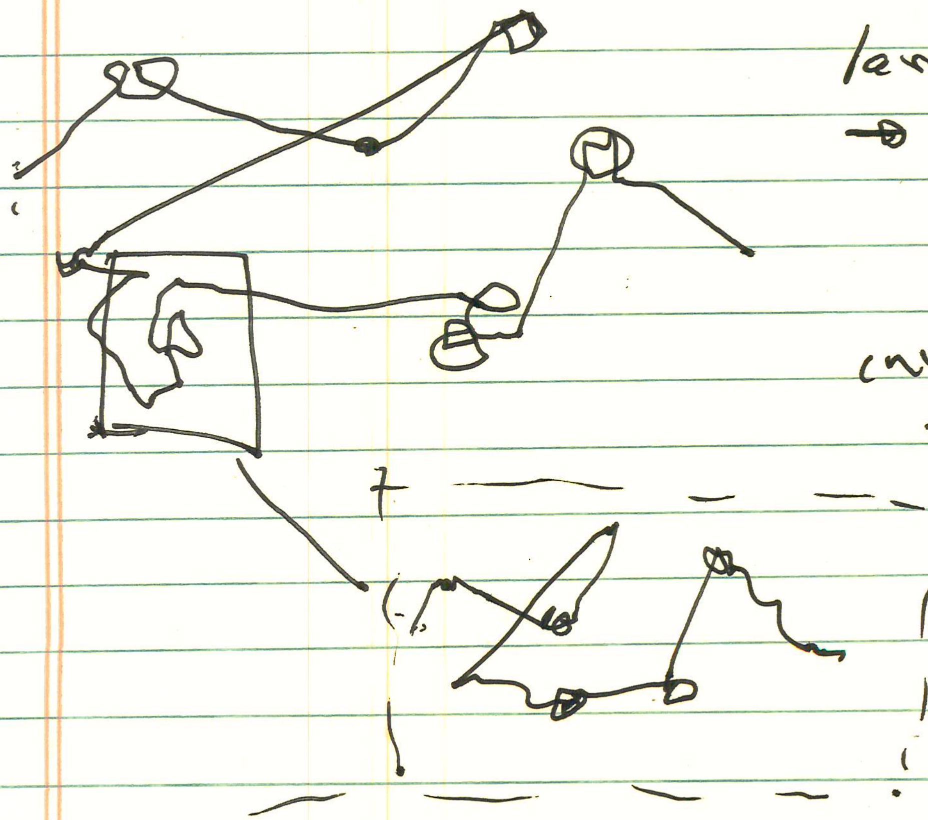
i.e. even diffn → $\frac{x^2}{t} \to \frac{\lambda^2 x^2}{\delta t}$

invariant under $\delta \sim \lambda^2$.

– some very large excursions (contrast standard random walk)

⇒ fat tail, large events weighted more

i.e.



large jumps → flights

invariant under zoom.

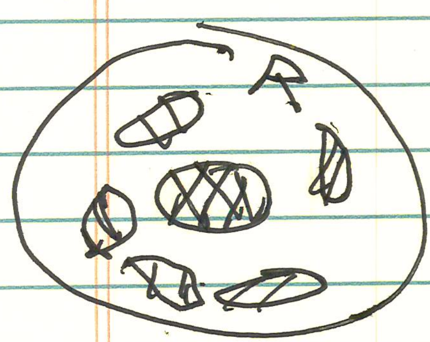
– a bit of physics:

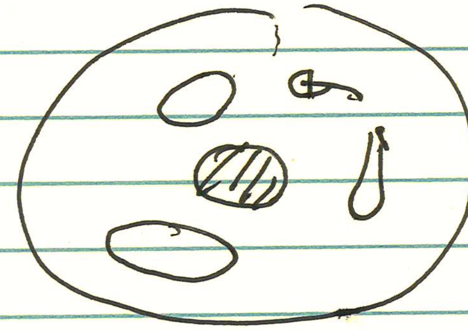
Solomon,
Weeks, ... Swinney et al. PRL '93
a must

→ rotating tank, water pumped in, out via bottom.

→ sheared, counter-rotating azimuthal jet.

→ passive tracers injected



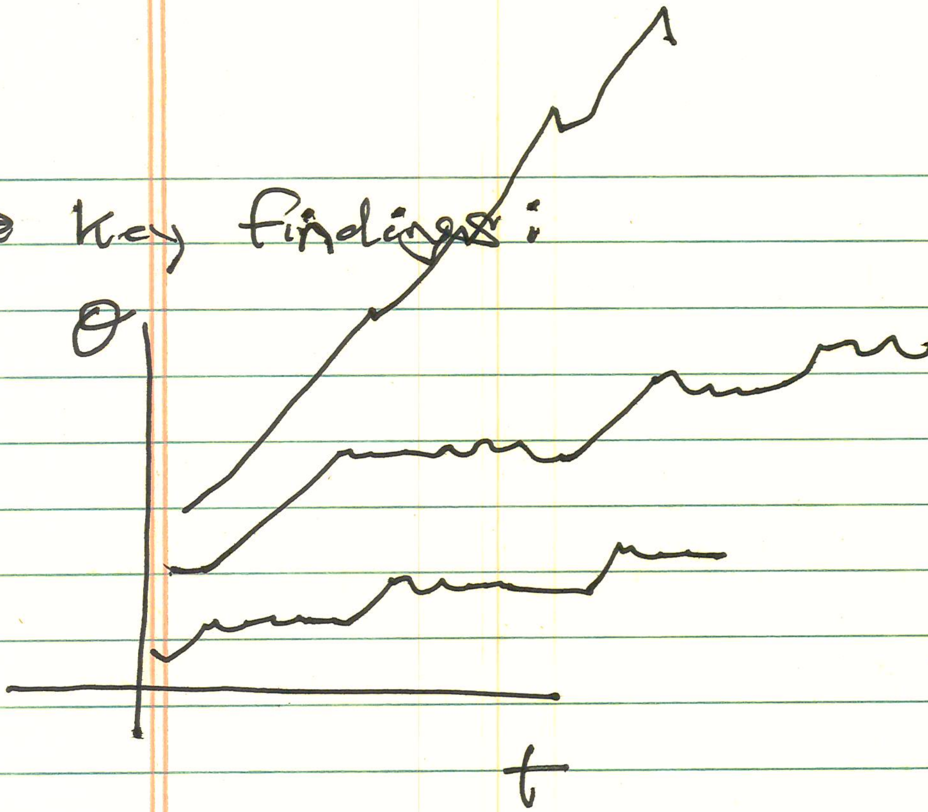


→ azimuthal vortex chain. (6)

→ follow tracer trajectories, which are chaotic,
but no "turbulence"

(flow chaotic, not turbulent)

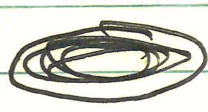
→ Key findings:

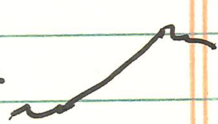


obviously:

→  are circulations in individual vortex.

tracer 'stuck' in individual vortex



→  are jumps or "flights" between vortices.

These are the large excursions of the Levy walk.

→ Can compute $\langle (\theta - \theta_0)^3 \rangle$

and $\langle (\theta - \theta_0)^2 \rangle \sim t^{1.6}$

→ Levy Flight

→ super-diffusive

$1/2 < H < 1$ (not calculated)

→ anomalous diffusion

N.B. → Anomalous Transport

$\Rightarrow D > D_{neo}$

Anomalous Diffusion

$\langle \delta x^2 \rangle \sim t^{\alpha}$

$\alpha \neq 1$

→ Analogy → marker particles

in - island chain (positrons)
(chaos)

- Zonal flow + vortex

(novelty?)

→ How Live in Levy World

– how calculate anything for diffusion processes, noise, etc.?

⇒ Fokker-Planck Theory. → yields pdfs

– What is F-P Eqn?

~ recipe for turning (micro) on step pdf into eqn.

for distribution

~ can calculate D, V, $P(x,t)$

$P(x,t)$
$\cancel{t \to \infty}$

of course:

~~first~~ 1 step ago.

$$P(x, t+\Delta t) = \int d(\Delta x)\, T(\underline{x}, \underline{\Delta x}, \Delta t)\, P(\underline{x} - \underline{\Delta x}, t)$$

transition probability

→ tacit assumption Δx Δt smaller than variation scale of pdf

→ $\langle (\Delta x)^2 T \rangle < \infty$
(dubious for Levy)

→ regular clock (Δt)

so

$$P(x,t) + \Delta t \frac{\partial P}{\partial t} = \int d[\Delta x] \left\{ T P - \frac{\partial}{\partial x}(\Delta x \, T P) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\Delta x^2 T P) \right\}$$

$$\int d\Delta x \; T = 1$$

$$\int d(\Delta x) \frac{\Delta x \, T}{\Delta t} = V$$ → mean convection velocity, drift

$$\int d(\Delta x) \frac{(\Delta x)^2 T}{2 \Delta t} = D$$ → diffusion

then

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left\{ \frac{\langle \Delta x \rangle}{\Delta t} P - \frac{\partial}{\partial x} \frac{\langle \Delta x \Delta x \rangle}{2\Delta t} P \right\}$$

$$= -\frac{\partial}{\partial x}\left\{ V P - \frac{\partial}{\partial x} D P \right\}$$

why $\frac{\partial}{\partial x}\{ \quad \}$? $\rightarrow$ conserve probability

note order. $\sim \nabla \cdot \Gamma$

and have ∞ of applications.

N.B. Often write diffusion in form:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial f}{\partial v}$$ as in QL

Can connect to F-P if

$$V = \frac{1}{2}\left(\frac{\partial D}{\partial v}\right)$$

Two big assumptions:

→ uniform, short time step Δt (fixed clock)

→ exception: Δt is statistically distributed ①

→ CTRW (Continuous Time Random Walk)

$\langle \Delta x^2 \rangle < \infty$ → no anomalous diffusion

② Fractional kinetics — ala' CTRW but Markovian with Fractional Derivatives

Pinned to scaling exponents:

$\langle \delta x^2 \rangle \sim t^{\gamma}$

for $\gamma \neq 1$ for anomalous diffusion / Levy walks.

References: Montroll, Klafter, Zaslavsky, Schlesinger .... posted.

- CTRW ⇒ accomodate flights, sticking in random walk model

→ ~~[illegible]~~ release time from dummy role, allow evolve dynamically, as walker position does.

→ solve the variance problem by allowing (divergent) steps in space to be accomplished in large step time

→ in CTRW:

$$\underline{r}_n = \underline{r}_0 + \underline{\Delta r}_1 + \dots \underline{\Delta r}_{n-1} + \Delta r_n$$

$$t_n = t_0 + \Delta t_1 + \Delta t_2 + \dots \Delta t_n$$

$\Downarrow$

times add, distributed series

so need PDF (Δt_i).

→ 2 approaches to CTRW
(Klafter)

⇝ <u>waiting time model</u>

- steps in position, time independent
- need specify 2 probabilities $\begin{cases} \Delta r \\ \Delta t \end{cases}$

idea: particle waits Δt at position — "sticking" — then jumps Δr in no time

⇝ velocity Model

- Δt → traveling time of particle

$$\Delta t = |\Delta r| / v$$

↳ const v

increments satisfy

$$\delta(\Delta t - |\Delta r|/v)\, P(\Delta r)$$

→ general – specify joint probability

To develop: C–K Eqn again
now have increments
Δx, Δt

joint pdf ↑ 2 randoms

$$Q(x,t) = \int d(\Delta x) \int_0^t d(\Delta t)\, P(\Delta x, \Delta t) \,*\, Q(x-\Delta x, t-\Delta t)$$

dist. of orbit/jump points

so for waiting → factorize joint pdf

$$P(\Delta x, \Delta t) = P(\Delta x)\, P(\Delta t)$$

– if $P(\Delta x)$ standard Gaussian expand

For $P(\Delta t) \to 1$ → recover F–P Eqn.

lowest order expansion in Δx

⟹

$$Q(x,t) = \int d(\Delta x) \int_0^t d(\Delta t)\; p(\Delta x)\, p(\Delta t) \times \left\{ Q(x, t-\Delta t) - \frac{\partial}{\partial x} \cancel{\Delta x\, Q\, p} \right\}$$

$$= \int_0^t d(\Delta t)\, p(\Delta t)\, Q(x, t-\Delta t)$$

and now cannot wait at least Δt

$$\phi_w(\Delta t) = \int_{\Delta t}^{\infty} dt'\, p(t')$$

$$\boxed{\begin{aligned} Q(x,t) &= \int_0^t d(\Delta t)\, \phi_w(\Delta t)\, Q(t-\Delta t) \\ \phi_w &= \int_{\Delta t}^{\infty} dt'\, p(t') \end{aligned}}$$

to go further, must specify $p(t')$ of time step

→ Note memory effect in evolution
⇒ eqn. is non-Markovian.

→ expand → ?

→ Velocity Model

$$P(\Delta x, \Delta t) = \delta\left(\Delta t - \frac{|\Delta x|}{v}\right) P(\Delta x)$$

and once again, plug into CK:

$$Q(x,t) = \int_{-vt}^{vt} d(\Delta x) \int_{0}^{\Delta t} d(\Delta t)\ Q(x-\Delta x, t-\Delta t) * \Phi_v(\Delta x, \Delta t)$$

Φ_v = probability to make a step of length $|\Delta x|$ in duration Δt.

$$\Phi_v(\Delta x, \Delta t) = \frac{1}{2}\delta(|\Delta x| - v\Delta t) * \int_{|\Delta x|}^{\infty} dx' \int_{\Delta t}^{\infty} dt\ P(x')\ \delta\left(t - \frac{|x'|}{v}\right)$$

need choose $P(x')$:

OV (Concluding) of CTRW:

- no magic results depend on input step probability

- Δt, as well as Δx, stochastic is key.

- small Δt small $\Delta x \rightarrow$ normal diffn / F-P Eqn

- need long $\Delta x \rightarrow$ flights for
  $\Delta t \rightarrow$ sticking
  anomalous diffusion

- sticking $\rightarrow$ sub-diffusion $\quad \gamma < 1$
  flights $\rightarrow$ super-diffusion $\quad \gamma > 1$
  Levy.

- CTRW models:
  - non-local
  - non-Markovian
  in space time

so — time history matters

— evolution by integral eqn.

build up from C-K eqn.

This brings us to fractional kinetics!
(see Zaslavsky)

— FK more systematic, though less intuitive than CTRW.

— FK relevant if:

1) phase space 'rough' or fractal

so need treat roughness / fractal structure

→ turbulence.

2) "sticky" domains in phase space.

→ Key is critical exponent - scaling!

∂_t^{α}, ∂_x^{β} instead ∂_t, ∂_x

relate $\langle \delta x^2 \rangle \sim t^{\mu}$!

→ Very technical → next time, an introduction.

but

Conceptually simple.

→ Key Open Issue → relate to basic system physics.