Physics 235

Prologue

What is this course about?

- transport in disordered / random / turbulent system, different regimes
- system evolution due such transport, i.e. relaxation

Topics:

- Diffusion & beyond:
  - → transport in stochastic magnetic fields
    - scattering + collisions → what is origin of irreversibility
    - $Ku < 1$ to $Ku > 1$
  - → shear dispersion, cellular arrays, random media
  - → Percolation and $Ku > 1$ transport
  - → Intermittency: (beyond Fokker-Planck)
    - → fractals, multi-scaling → scaling, power laws
    - → Hurst exponent (correlations)
    - → Fat tails, Levy Flights

→ CTRW and Fractional Kinetics

→ Relaxation → Avalanching.

→ Self-Organized Criticality (SOC)

→ Traffic Flow, Jams

→ Models of SOC

→ Turbulence spreading and avalanching

→ Selected Topics, TBD.

"How many magnetic field lines in the universe?" → 1.

Turbulent Transport

I) Case Study: Transport in Stochastic Fields (Buried bodies in QLT).

A) Review – Basics of Hamiltonian Chaos (cf. OH, and other supplementary material)

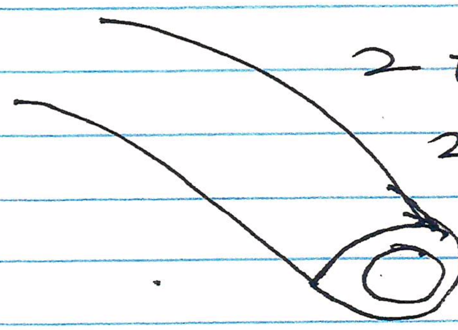
OH, Chapt. 7

If integrable system, can write:

$H = H_0(\underline{J})$ $\underline{J}$ ≡ action variable
$\underline{\theta}$ ≡ angle variable

so $\frac{d\underline{\theta}}{dt} = \frac{\partial H}{\partial \underline{J}} = \underline{\omega}(\underline{J})$

2 torus
2 D-O-F



$\frac{d\underline{J}}{dt} = 0$

trajectories lie on toroidal surfaces.

For 2-torus, have:

$\omega_1/\omega_2 = P/Q$ → rational number
closed trajectory

$\omega_1/\omega_2 =$ irrational → ergodic trajectory, fills surface

recall: Poincaré recurrence....

Surfaces where $\omega_1/\omega_2 = p/q$ are rational surfaces, and define natural resonances of system

Now if perturb:

$$H = H_0(\underline{J}) + \epsilon H_1(\underline{J}, \underline{\theta})$$

then must implement perturbation theory such that canonical structure maintained, so ΔS (correction to action) needed: → perturbation in Liouville eqn.

ΔS of

and $\Delta S \sim \epsilon H_1(\underline{J})_{\underline{m}} / \underline{\omega}\cdot\underline{m}$

$\underline{m}\cdot\underline{\omega} = 0$ → "small denominator problem"

what happens? → central issue in chaos theory

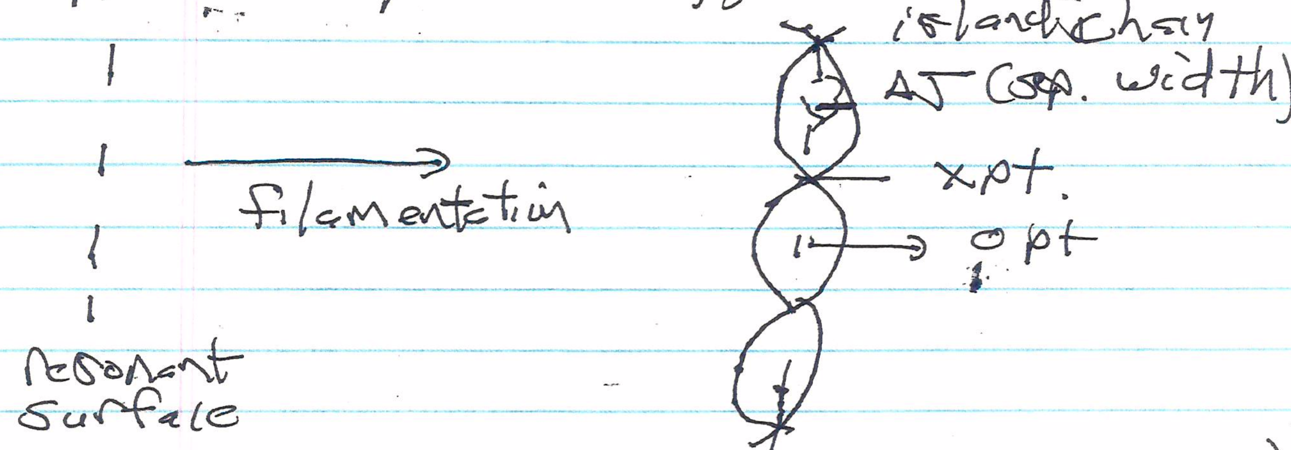
Small denominator problem ⟺ resonance phenomena! (n.b. akin Landau resonance)

i.e. $m\omega_1 + n\omega_2 = 0$

$$m/n = -\omega_2/\omega_1 = -q/p$$

pitch of perturbation　　pitch of trajectory

Now, can (for single resonance) resolve small denominator problem by secular perturbation theory (see supplementary notes), so



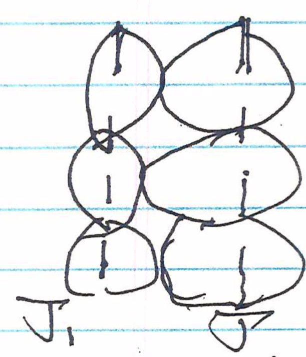
lines on perturbed surfaces

$$\Delta J \sim \left(\epsilon H_1 / \partial \omega / \partial J \right)^{1/2}$$

ϵH_1: perturbation strength

$\partial \omega / \partial J$: shear (diff'nt'l rotation in phase space)

Now, this fix-up works in the region of a single resonance. But if resonances overlap, i.e.



trajectories:
- wander in radius
- fill volume, not surface
- chaos results

Chaos:

- trajectory separation exhibits linear instability, exponentially growing

i.e. $J_1 - J_2 = \Delta J_0\, e^{\gamma t}$

$\Rightarrow$ 1 (at least) Lyapunov exponent > 0

- chaotic motion $\Rightarrow$ statistical approach for prediction / characterization

$\Rightarrow$ Fokker-Planck Eqn.

or

$\Rightarrow$ Hamiltonian dynamics (Liouville Thm) + chaos

$\therefore$ Quasilinear eqn. ($f \rightarrow$ pdf)

(F-P and QLT equiv. for Hamiltonian) $\nabla \cdot V_{\Gamma} = 0$

N.B.: Approaches limited to $ku \ll 1$

- criterion (working) for chaos:

Chirikov overlap!

$\overset{\rightarrow \text{ island width}}{}$

$$\frac{\Delta J_1 + \Delta J_2}{|J_1 - J_2|} > 1 \qquad \text{(good working criterion)}$$

$\hookrightarrow$ spacing

i.e. islands $\dfrac{\Delta w_1 + \Delta w_2}{|\Omega_2 - \Omega_1|} > 1$

- KAM theorem is concerned with ruggedness of irrational surfaces but chaos onset concerned with rational surfaces.

Prime example: Field Lines in Torus

- magnetic field lines + perturbation

$$\hat{B}_r = \sum_{m,n} B_{m,n}\, e^{i(m\theta - n\phi)}$$

- seek D_M → diffusivity of field lines in chaotic regime

but who cares about "lines"? → seek impact on

- heat, particle, momentum transport

and

- is chaotic dynamics always diffusive ?!

i.e. $Ku \equiv l_{ac}\, \delta B/B \,/\, \Delta r$ $\begin{matrix} < 1 \\ > 1 \end{matrix}$

$\downarrow$

Kubo #. what of $Ku > 1$?

Line Wandering / Diffusion

if $f = f(r, \theta, z) \rightarrow$ line density
i.e. magnetic flux

then, $\underline{B} \cdot \underline{\nabla} f = 0$ i.e. $\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta} = \frac{dz}{B_0}$
z as time

so if $\underline{B} = B_0 \hat{z} + B_\theta(r)\hat{\theta} + \tilde{B}_r \hat{r} + \tilde{B}_\theta \hat{\theta}$

B_0 $\downarrow$ toroidal, strong; $B_\theta(r)$ $\downarrow$ poloidal

then
- Hamiltonian system
- $f \rightarrow$ phase space density.

$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_0}$

$r\,d\theta/dz = \frac{\langle B_\theta \rangle + \tilde{B}_\theta}{B_0}$

$$B_0 \partial_z f + \frac{B_\theta(r)}{r} \partial_\theta f + \tilde{\underline{B}} \cdot \nabla \tilde{f} = 0$$

$$\partial_z f + \frac{B_\theta(r)}{B_0 r} \partial_\theta f + \frac{\tilde{\underline{B}}_\perp}{B_0} \cdot \underline{\nabla} \tilde{f} = 0$$

$$\Rightarrow \partial_z f + \frac{1}{R q(r)} \partial_\theta f + \frac{\tilde{\underline{B}}}{B_0} \cdot \underline{\nabla} f = 0$$

N.B.: z → plays role of time
- periodicity of fast scale perturbations
- irreversibility of $\langle f \rangle$ evolution

θ → periodic

so, for $\langle f \rangle$,

$$\partial_z \langle f \rangle + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0$$

$$\Gamma_{r,B} = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$ so Fick's Law.

⇓

flux of line density

How close?

Now, characteristics of Liouville Eqn.
⇒ equations of lines

$$\frac{dr}{B_r} = \frac{r\,d\theta}{\langle B_\theta(r) \rangle + \cancel{\tilde{B}_\theta}} = \frac{dz}{B_{z_0}}$$

so radial excursion given by:

$$dr/dz = \tilde{B}_r/B_0$$

so $$\delta r \cong \int_0^{l} (\tilde{B}_r/B_0)\, dz$$

Now, line trajector de-coheres from perturbation for $l > l_{ac}$

↳ autocorrelation length

$l_{ac} \cong 1/|\Delta(k_\parallel)|$ i.e. inverse spatial bandwidth

∴ $\boxed{\delta r \cong l_{ac}\, \tilde{B}_r/B_0}$ ⇒ { line excursion in ~~I~~ l_{ac}

Can identify $\Delta_r \equiv$ scatterer radial correlation length (i.e. spatial spectral width) — radial coherence length.

then:

$$Ku \cong \delta r/\Delta_r \cong \frac{l_{ac}}{\Delta_r}\tilde{B}_r/B_0 \Rightarrow \text{Kubo \#}$$

and can then posit:

⇒ $Ku < 1$ ⇒ many kicks in coherence length

⇒ diffusion process

$\left\{ \begin{array}{l} Ku \sim 1 \rightarrow \text{B.B.K. "natural state of EM turbulence"} \\ Ku \simeq 1 \rightarrow \text{critical balance.} \end{array} \right.$

$\rightarrow Ku > 1 \rightarrow$ more than one Δ_r in Kib
$\rightarrow$ strong scattering $\leftrightarrow$ percolation.

Here $Ku \lesssim 1$, at first. So, proceed via Quasilinear theory.

$$\Gamma_M = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$

$$= \sum_{\underline{k}} \frac{\tilde{B}_{r-\underline{k}}}{B_0} \tilde{f}_{\underline{k}}$$

$$-i\left(k_z - k_\theta \frac{B_\theta}{B_0}\right) \tilde{f}_{\underline{k}} = -\tilde{B}_{r_{\underline{k}}} \frac{\partial \langle f \rangle}{\partial r}$$

So

$$\Gamma_M = -D_M \, \partial \langle f \rangle / \partial r$$

$$D_M = \sum_{\underline{k}} \left| \frac{\tilde{B}_{r\underline{k}}}{B_0} \right|^2 \pi \delta(k_z - k_\theta B_\theta / B_0)$$

magnetic diffusivity

$$= \sum_{\underline{k}} \left| \frac{\tilde{B}_{r\underline{k}}}{B_0} \right|^2 \pi \delta(k_\parallel)$$

(RSTZ '66)

$$\simeq \left\langle \left(\frac{\delta B_r}{B_0} \right)^2 \right\rangle l_{ac}$$

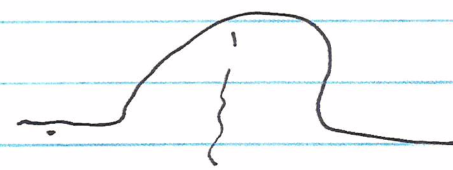
N.B.: $\sum_{\underline{n}} = \sum_{m,n}$

$q \neq 0$

$n = \frac{m}{q}$, $dn = -\frac{m}{q^2} q' dx$

$\Rightarrow$ spatial scale of spectral width (Δr) sets $|k_\parallel| \sim \left|\frac{k_\theta \Delta r}{L_s}\right|$

$$\boxed{l_{ac} \sim L_s / |k_\theta \Delta r|}$$



Lines then diffuse as:

$\langle \delta r^2 \rangle \sim D_M z$

N.B. Line Liouville eqn can be obtained by reducing / simplify in DKE

(*)

$$\cancel{\frac{\partial f}{\partial t}} + v_\parallel \hat{n}_0 \cdot \underline{\nabla} f + \cancel{\underline{v}_D \cdot \underline{\nabla} f} - \cancel{\frac{c}{B} \underline{\nabla} \phi \times \hat{z} \cdot \underline{\nabla} f}$$

$$+ v_\parallel \frac{\delta \underline{B}_\perp}{B_0} \cdot \underline{\nabla} f - \cancel{\frac{|e|}{m_e} E_\parallel \frac{\partial f}{\partial v_\parallel}} = \cancel{C(f)}$$

$$\Rightarrow \quad n_0 \cdot \nabla f' + \frac{\delta \underline{B}_\perp}{B_0} \cdot \underline{\nabla} f = 0 \quad \checkmark$$

Now, scales:

$l_{ac} \to$ (scatters)

$\to$ field line memory length.

$l_c \to$ line decorrelation length
(length over which line scattered from y_0).

i.e. $$r \frac{d\theta}{dz} = \frac{B_\theta(r)}{B_0}$$

but r scattered, $\Rightarrow$

$$\frac{dy}{dz} = B_\theta(r_0) + \frac{B_\theta'(r_0)}{B_0} \delta r$$

$\underline{\underline{so}}$

$$\frac{d}{dz} \delta y \cong \frac{B_\theta'(r_0)}{B_0} \delta r$$

$$\langle \delta y^2 \rangle = \left\langle \left(\int^z \left(\frac{B_\theta'}{B_0} \right) \delta r \, dz' \right)^2 \right\rangle$$