Notes 11 — L-stable Distributions, Levy Walks etc.

Recall many new elements:

— H-parameter ("random" stationary series)

$0 < H < 1/2$ — subdiffusion (sticky)

$1/2 < H < 1$ — superdiffusion (flights, ballistic)

$H = 1/2$ — diffusion (Gaussian Distrib.)

also: $\left\{ \begin{array}{l} 1/f \text{ noise} \Leftrightarrow H \sim 1. \\ \text{Zipf's Law} \quad P \sim 1/\Delta x \end{array} \right.$

— Range of $H \Rightarrow$ should broaden our view of random processes.

Fundamental element of conventional wisdom on random processes is Central Limit Theorem.

But range of $H \rightarrow$ more than Gaussians ...

Recall Central Limit Theorem:

(c.f. Chandrasekhar, Kubo, et. al. "Statistical Physics II")

- Consider a sum of n independent random variables $\Delta x_1, \Delta x_2, \ldots \Delta x_n$ $(n \gg 1)$

Let $x_n = \Delta x_1 + \Delta x_2 + \ldots \Delta x_n$

- For Δx_i;

$$\langle \Delta x_i \rangle = 0$$

$$\langle \Delta x_j^2 \rangle = \sigma_i^2$$

n.b. Note only variance (i.e. second moment) assumed to exist.

Let $s_n^2 = \sigma_1^2 + \sigma_2^2 + \ldots \sigma_n^2$

Then $\Rightarrow$

- CLT states that pdf of sum:

$$\underset{n \to \infty}{\mathrm{Pdf}(y)} \rightarrow 1/\sqrt{2\pi}\, e^{-y^2/2}$$

$y_n = x_n / s_n$

$\rightarrow$ Gaussian.

i.e.
$$Pdf(X_n) \underset{n \gg 1}{\cong} \frac{1}{\sqrt{2\pi}\, s_n} \exp\left(-X_n^2 / 2 s_n^2\right)$$

- Key points:

  i.) ΔX_i all alike, sum not dominated by few.

  ii.) finite variance of step prob.

Buried bodies:

- what of higher moments?, i.e.

  $$\langle \Delta X_i^2 \rangle < \infty \not\Rightarrow \langle \Delta X_i^4 \rangle < \infty$$

  $\Rightarrow$ kurtosis can induce heavy tails.

- how much?

Now, observe that CLT effectively states that, modulo conditions,

$X_i \to$ statistically distributed consistent with CLT then if X_i, y_i follow CLT,

then $(a X_i + b) + (a' y_i + b') = a'' Z_i + b''$ follows CLT.

Z_i Gaussian

$a, b > 0$, $a', b' > 0$ non-stoch consts.

Adding two Gaussian distributed series ⇒ sum is Gaussian distributed

⇒ $\boxed{\text{CLT} \Rightarrow \text{Gaussian, modulo conditions is attractor in function space}}$

The point :- Gaussian is a very special case !

- more generally, a class of such distributions exist, which are said to be L-stable (L for Levy).

$\boxed{\text{Gaussian is one particular case, indeed the only which has finite variance}}$

So, need consider $\left.\begin{matrix} x_i \\ y_i \end{matrix}\right\}$ L-stable

$(a x_i + b) + (a' y_i + b') = a'' z_i + b''$

z_i L-stable. → i.e. class of attractors in function space.

→ Big expansion in family of allowed distributions!

→ Encompass super-Diffusion, etc. for $H > 1/2$

↔ Levy Flight!

How understand L-stable Processes?

First, re-visit Central Limit Theorem in depth.

→ How is Central Limit Theorem proved?

Central ideas are:

- Markov property → Chapman-Kolmogorov Equation
- Convolutions

— "Generating" or "Characteristic" function.

Point: Fourier transform of step probability is more significant than probability.

i) C-K Eqn:

$$P_N(\underline{x}) = \int dy \; P_{N-1}(y) \, P_N(x|x-y)$$

(convolution)

i.e. $\left\{ \begin{array}{l} x-y \to x \\ \text{don't expand} \end{array} \right.$

Then, if F.T., and noting that F.T. of ~~[illegible]~~ convolution = Product of functions convolved,

$$P_N(\underline{k}) = \hat{p}_1(\underline{k}) \, \hat{p}_2(\underline{k}) \cdots \hat{p}_N(\underline{k})$$

$$= \prod_{n=1}^{N} \hat{p}_n(k)$$

$$P_N(x) = \int dk \; e^{i\underline{k}\cdot\underline{x}} \prod_{n=1}^{N} \hat{p}_n(k)$$

applies for identical steps.

iii) Then, also have, for moments:

$$m_1 = \langle x \rangle$$
$$m_2 = \langle x^2 \rangle$$
$$\vdots$$
$$m_n = \langle x^n \rangle$$

so

$$\hat{P}(k) = \sum_{n=0}^{\infty} (-i)^n \frac{k^n m_n}{n!}$$

(Prob.)

$$m_n = i^n \frac{\partial^n \hat{P}}{\partial k^n}\Big|_{k=0} \quad \text{, from F.T.}$$

$$\hat{P}(k) = 1 - i m_1 k - \frac{1}{2} k^2 m_2$$

→ easily generalized to higher dimensions

iii) Cumulants

- nonlinear combinations of moments.

$$P(x) = \int e^{ikx} \frac{dk}{(2\pi)} \hat{P}(k)$$

$$= \int e^{[ikx + \psi(k)]} \frac{dk}{2\pi}$$

$$\psi(k) = \ln \hat{P}(k)$$

and,

$$\psi(k) = -i c_1 k - \frac{1}{2} c_2 k^2 + \cdots$$

$$\begin{cases} c_1 = M_1 \\ c_2 = M_2 - M_1^2 = \sigma^2 \end{cases} \quad \text{etc.}$$

(assumes exist!)

generally,

$$i^n C_n^{(j_1, j_2 \ldots)} = \frac{\partial^n \psi}{\partial k_{j_1} \partial k_{j_2} \cdots \partial k_{j_n}}$$

Now, cumulants additive - assuming identical, independently distributed (IID) steps:

$$P_N(x) = \int e^{ikx} \frac{dk}{2\pi} \, P_N(k)$$

$$= \int e^{ikx} \frac{dk}{2\pi} \left(P(k)\right)^N$$

$$= \int e^{ikx} \frac{dk}{2\pi} \left(e^{\psi(k)}\right)^N$$

$$= \int e^{ikx} \frac{dk}{2\pi} \, e^{N\psi(k)}$$

$$\boxed{\psi_N(k) = N\psi(k)}$$

So, for CLT:

$\left\{ \begin{array}{l} \text{consider as} \\ N \to \infty \end{array} \right.$

$$P_N(x) = \int_{-\infty}^{+\infty} e^{ikx} \prod_{n=1}^{N} \hat{P}_n(k) \frac{dk}{2\pi}$$

$$= \int_{-\infty}^{+\infty} e^{ikx} \frac{dk}{2\pi} \left(\hat{P}(k)\right)^N$$

$$= \int_{-\infty}^{+\infty} e^{ikx} \frac{dk}{2\pi} \, e^{N\psi(k)}$$

Now, additivity ⇒

$$\psi_N = N\psi(k)$$

$$\psi(k) = -i c_1 k - \frac{1}{2} k^2 c_2 + \cdots$$

Thus,

$$P_N(x) = \int \frac{dk}{2\pi} e^{ikx} e^{N\psi(k)}$$

For large N, only region near $k=0$ contributes, so: (Laplace method)

$$P_N(x) = \int_{|k|<\epsilon} e^{ikx} e^{N\psi(k)} \frac{dk}{2\pi}$$

$$= \int_{|k|<\epsilon} \frac{dk}{2\pi} e^{ikx} \exp\left[-iNc_1 k - \frac{N}{2} k^2 c_2 + \cdots\right]$$

and can re-expand integration domain to:

$c_1 \equiv 0$, $e < \infty$

$$P_N(x) = \int \frac{dk}{2\pi}\, e^{ikx}\, \cancelto{1}{e^{-iNc_1 k}}\, e^{-\frac{1}{2} N c_2 k^2}$$

F.T. of Gaussian is Gaussian, so $\Rightarrow$

$$\boxed{P_N(x) = \frac{1}{2\pi (N c_2)^{1/2}}\, e^{-x^2/N c_2}}$$

C.L.T.

A few points:

- no questions asked about higher moments. Indeed these need <u>not</u> be well behaved, and induce <u>Fat Tails</u>.

- i.e. $p(x) = 1 / 1+x^2$ has diverging second moment, but

$$p(x) = 2 / \sqrt{\pi}\,(1+x^4)$$

meets C.L.T., but kurtosis diverges $\Rightarrow$ Fat Tail.

Nominally, low probability events have large influence.

N.B. Interesting to examine consequences for resonance broadening theory

- Can show:

  - Gaussian eroded (fat tail)

  - large x (how?)

$$P_N(x) \sim NA/x^4$$

Fat tail (power, not Gaussian).

Issue of CLT with fat tails is interesting and merits more analysis. Good paper topic!

Now, (Integrable)

→ Levy Stability and Distributions
(B.D. Hughes, Random Walks and Random Environments)

Now, x_i → random variables

$$X_N = \sum_{n=1}^{N} x_n$$

generalizes CLT

i.e. $p(c_1 z)\, p(c_2 z) = p(c z)$
condition.

Now, $\hat{p}_n(k) = [\hat{p}(k)]^n$ — characteristic fctn

$Z_N =$ [redacted] x_N / a_N → rescaling

$P_{df}(Z_N) = F_N(x) / a_N$

$x = x_N / a_N$.

and:

— $\hat{F}_N(a_N k) = \hat{p}_N(k)$

some fixed fctn. (attractor)

Now, want $F_n(k) \xrightarrow[n \to \infty]{} F(k)$. [redacted]

let $\lim\limits_{m \to \infty} \frac{a_{nm}}{a_m} = c_n$ (not cum)

~~Condition for function as~~ limiting case.

$$\boxed{\hat{F}(k\, c_n) = [\hat{F}(k)]^n}$$

self-sim.

scale.

so, need solve:

$$\boxed{\hat{F}(k\, u(\lambda)) = [\hat{F}(k)]^\lambda}$$

fixed fctn. condition

$$\psi = \ln \hat{F}(k)$$

so

$$\psi(k\mu(\lambda)) = \lambda \psi(k)$$

$$k\left(\frac{d\mu}{d\lambda}\right)\psi'(k\mu(\lambda)) = \psi(k)$$

$$\mu(\lambda = 1) = 1 \qquad (\text{must.})$$

$$k\mu'\psi' = \psi(k)$$

$$\boxed{\frac{d\psi}{dk} = \psi / \mu'(1)k}$$

$$\frac{k}{\psi}\frac{d\psi}{dk} = 1/\mu$$

$$k\frac{d\ln\psi}{dk} = 1/\mu$$

self-sim.

$$\psi(k) = \begin{cases} v_1 |k|^\alpha & k>0 \\ v_2 |k|^\alpha & k<0 \end{cases}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1 |k|^\alpha) & k>0 \\ \exp(v_2 |k|^\alpha) & v<0 \end{cases}$$

or, in more detail,

$$\boxed{\hat{F}(k) = \exp\left[-a|k|^{\alpha}\left(1 - i\beta \tan\frac{\alpha\pi}{2}\,\mathrm{sgn}(k)\right)\right]}$$

skewness

· Now, take $\beta = 0$ → Levy Distribution

$$\boxed{L_\alpha(a,k) = \hat{F}(k) = \exp(-a|k|^{\alpha})}$$

$\alpha = 2$ $\quad \hat{F}(k) = \exp\left[-ak^2\right]$

→ Gaussian

$\alpha = 2$ is case of CLT

$$P(x) = \exp\left[-x^2/a\right]$$

$\alpha = 1$ $\quad \hat{F}(k) = e^{-a|k|}$

→ Cauchy, Lorentzian

$$P(x) = 1/a^2 + x^2.$$

Note $\alpha = 2$ is max, and only Levy stable distribution with 2nd moment finite.

→ Large x expansion: $0 < \alpha < 2$

$$L_\alpha(a, x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx} e^{-a|k|^\alpha}\, dk$$

$$= \frac{1}{\pi}\int_0^\infty \cos(kx)\, e^{-a|k|^\alpha}\, dk$$

ibp

$$= \frac{a\alpha}{\pi}\int_0^\infty dk\, \frac{\sin k|x|}{|x|} \exp\left[-ak^\alpha\right] k^{\alpha-1}\, dk$$

$$= \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^\infty \varepsilon^{\alpha-1} \sin\varepsilon \exp\left(\frac{-a\varepsilon^\alpha}{|x|^\alpha}\right) d\varepsilon$$

Now, $\varepsilon = k|x|$

$d\varepsilon = dk|x|$

$$L_\alpha(a, x) \underset{x\to\infty}{=} \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^\infty \varepsilon^{\alpha-1} \sin\varepsilon\, d\varepsilon$$

so

$$L_\alpha(a, x) \sim (\;)\,/\,|x|^{1+\alpha}$$

$$(\;) \sim \frac{a\alpha}{\pi} \Gamma(\alpha) \sin \pi\alpha$$

Point: Levy Distributions $L_\alpha(a, x)$ have power law tails (fixed pt. fctn $\leftrightarrow$ self-similarity)

$$L_\alpha(a, x) \simeq 1 / |x|^{1+\alpha}$$

$$\underline{0 < \alpha < 2}$$

obviously need $\alpha = 2$ for convergent 2nd moment.

- width, for N steps, $\sim N^{1/\alpha}$
- $\alpha > 2 \rightarrow$ super-diffusive.

$\rightarrow$ Now, obvious analogy question arises:

CLT : Diffusion :: Levy Distribution : ?

Gaussian $\qquad L_\alpha(a, x)$

→? Levy Process !

see Zaslovsky, Chapt. 15

- analogue / generalization of diffn.
  → Levy Flights

- time dependent

- Levy dist. at infinitesimal time

Now, C-K Eqn:

(t understood)

$$P(x_0, t_0 \mid x_N, t_N) = \int dx_1 \int dx_2 \dots \int dx_{N-1}$$

$$* \left[P(x_0, t_0; x_1, t_1)\, P(x_1, t_1; x_2, t_2) \dots P(x_{N-1}, t_{N-1}; x_N, t_N) \right]$$

now $\quad t_{j+1} - t_j = \Delta t$

$\quad t_N - t_0 = N \Delta t \qquad N \gg 1$

and assume process uniform in space and stationary in time, so:

so

$$P(x_j, t_j; x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j, \Delta t)$$

$\Rightarrow$

$$P(x_N - x_0; N\Delta t) = \int dy_1 \cdots \int dy_N \; P(y_1 \Delta t) \cdots P(y_N, \Delta t)$$

Now,

$$P(q) = \int dy_j \; e^{iqy_j} P(y_j, \Delta t)$$

$$P_N(q) = \int dy^N \cdots \; e^{iqy^N} P(y^N, N\Delta t)$$

$$y^N = \sum_{i=1}^{N} y_j \quad = y_N - y_0$$

so

$$P_N(q) = \left[P(q)\right]^N. \quad , \text{ as before.}$$

(identical steps)

Now, take generating fctn. to be Levy Distribution.

Now,

$$P(\xi) = P(\xi \mid \alpha, c)$$

here:

$$P(\xi) = P_\alpha(\xi, \Delta c)$$

$$= L_\alpha(\xi, \Delta c)$$

and need

$$P_N(\xi) = P_\alpha(\xi, c_N)$$

above consistent if:

$$c_N = N \Delta c = N \Delta t \frac{\Delta c}{\Delta t}$$

$$= (N \Delta t) \overset{\text{const}}{c}$$

$$= tc = ct$$

$$\boxed{\begin{aligned} P_N \to P_\alpha(\xi, ct) &= \exp\left[-c N \Delta t \, |\xi|^\alpha\right] \\ &= \exp\left[-ct \, |\xi|^\alpha\right] \end{aligned}}$$

characteristic fctn of Levy Process

Characteristic fctn:

$$\hat{P}_\alpha(\xi, t) = \exp\left(-ct|\xi|^\alpha\right)$$

$$\boxed{P_\alpha(x, t) = \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, e^{-ct|\xi|^\alpha}}$$

- $\alpha = 2 \Rightarrow$ diffusion ✓
- $\langle x^2 \rangle \to \infty$, for $\alpha < 2$, at any t.

for $x \to \infty$

$$\boxed{P_\alpha(x, t) \sim t \,/\, |\Delta x|^{\alpha+1}}$$

- 'accelerating tail' distribution
  'expanding tail'
- i.e. P equal at $t > t'$
  $\Rightarrow (\Delta x) > (\Delta x)'$

Next:

- Physics: Weeks, Swinney experiment
- how extend Fokker-Planck Theory?
  - ~ CTRW
  - ~ Fractional Kinetics.