Notes 10 : Intermittency, Fractional Brownian Motion, Hurst Parameter

→ Here : Consider → FBM {Fractional / Fractal Brownian Motion}

{ → Hurst Parameter
→ Temporal Intermittency.

Fractal character T.B.D.

- Harold E. Hurst's story

- hydrological engineer
- meticulous record keeper, observer
- active in Aswan Dam's construction

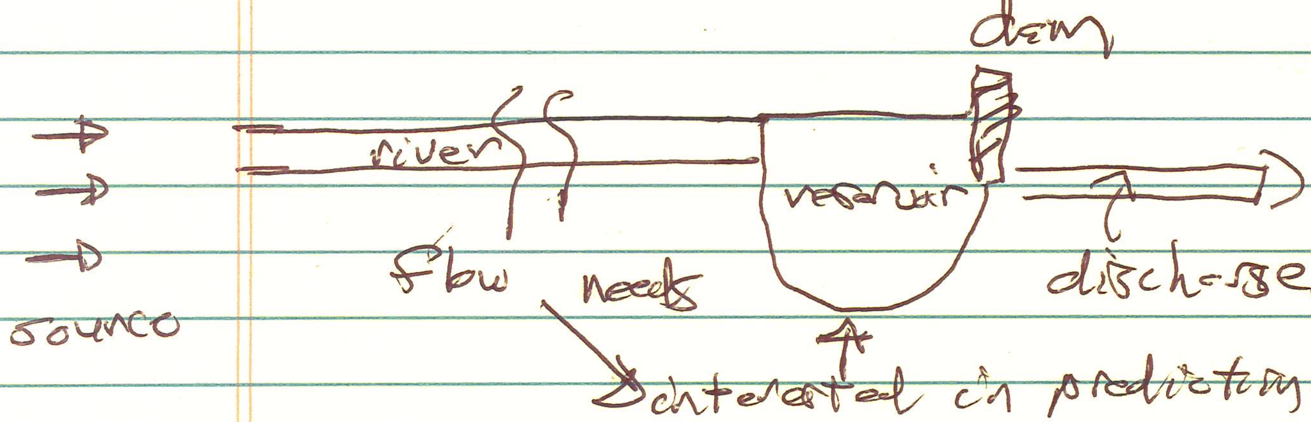
Problem?

- statistical description of flow, discharge of Nile

- prediction, modelling based on time series

i.e. how big should the Aswan reservoir be?

i.e. schematic:



Challenge:

- characterize the time variation of the river discharge.
- how big a reservoir?

→ analogous to machine construction.

Data - Time Series

- For long time average in stationary state, might expect random walk, i.e.

i.e. $B_H(t)$ → some general, stationary time series
(i.e. river flow due precipitation)

then expect:

$$E\left\{\left(B_H(t+\tau) - B_H(t)\right)^2\right\} \simeq \tau^{(1)}$$

but instead get:

- periods of sustained precipitation, discharge:
  → "Joseph effects" – persistence

c.f. as in '7 years of feast, famine', in Bible

→ "Noah effect" (large outlier "Black Swan")

→ anomalously large event / flood

c.f. Great Flood

see: – Mandelbrot and Wallis '68
+ terminology.
– Mandelbrot and Van Ness

i.e. not random... instead B M.

$$\boxed{E\left\{\left(B_H(t+\tau) - B_H(t)\right)^2\right\} = \tau^{2H}}$$

H → $\left.\begin{array}{l}\text{Hurst}\\ \text{Holder}\end{array}\right\}$ exponent

$H = 1/2$ → Brownian Motion

instead find $0 < H < 1$

→ $1/2 < H < 1$

- memory, positive correlation
- long term persistence
- "Joseph" - avalanche
- super-diffusion

i.e. $H = 1$

$\langle \delta B^2 \rangle \sim \tau^2$

$\delta B \sim \tau$

"ballistic"

→ pulse

on

→ $0 < H < 1/2$

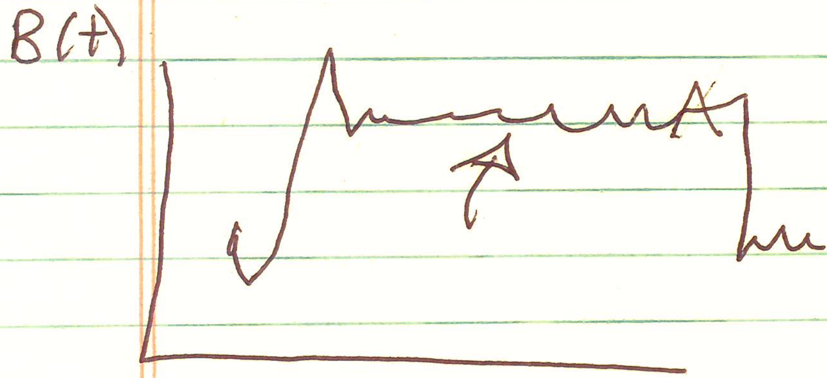
- temporal anti-correlation (weak excursion)
  → i.e. not much excursion
- Hi/Low value switching
- sticking
- sub-diffusion

$\Delta B \sim \tau^{\alpha}$

$\alpha < 1/2$

→ $H = 1/2$, Wiener Process
Brownian Motion

i.e.

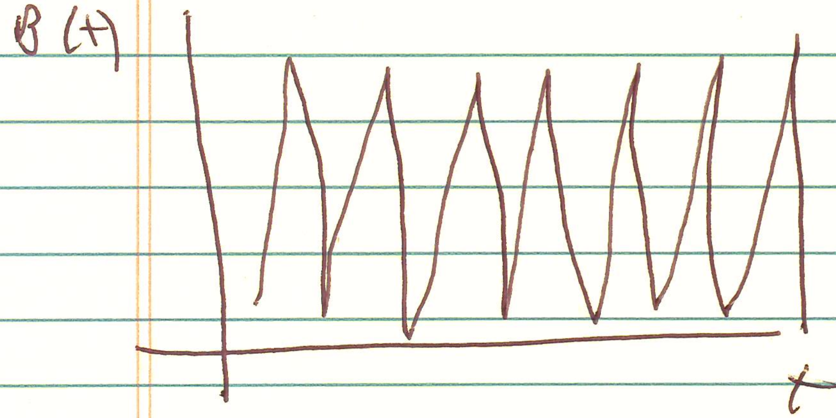


($H > 1/2$)

$H = .75$

long term persistence..

pulse → avalanche



$(H < 1/2)$

$H \sim 0.04$

Cycling $\rightarrow$ anti-persistent

Point: H measures, represents memory in dynamics.

(N.B., Recall: D measures ℓ/ℓ_0 scaling (indep ϵ) $\leftrightarrow$ memory.)

Note: Can re-write as:

$$E\{(\Delta B)^2\} = (\Delta t)^{2H}$$

so

$$\boxed{\ln|\Delta B| / \ln|\Delta t| = H}$$

and recall:

$$D_0 = \ln N / \ln(1/\epsilon)$$

box counting dimension

$$\boxed{H \Leftrightarrow D_0 \quad \text{(hence Holder exponent).}}$$

similarity

so H is (obviously) related to dimensionality of "Brownian" process

→ hence "Fractional Brownian Motion" (generalization).

- Generalization of B.M. can't account for Noah, Joseph phenomena.

A bit more:

→ Hurst observed (empirically) the ideal reservoir capacity

(i.e. big enough to hold "bigger!")

seek characterize, predict

$\downarrow$

i.e. $R(\delta) \equiv$ ideal capacity

$S(\delta) \equiv$ standard deviation of discharges

$\delta \equiv$ # of successive discharges

$$\boxed{R(\delta)/S(\delta) \sim \delta^H}$$

i.e. pile
$\begin{cases} R \rightarrow \text{pile content} \\ S \rightarrow \text{standard Deviation} \\ \delta \rightarrow \text{time} \end{cases}$

empirically, $\boxed{H \sim .7 \rightarrow .85}$ for Nile (Hurst)

(significant deviation from B.M.)

River flow $\rightarrow$ series of pulses.

(see 8a.)

$\rightarrow$ How to H?

Consider a time series:

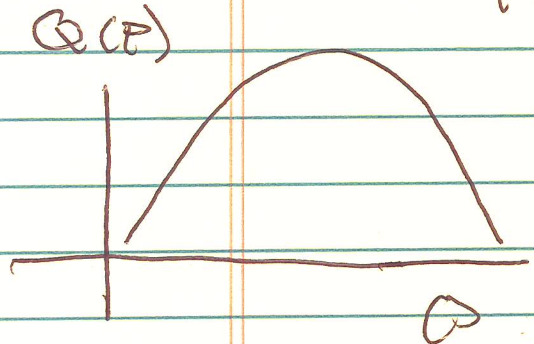
$X_1, X_2, \ldots\ldots X_N$

N.B. Can apply kinematic wave theory to problem of dynamics of river,

i.e. traffic: $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial q}{\partial x} = 0$ cf Whitham

$$q = Q(\rho)$$

$$\partial_t \rho + c(\rho)\frac{\partial \rho}{\partial x} = 0$$



etc.

Then, for floods:

$\rho \to A(x,t) \to$ cross sectional area of river-bed.

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} Q = 0$$

$$Q = Q(A, x)$$

so $$\frac{\partial}{\partial t} A + \frac{\partial Q}{\partial A}\frac{\partial A}{\partial x} = -\frac{\partial Q}{\partial x}$$

$$C = \frac{\partial Q}{\partial A} = \frac{1}{\underset{\substack{\downarrow \\ \text{width}}}{b}} \frac{\partial Q}{\underset{\substack{\downarrow \\ \text{ht.}}}{\partial h}}$$

and can use relations (empirical)

$$V = Q/A$$

$$\begin{cases} v = \left(\dfrac{A}{p} \dfrac{g \sin\alpha}{C_f}\right)^{1/2} \\ Q = vA \end{cases}$$

i.e. $$F_f = \underset{\substack{\downarrow \\ \text{coeff}}}{C_f} \overset{\substack{\text{wetted perimeter} \\ \downarrow}}{\rho} V^2 P_o$$

and $$F_g = \rho g A \sin\alpha$$

balance $\rightarrow V$.

so $Q \sim A^{3/2}$ etc.

$\Rightarrow \quad \partial_t A + c(A) A_x = 0$

etc.

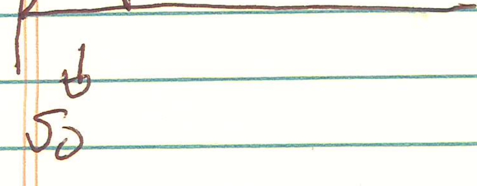
can extend to:

$$\begin{cases} \partial_t h + v \partial_x h + h v_x = 0 \\ \partial_t v + v \partial_x v + g' h_x = g' S - \dfrac{C_f v^2}{h} \end{cases}$$

etc. $\rightarrow$ similar to dynamic traffic flow, and shallow water.

then, consider localized, noisey source in h equation

$$\partial_t h + v \partial_x h + h v_x = S_0(x, t)$$



$$S_0 = \tilde{\nu}\, \delta(t - t')\, F_0(x)$$

etc.

What is/are { correlation, statistics }, H,

of downstream flow, ht, etc. ? ?

Pt: NL + Noise

Flooding ?

What is sensitivity to excitation ?

then:

$$c\, n^H = \overset{\text{expectation}}{E}\left[\frac{R(n)}{S(n)}\right]$$

$R(n) \equiv$ Range of first n values

$S(n) \equiv$ std Deviation first n.

More quantitatively:

1) mean, $m \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$

2) adjust series to mean, (de-mean).

$$Y_t = X_t - m \;; \quad t = 1, \ldots n$$

3) calculate cumulative deviate from mean

$$Z_t = \sum_{i=1}^{t} Y_i$$

4) Compute $\boxed{\underline{\text{Range}}}$ of deviate:

$$R(n) = \max(Z_1, \ldots Z_n) - \min(Z_1, \ldots Z_n)$$

5) Computed Std. Dev.

$$\sigma(n) = \left(1/2 \sum_{i=1}^{n} (x_i - m)^2 \right)^{1/2}$$

$R(n) / S(n) \xrightarrow{\text{gives}} H$ {avg over partial series

i.e.

$$\boxed{H = \ln \left[R(n)/S(n) \right] / \ln(n)}$$

– $H \rightarrow$ "R/S analysis"

– Key: $R(n) \rightarrow$ range

Measures 'dispersion' in time series.

Some similarity to Gini Parameters in Econ.

– Gini Coeff – Measures dispersion/
Concentration of Wealth
in Population

For wealths x_i, j

$$G = \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| \Big/ 2n \sum_{i=1}^{n} x_i$$

dispersion

$G = 1$ if 1 person has all \$.

Some similarity R and G.

→ Gini or analogue can be useful as measure of concentration in a time series.

Details re H:

– Can ask, is high H ('wild' randomness) a masquerade for $D(t)$ → time varying diffusion.

– Remedy N series
divide into shorter
$n = N, N/2, N/4 \ldots$

then rescaled range calculated for each n.

Related Issues

- $B(t)$ self-similar / self-affine with Holder (Hurst) dimension H if

- Can parallel fractal / multi-fractal:

$$\left\{ E\left[\left(B(t+T) - B(t) \right)^q \right] \right\}^{1/q} \sim \text{const } T^H$$

$\Rightarrow$ unifractal scaling (applies all q moments)

vs.

$$\left\{ E\left[\left(B(t+T) - B(t) \right)^q \right] \right\}^{1/q} \sim T^{H(q)}$$

i.e. dependent on q

$\rightsquigarrow$ multi-fractal

→ H defines "roughness" of series.
↕
"randomness"

→ Can define $\langle B^2(\omega)\rangle \simeq B_\omega^2 = \int_t^{t+\tau} d\tau \, \langle B(t)\, B(t+\tau)\rangle$

Result:

$\langle B^2\rangle_\omega \simeq \omega^{-\beta}$

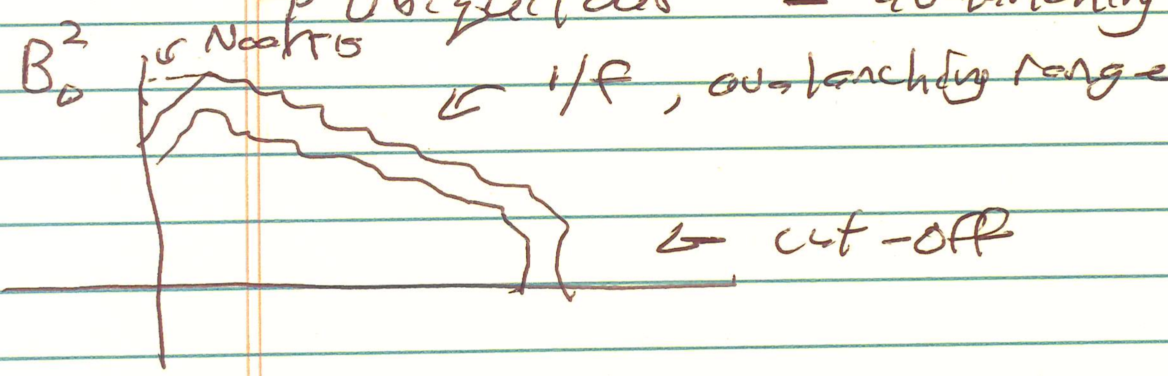
$\beta = 2H-1$

~ $H=1$ $\langle B^2\rangle_\omega \sim 1/\omega$

"1/f noise"

→ power at low frequencies – persistence

→ ubiquitous – avalanching



~ $H = 1/2$ (BM)

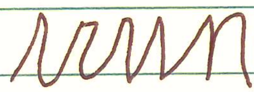
$\langle B^2\rangle_\omega \sim 1$ white noise

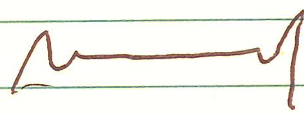
$\sim \quad H = 0$

$\langle B^2 \rangle_\omega \sim \omega \qquad \longrightarrow$ coherent

⇒ Can define fractal dimension of time series:

$$D = 2 - H$$

i.e. $H = 0$  2D

$H = 1$  1D

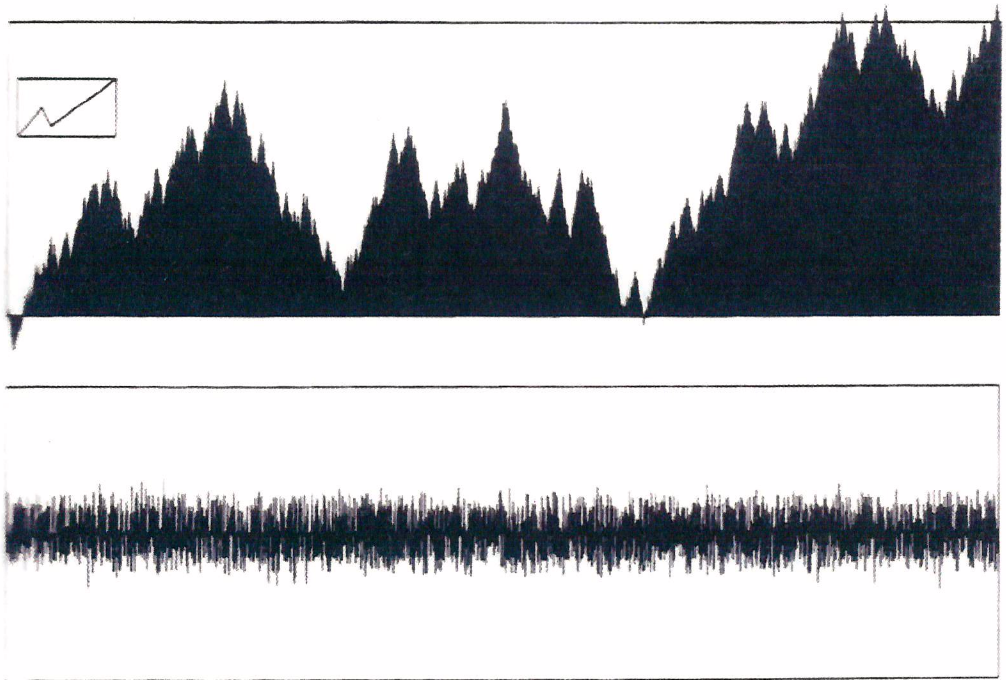
⇒ $H \sim 1/2 \quad \rightarrow$ "mild"

$H \sim 1 \quad \rightarrow$ "wild" large variation.

Mild Variation

15.

Wiener Process



Wiener cumulative

increment

~~[illegible]~~

White noise

FIGURE E6-6. The top line illustrates a cartoon of Wiener Brownian motion carried to many recursion steps. The generator, shown in a small window, is identical to the generator A2 of Figure 2. At each step, the three intervals of the generator are shuffled at random; it follows that, after a few stages, no trace of a grid remains visible to the naked eye.

The second line shows the corresponding increments over successive small intervals of time. This is for all practical purposes a diagram of Gaussian "white noise" as shown in Figure 3 of Chapter E1.

Mandelbrot on Finance

Wild Variation

16

Wild variation

Wiener process in multi-fractal trading time

Increment

non-Gaussian serial dependence

high variability of increment → wild

diffnt seed

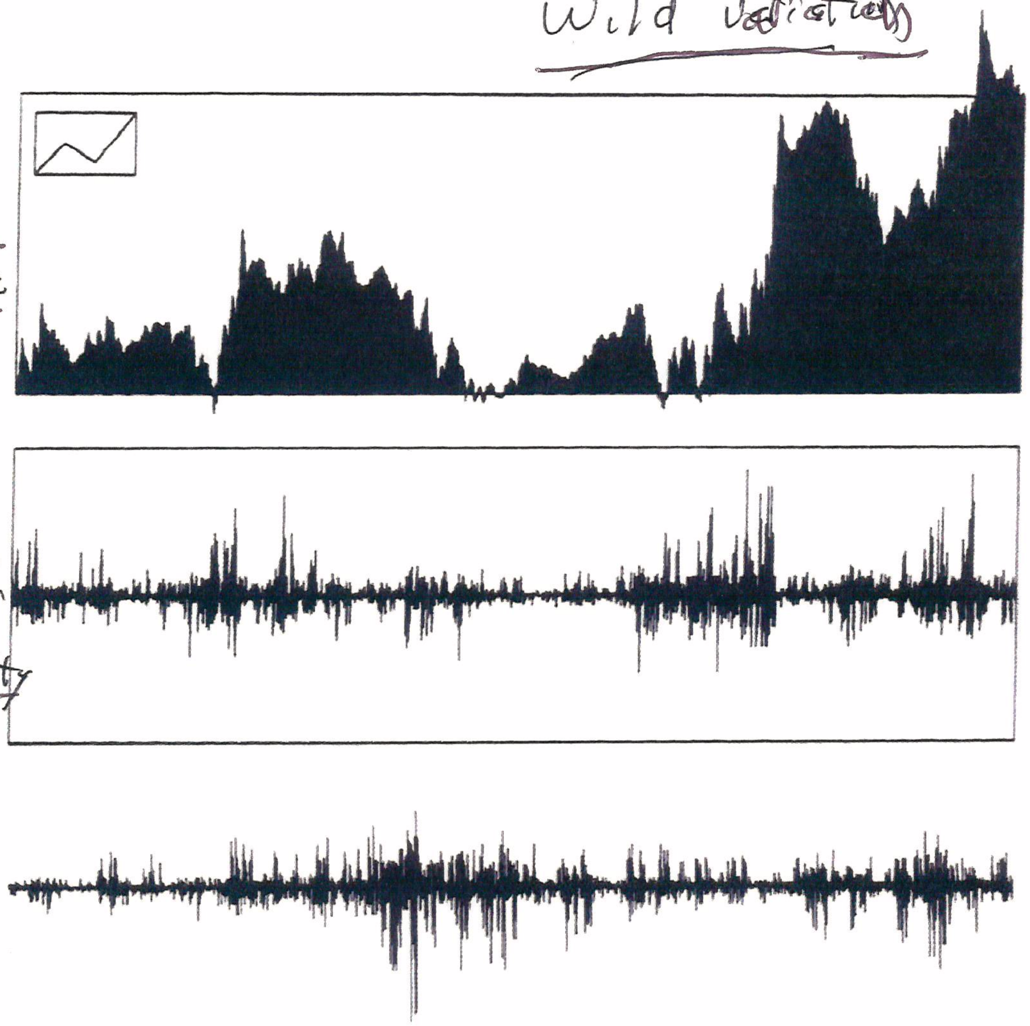


FIGURE E6-7. This figure reveals – at long last – the construction of Figure 2 of Chapter E1. The top line illustrates a cartoon of Wiener Brownian motion followed in a multifractal trading time. Starting with the three-box generator used in Figure 6, the box heights are preserved, so that D_T is left unchanged at $D_T = 2$ (a signature of Brownian motion), but the box widths are modified. (Unfortunately, the seed is not the same as in Figure 6.)

The middle line shows the corresponding increments. Very surprisingly, this sequence is a "white noise," but it is extremely far from being Gaussian. In fact, serial dependence is conspicuously high. The bottom line repeats the middle one, but with a different "pseudo-random" seed. The goal is to demonstrate once again the very high level of sample variability that is characteristic of wildly varying functions.

The resemblence to actual records exemplified by Figure 1 of Chapter E1 can be improved by "fine-tuning" the generator.

→ $1/f$ noise ($1/f^{\alpha}$ $\alpha \sim 1$)

~ $1/f$ noise is ubiquitous in physical systems

~ was a driver for SOC theory

~ low ~~[illegible]~~ frequency power suggestive of large avalanches (percolation clusters).

~ suggestive of scale invariance

→ Getting $1/f$

— not easy, i.e. usual:

$$\langle \vec{\phi}(t_1)\vec{\phi}(t_2)\rangle = |\phi|^2 e^{-|(t_2-t_1)/\tau_0|}$$

$$\Rightarrow S(\omega) = \frac{1/\tau_0}{\omega^2 + 1/\tau_0^2} \sim 1/\omega^2$$

i.e. $1/\tau_0$ imposes scale, but $1/f$ noise scale free (self-affinity).

- if conserved order parameter,

$$\partial_t \phi + D\phi \times \hat{z} \cdot \nabla \phi = 0$$

$$\langle \phi(0)\phi(\tau)\rangle_k = |\phi|^2 e^{-k^2 D\tau}$$

(follows Taylor + McNamara)

$$S(\omega, k) = |\phi|^2 \frac{k^2 D}{\omega^2 + (k^2 D)^2}$$

$\tau_{c_{eff}} \to \infty$, $k \to 0$.

can recover scale invariance.

- Alternative: <u>Montroll</u> (postal)

  - now consider ensemble of random processes, each with τ_c (i.e. distribution)

probability τ_c.

so

$$S(\omega)_{eff} = \int_{\tau_{c_1}}^{\tau_{c_2}} P(\tau_c)\, S_{\tau_c}(\omega)\, d\tau_c$$

- Demand $P(\tau_c)$ scale invariant

$P(\tau_c) = 1/\tau_c$ (dims).

$$S(\omega) = \frac{\tan^{-1}\omega\tau_c}{\omega}\Big|_{\tau_{c_1}}^{\tau_{c_2}}$$

$\sim 1/\omega \Rightarrow 1/f$

N.B.: $P \approx 1/\tau_c$

- long events rare
- short events numerous

$\Rightarrow$ Zipf's Law

$P(\Delta x) \sim 1/|\Delta x| \Leftrightarrow 1/f$

N.B. Lognormal well approximated by $P \sim 1/x$ over finite range

Gini ⟺ R/S ⟺ Kurtosis — A Broader vision of PDFs 9.

Now ⟹ characterizing "wild"

→ How characterize "wild" randomness? ⟹ prob. distribution

– $\boxed{\text{Levy flights}}$ are prime example of wild randomness

– Levy flight / Levy process (pioneered by Paul Levy, coined – Mandelbrot) is random walk in which ΔX distributed along $P(\Delta X)$ where $P(\Delta X)$ has "heavy" tail (→ power law).

e.g. Cauchy flight, $P(u) \sim A/1+u^2$

– Specific example: ~~[illegible]~~ probability

Consider $P(U) = P(U/u)$

U ↓ step size

$$P(U>u) = \begin{cases} 1 & : u<1 \\ u^{-D} & : u \geq 1 \end{cases}$$

↓ power law

derived from Pareto distribution of incomes (power law).

More generally; density

$$\begin{cases} P(|X|>u) = O(u^{-k}) \\ 1 < k < 3 \end{cases}$$

Brings us to:

OV: Pareto-Levy Law
Mandelbrot 1960.

→ emerged from economics, concerned with income distributions, especially tail.

→ Pareto (1897), Levy (1925) — observed power law (P-L) ⇒ wild-flights / Levy
— noted that P-L distribution satisfies a Limit Theorem (but not Gaussian)

→ Strong Pareto Law:

$P(u) \equiv$ % of indiv. with income $u > u_0$.

$$P(u) = \begin{cases} (u/u_0)^{-\alpha}, & u > u_0 \\ 1, & u < u_0 \end{cases}$$

Power Laws

then density → $p(u) = -dP(u)/du$:
(pdf)

→ power law

$$\boxed{p(u) = \begin{cases} \alpha (u_0) u^{-(\alpha+1)} & , \; u > u^0 \\ 0 & , \; u < u^0 \end{cases}}$$

$p(u)$ characterized by $\begin{cases} u_0 \Rightarrow \text{scale factor} \\ \alpha \Rightarrow \text{inequality index} \end{cases}$

$P(u)$ fits broad range of populations
(US tax payers, Renaissance towns etc)
(debated) (hint of robustness) ⟺ pdf is attractor in fctn space?

⇒ Weak Pareto Law (more robust)

$P(u)$ "behaves like" $(u/u_0)^{-\alpha}$, $u \to \infty$

⇒ $\begin{cases} p(u) \cong (u/u_0)^{-(\alpha+1)} & , \; \boxed{\alpha < 2 \Rightarrow \text{fat tail}} \\ \text{need } \alpha > 2 \text{ for } 2^{nd} \text{ moment convergence} \end{cases}$

N.B. Competitors for Pareto:

— exponential tail:

$$p(u) = k u^{-(\alpha+1)} e^{-bu}$$

$b \to ?$

$b \to 0$

- log-normals (why log normal relevant to incomes)

u.e

⇒ Thermodynamic Theories (?!)

- noting that Gaussian arises from Brownian motion ⇒ many small kicks in velocity,

qs4

- can economic interactions exchange increments of money leading to P-L [redacted] in equilibrium?
  Is P-L result of [redacted] a Limit Theorem?

⇒ NO! / Yes!

large events dominate.

- P(u) decreases too slowly, large u.

- might try $\ln u \equiv v$ ⇒ heads to lognormal (can speak of additivity of log u increments, and convergence),

all debatable

Percolation → mild → wild transition ?!

but

→ Pareto - Levy Random Variables

- Issue: Pareto law resilient to how income computed!

⇒

- Law emerges as a Limit Theorem.

i.e

Levy Stable Distributions

(attractors in fctn. space)

- U_i → statistically indep. incomes (up to scale, origin)
- U', U'' follow P-L, then:

$U' \oplus U''$ follows law [illegible], where:

⊕ → addition random variable

linear comb. → on PL

non random coeffs

i.e.

$$(a'U + b') \oplus (a''U + b'') = aU + b$$

$$a', a'' > 0,\ b', b'' > 0 \Rightarrow \exists\, a > 0,\ b.$$

i.e. adding two P-L Law incomes ⇒ income "on" P-L Law.

∴ $\boxed{\text{P-L law is an example of an L-stable process!}}$

P-L densities 23.
$\alpha = 1.2, 1.5, 1.8$

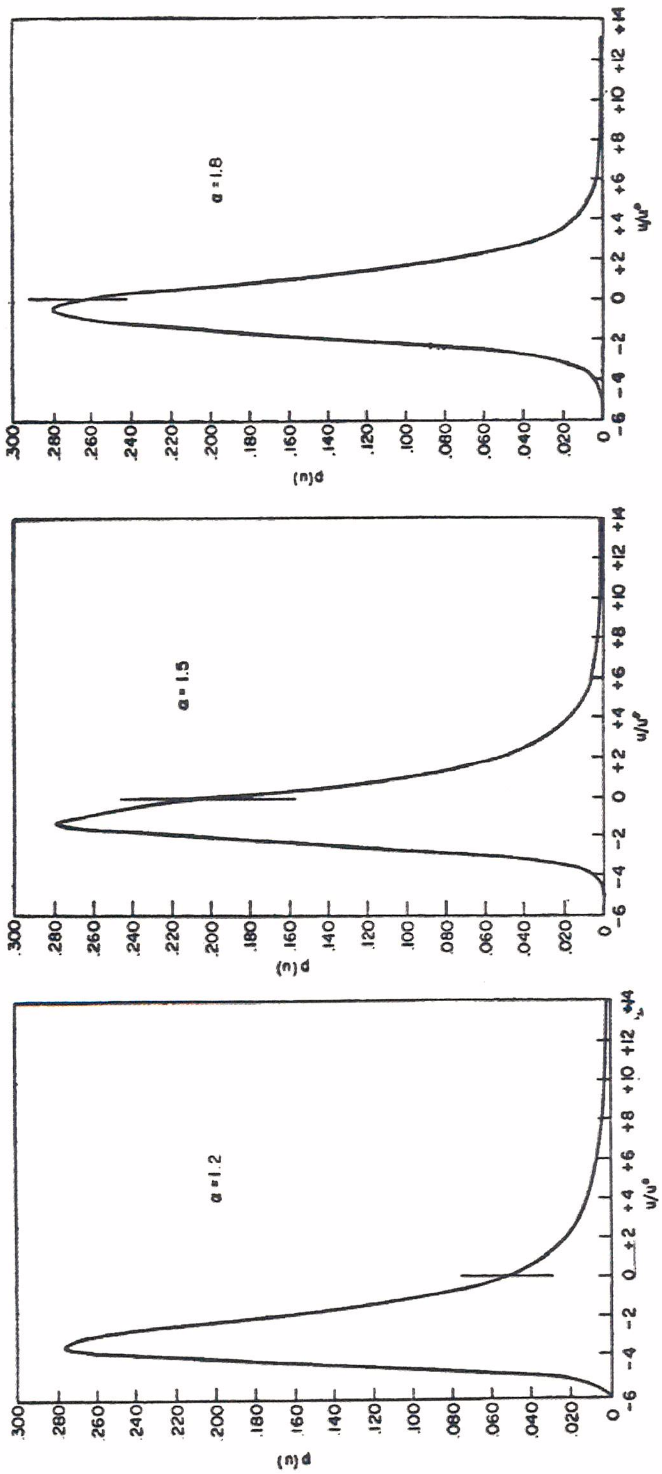


FIGURE 1: DENSITIES OF REDUCED P-L VARIABLES, ROF $M=0$ AND $\alpha=1.2$, 1.5, 1.8

Levy Flight

23a

Levy

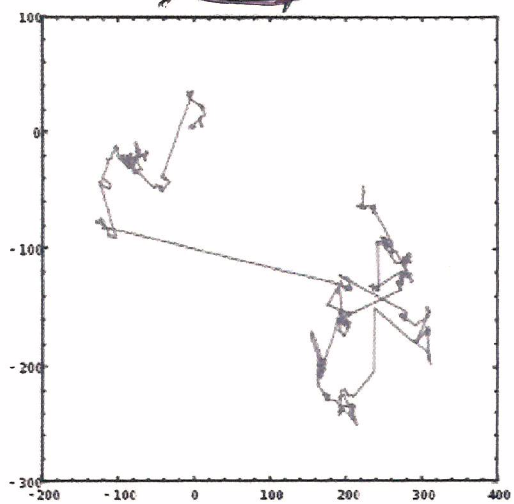


Figure 1. An example of 1000 steps of a Lévy flight in two dimensions. The origin of the motion is at [0,0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 1$ and $\beta = 0$ which is a Cauchy distribution. Note the presence of large jumps in location compared to the Brownian motion illustrated in Figure 2.

Brownian

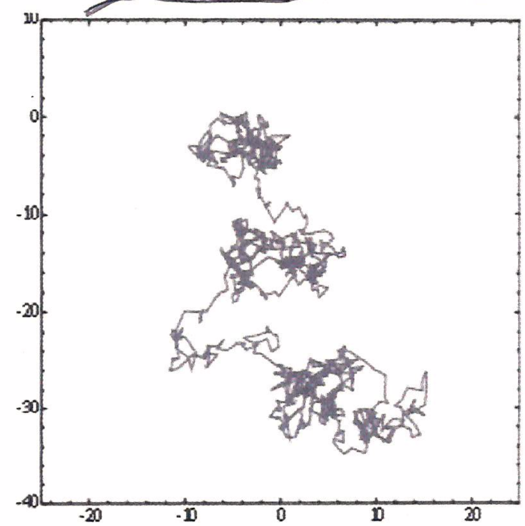


Figure 2. An example of 1000 steps of an approximation to a Brownian motion type of Lévy flight in two dimensions. The origin of the motion is at [0, 0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 2$ and $\beta = 0$ (*i.e.*, a normal distribution).

Applications

The definition of a Lévy flight stems from the mathematics related to chaos theory and is useful in stochastic measurement and simulations for random or pseudo-random natural phenomena. Examples include earthquake data analysis, financial mathematics, cryptography, signals analysis as well as many applications in astronomy, biology, and physics.

Another application is the Lévy flight foraging hypothesis. When sharks and other ocean predators can't find food, they abandon Brownian motion, the random motion seen in swirling gas molecules, for Lévy flight — a mix of long trajectories and short, random movements found in turbulent fluids. Researchers analyzed over 12 million movements recorded over 5,700 days in 55 data-logger-tagged animals from 14 ocean predator species in the Atlantic and Pacific Oceans, including silky sharks, yellowfin tuna, blue marlin and swordfish. The data showed that Lévy flights interspersed with Brownian motion can describe the animals' hunting patterns.[7][8][9][10] Birds and other animals[11] (including humans)[12] follow paths that have been modeled using Lévy flight (e.g. when searching for food).[13] Biological flight data can also apparently be mimicked by other models such as composite correlated random walks, which grow across scales to converge on optimal Lévy walks.[14] Composite Brownian walks can be finely tuned to theoretically optimal Lévy walks but they are not as efficient as Lévy search across most landscapes types, suggesting selection pressure for Lévy walk characteristics is more likely than multi-scaled normal diffusive patterns.[15]

Wikipedia

→ $\boxed{\text{Class of L-stable processes is thus "stable", as above, under addition.}}$

includes:

~ Lorentzian → Gaussian (only stable distribution with finite variance!)

→ weak (wild) P-L laws with $1 < \alpha < 2$

→ {only possible limit laws of weighted sums of identical and ~~in~~ ~~tes~~ independent random variables}

→ density p of P-L laws (see pics)

$$G(b) = \int_{-\infty}^{+\infty} e^{-bu} p(u)\, du$$

$$= \exp\left[(bu^*)^\alpha + Mb\right]$$

and Laplace transform.

$\left\{ \begin{array}{l} \alpha \\ u_* \\ M \to E(u) \end{array} \right.$

→ Working Principle:

if ⇒
- sum of many components non-Gaussian
- skewed
- $E(u) < \infty$

⇒ $\boxed{\text{reasonable assumption that follows P-L.}}$