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# Chapter 3

## Fractional Quantum Hall Effect

### 3.1 Many-Body States in the Lowest Landau Level

#### 3.1.1 Introduction

Transport experiments in the quantum Hall regime reveal QH plateaus at fractional values of  $\sigma_{xy} = \nu e^2/h$  with  $\nu = p/q$  a rational fraction, principally with  $q$  odd. This corresponds to fractional filling of a Landau level. Recall that electrons are fermions, and a many-body fermionic wavefunction must be totally antisymmetric with respect to change of labels, *viz.*

$$\Psi(\xi_{\sigma(1)}, \dots, \xi_{\sigma(N)}) = \text{sgn}(\sigma) \Psi(\xi_1, \dots, \xi_N) \quad , \quad (3.1)$$

where  $\xi_j = (\mathbf{r}_j, \tau_j)$  is a compound variable including the spatial coordinates and spin polarization ( $\tau_j = \pm 1$ ) for the  $j^{\text{th}}$  electron, and where  $\text{sgn}(\sigma) \equiv (-1)^\sigma$  is the sign of the permutation  $\sigma \in \mathcal{S}_N$ . Initially we will presume that the Zeeman field polarizes all the electrons into the same spin state with  $\tau_j = +1$  for all  $j$ . In this case we only need concern ourselves with the spatial coordinates  $\{\mathbf{r}_j\}$ .

One way to construct such a totally antisymmetric state is via the Slater determinant,

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \det \{\varphi_i(\mathbf{r}_j)\} = \det \begin{pmatrix} \varphi_1(\mathbf{r}_1) & \varphi_1(\mathbf{r}_2) & \cdots & \varphi_1(\mathbf{r}_N) \\ \varphi_2(\mathbf{r}_1) & \varphi_2(\mathbf{r}_2) & \cdots & \varphi_2(\mathbf{r}_N) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_N(\mathbf{r}_1) & \cdots & \cdots & \varphi_N(\mathbf{r}_N) \end{pmatrix} \quad , \quad (3.2)$$

Here  $\{\varphi_i(\mathbf{r})\}$  is a basis of single particle wavefunctions. Recall that in the LLL, in the symmetric gauge  $\mathbf{A} = \frac{1}{2}B(y, -x)$ , for which  $\mathbf{B} = -B\hat{z}$ , all the wavefunctions are of the restricted form  $\psi(\mathbf{r}) = f(z) \exp(-z\bar{z}/4\ell^2)$ , where  $f(z)$  is an analytic function in  $z = x + iy$ , meaning  $\bar{\partial}f(z) = 0$ ,

where  $\bar{\partial} \equiv \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ . In the LLL, then, the most general  $N$ -electron wavefunction is of the form

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = F(z_1, \dots, z_N) \prod_{j=1}^N \exp(-z_j \bar{z}_j / 4\ell^2) \quad , \quad (3.3)$$

where  $F$  is analytic in all its arguments. Now recall the angular momentum basis for the LLL,

$$\varphi_m(\mathbf{r}) = \frac{1}{\sqrt{2\pi\ell^2 m!}} \left( \frac{z}{\sqrt{2}\ell} \right)^m e^{-z\bar{z}/4\ell^2} \quad . \quad (3.4)$$

Without normalization, we have the analytic factor  $f_m(z) = z^m$ , and forming a Slater determinant among  $N$  electrons in the angular momentum states  $m \in \{0, 1, \dots, N-1\}$ , we have

$$F(z_1, \dots, z_N) = \det\{z_j^m\} = \det \begin{pmatrix} z_1^0 & z_2^0 & \cdots & z_N^0 \\ z_1^1 & z_2^1 & \cdots & z_N^1 \\ \vdots & & \ddots & \vdots \\ z_1^{N-1} & \cdots & \cdots & z_N^{N-1} \end{pmatrix} \quad . \quad (3.5)$$

Clearly  $F(Z)$  is a homogeneous polynomial in its arguments  $Z = \{z_1, \dots, z_N\}$ , which says  $F(\lambda Z) = \lambda^{\deg F} F(Z)$ . Since the  $k^{\text{th}}$  row of  $F(\lambda Z)$  is multiplied by  $z^{k-1}$ , we have

$$\deg F = \sum_{k=1}^N (k-1) = \frac{1}{2}N(N-1) \quad . \quad (3.6)$$

Furthermore, since  $F(Z)$  is totally antisymmetric, it must vanish whenever  $z_i = z_j$  for all  $i \neq j$ . Thus, the product

$$V(Z) \equiv \prod_{i>j} (z_i - z_j) \quad (3.7)$$

must be a factor of  $F(Z)$ . But since there are  $\frac{1}{2}N(N-1)$  terms in the product for  $V(Z)$ , we must have that  $F(Z) = CV(Z)$ , where  $C$  is a constant. Since the coefficient of the term  $z_1^0 z_2^1 \cdots z_N^{N-1}$  in both  $F(Z)$  and  $V(Z)$  is 1, we conclude  $C = 1$  and hence  $F(Z) = V(Z)$ , which is called the *Vandermonde determinant*. It corresponds to the holomorphic part of the  $N$ -body LLL wavefunction where each of the lowest  $N$  angular momentum states, *i.e.* with  $m \in \{0, \dots, N-1\}$ , is filled, with no holes. *The Vandermonde determinant holomorphic factor corresponds to a filled Landau level.* The many-body normalization integral is

$$\int d^2r_1 \cdots \int d^2r_N |V(z_1, \dots, z_N)|^2 \exp\left(-\frac{1}{2\ell^2} \sum_{i=1}^N |z_i|^2\right) = N! \prod_{m=0}^{N-1} \left[2\pi\ell^2 (\sqrt{2}\ell)^m m!\right] \quad . \quad (3.8)$$

### 3.1.2 Second quantization

With an orthonormal set of single particle wavefunctions  $\{\varphi_\alpha(\mathbf{r}_i)\}$ , the normalized Slater determinant state is given by

$$\Psi_{\alpha_1 \dots \alpha_N}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) \varphi_{\alpha_{\sigma(1)}}(\mathbf{r}_1) \cdots \varphi_{\alpha_{\sigma(N)}}(\mathbf{r}_N) \quad . \quad (3.9)$$

We define the state

$$|\alpha_1, \dots, \alpha_N\rangle = \frac{1}{\sqrt{N!}} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) |\alpha_{\sigma(1)}\rangle \otimes \cdots \otimes |\alpha_{\sigma(N)}\rangle \equiv c_{\alpha_N}^\dagger \cdots c_{\alpha_1}^\dagger |0\rangle \quad (3.10)$$

in which case  $\Psi_{\alpha_1 \dots \alpha_N}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \langle \mathbf{r}_1, \dots, \mathbf{r}_N | \alpha_1, \dots, \alpha_N \rangle$ . Here  $\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta}$  are the canonical anticommutation relations for fermionic annihilation ( $c_\alpha$ ) and creation ( $c_\beta^\dagger$ ) operators.

The second quantized Hamiltonian is written as  $\hat{H} = \hat{T} + \hat{U} + \hat{V}$ . The kinetic energy is

$$\hat{T} = \sum_{\alpha, \beta} \langle \alpha | t | \beta \rangle c_\alpha^\dagger c_\beta \quad , \quad (3.11)$$

where

$$\langle \alpha | t | \beta \rangle = \int d^d r \varphi_\alpha^*(\mathbf{r}) t(\mathbf{r}, \nabla) \varphi_\beta(\mathbf{r}) \quad , \quad (3.12)$$

where  $t(\mathbf{r}, \nabla)$  is the single particle kinetic energy operator, and is often a function of the vector derivative  $\nabla$  alone, is  $t = -\frac{\hbar^2}{2m} \nabla^2$ . Of course for a particle in a magnetic field, we have that  $t = \frac{\hbar^2}{2m} (-i\nabla + \frac{e}{\hbar c} \mathbf{A})^2$ . A single particle potential  $u(\mathbf{r})$  gives rise to the second quantized contribution

$$\hat{U} = \sum_{\alpha, \beta} \langle \alpha | u | \beta \rangle c_\alpha^\dagger c_\beta \quad , \quad (3.13)$$

where

$$\langle \alpha | u | \beta \rangle = \int d^d r \varphi_\alpha^*(\mathbf{r}) u(\mathbf{r}) \varphi_\beta(\mathbf{r}) \quad , \quad (3.14)$$

Finally, the two-body potential is given in second quantized form as

$$\hat{V} = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \langle \alpha\beta | v | \gamma\delta \rangle c_\alpha^\dagger c_\beta^\dagger c_\delta c_\gamma \quad , \quad (3.15)$$

where

$$\langle \alpha\beta | v | \gamma\delta \rangle = \int d^d r_1 \int d^d r_2 \varphi_\alpha^*(\mathbf{r}_1) \varphi_\beta^*(\mathbf{r}_2) v(\mathbf{r}_1 - \mathbf{r}_2) \varphi_\delta(\mathbf{r}_2) \varphi_\gamma(\mathbf{r}_1) \quad . \quad (3.16)$$

The field operator is given by

$$\psi(\mathbf{r}) = \sum_{\alpha} \varphi_\alpha(\mathbf{r}) c_\alpha \quad , \quad \psi^\dagger(\mathbf{r}) = \sum_{\alpha} \varphi_\alpha^*(\mathbf{r}) c_\alpha^\dagger \quad . \quad (3.17)$$

Thus,

$$\begin{aligned} \{\psi(\mathbf{r}), \psi^\dagger(\mathbf{r}')\} &= \sum_{\alpha} \varphi_{\alpha}^*(\mathbf{r}) \varphi_{\alpha}(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') && \text{(entire Hilbert space)} \\ &= \frac{1}{2\pi\ell^2} e^{i\text{Im}(\bar{z}z')/2\ell^2} e^{-|z-z'|^2/4\ell^2} && \text{(LLL only)} \end{aligned} \quad (3.18)$$

As an example of the second quantized formalism, consider the density operator

$$n(\mathbf{r}) = \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) = \sum_{m_1} \sum_{m_2} \varphi_{m_1}^*(\mathbf{r}) \varphi_{m_2}(\mathbf{r}) c_{m_1}^\dagger c_{m_2} \quad (3.19)$$

where we use the angular momentum basis. Let  $|\Psi_1\rangle = \prod_{m=0}^{N_\phi-1} c_m^\dagger |0\rangle$  denote the filled Landau level, where  $N_\phi$  is the Landau level degeneracy and  $N = N_\phi$  is the number of electrons. Then

$$\langle \Psi_1 | c_{m_1}^\dagger c_{m_2} | \Psi_1 \rangle = \delta_{m_1, m_2} \quad (3.20)$$

and therefore

$$n(\mathbf{r}) = \langle \Psi_1 | \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) | \Psi_1 \rangle = \sum_{m=0}^{N_\phi-1} |\varphi_m(\mathbf{r})|^2 = \frac{1}{2\pi\ell^2} \sum_{m=0}^{N_\phi-1} \frac{1}{m!} \left( \frac{|z|^2}{2\ell^2} \right)^m e^{-|z|^2/2\ell^2} \quad (3.21)$$

In the limit  $N_\phi \rightarrow \infty$ , we have  $n(\mathbf{r}) \rightarrow 1/2\pi\ell^2$ , the number density of a filled Landau level. For finite  $N_\phi$ , the electron density is described by a droplet of radius  $R$ , where  $\pi R^2 = 2\pi\ell^2 N_\phi$ . To see this, let  $\zeta \equiv |z|^2/2\ell^2$ , so  $\nu(\zeta) \equiv 2\pi\ell^2 n(\mathbf{r}) = e^{-\zeta} \sum_{m=0}^M \zeta^m/m!$  where  $M \equiv N_\phi - 1$ . Thus we have  $d\nu/d\zeta = -e^{-\zeta} \zeta^M/M!$  which is maximized in magnitude at  $\zeta = M$ , where for large  $M$  it takes the value  $-1/\sqrt{2\pi M}$ . Now using the chain rule we obtain  $(d\nu/dr)_{\min} = -1/2\sqrt{\pi}\ell$ . Thus,  $\nu(r)$  drops from  $\nu \approx 1$  inside the droplet, *i.e.*  $r < R = (2N_\phi)^{1/2}\ell$ , to  $\nu \approx 0$  outside the droplet on a distance scale  $\Delta r \sim \ell$ .

We can carry out the same computation in the Landau basis, where in the  $n = 0$  LL

$$\psi_k(x) = (\sqrt{\pi}\ell L)^{-1/2} e^{iky} e^{-(x-k\ell^2)^2/2\ell^2} \quad (3.22)$$

Suppose we fill all states with  $k < 0$ , so

$$n(x) = L \int_{-\infty}^0 \frac{dk}{2\pi} |\psi_k(x)|^2 = \frac{1}{4\pi\ell^2} \text{erfc}(x/\ell) \quad (3.23)$$

where

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty dt e^{-t^2} = 1 - \text{erf}(z) \quad (3.24)$$



Note  $\text{erfc}(-\infty) = 1$  while  $\text{erfc}(0) = \frac{1}{2}$  and  $\text{erfc}(\infty) = 0$ . Thus there is an edge at  $x = 0$ , across which the electron density drops from  $n(-\infty) = 1/2\pi\ell^2$  to  $n(\infty) = 0$  within an interval  $\Delta x \sim \ell$ . If we further assume a neutralizing background of number density  $\Theta(-x)/2\pi\ell^2$ , then the total charge density in units of the electron charge is given by

$$\rho(x) \equiv n(x) - \frac{\Theta(-x)}{2\pi\ell^2} = \frac{1}{4\pi\ell^2} \text{erfc}(x/\ell) \text{sgn}(x) \quad . \quad (3.25)$$

Thus there is overall charge neutrality, *i.e.*  $\int_{-\infty}^{\infty} dx \rho(x) = 0$ , and we may define a dipole moment per unit length

$$\delta = \int_{-\infty}^{\infty} dx x \rho(x) = \frac{1}{8\pi} \quad . \quad (3.26)$$

For our next trick, let's evaluate the expression

$$\begin{aligned} n_2(\mathbf{r}, \mathbf{r}') &= \langle \Psi_1 | \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) | \Psi_1 \rangle \\ &= N(N-1) \int d^2r_3 \cdots \int d^2r_N |\Psi_1(\mathbf{r}, \mathbf{r}', \mathbf{r}_3, \dots, \mathbf{r}_N)|^2 \\ &= \sum_{m_1} \sum_{m_2} \sum_{m_3} \sum_{m_4} \varphi_{m_1}^*(\mathbf{r}) \varphi_{m_2}^*(\mathbf{r}') \varphi_{m_3}(\mathbf{r}') \varphi_{m_4}(\mathbf{r}) \langle \Psi_1 | c_{m_1}^\dagger c_{m_2}^\dagger c_{m_3} c_{m_4} | \Psi_1 \rangle \end{aligned} \quad (3.27)$$

Now

$$\langle \Psi_1 | c_{m_1}^\dagger c_{m_2}^\dagger c_{m_3} c_{m_4} | \Psi_2 \rangle = \delta_{m_1, m_4} \delta_{m_2, m_3} - \delta_{m_1, m_3} \delta_{m_2, m_4} \quad (3.28)$$

and therefore

$$\begin{aligned} n_2(\mathbf{r}, \mathbf{r}') &= \sum_m \sum_{m'} \left( |\varphi_m(\mathbf{r})|^2 |\varphi_{m'}(\mathbf{r}')|^2 - \varphi_m^*(\mathbf{r}) \varphi_m(\mathbf{r}') \varphi_{m'}^*(\mathbf{r}') \varphi_{m'}(\mathbf{r}) \right) \\ &= \frac{1}{(2\pi\ell^2)^2} \left( 1 - e^{-(\mathbf{r}-\mathbf{r}')^2/2\ell^2} \right) \equiv n_0^2 g(|\mathbf{r}-\mathbf{r}'|) \equiv n_0^2 \left( 1 + h(|\mathbf{r}-\mathbf{r}'|) \right) \quad , \end{aligned} \quad (3.29)$$

where we have taken the  $N_\phi \rightarrow \infty$  limit. Here  $n_0 = \nu/2\pi\ell^2$  is the droplet density for the filled LL ( $\nu = 1$ ),  $g(r) = 1 - \exp(-r^2/2\ell^2)$  is the *pair distribution function* and

$$h(r) = g(r) - 1 = -\exp(-r^2/2\ell^2) \quad (3.30)$$

is the *pair correlation function*. The Coulomb energy per particle, once a neutralizing background is introduced, is given by

$$\frac{\langle V \rangle_{\text{corr}}}{N} = \frac{1}{2} n \int d^2r v(r) h(r) = -\sqrt{\frac{\pi}{8}} \frac{e^2}{\epsilon \ell} \quad . \quad (3.31)$$

### 3.1.3 LLL projection

Consider the matrix element of a function  $V(\mathbf{r})$  between two LLL states,  $f(z) \exp(-z\bar{z}/4\ell^2)$  and  $g(z) \exp(-z\bar{z}/4\ell^2)$ . We define

$$\langle g | V | f \rangle = \int d^2r \overline{g(z)} V(\mathbf{r}) f(z) \exp(-z\bar{z}/2\ell^2) \quad . \quad (3.32)$$

With  $f(z) = \sum_{m=0}^{\infty} f_m z^m$  and  $g(z) = \sum_{m=0}^{\infty} g_m z^m$ , we have  $\overline{g(z)} = \sum_{m=0}^{\infty} \bar{g}_m \bar{z}^m$ , i.e.  $\overline{g(z)} = \bar{g}(\bar{z})$ . Now define the *normal ordered* operator  $:V(\bar{z}, z):$  to be the function  $V(\mathbf{r})$  expressed in terms of  $z$  and  $\bar{z}$ , but with all  $\bar{z}$  factors to the left of all  $z$  factors. Thus,  $:r^2: = \bar{z}z$ . When  $z$  and  $\bar{z}$  commute, normal ordering accomplishes nothing. But note that

$$\langle g | V | f \rangle = \int d^2r \exp(-z\bar{z}/2\ell^2) \bar{g}(\bar{z}) :V(2\ell^2\partial, z): f(z) \quad (3.33)$$

because we can integrate by parts, acting with  $-2\ell^2\partial$  to the left, where it has no effect on  $\bar{g}(\bar{z})$ , which is holomorphic in  $\bar{z}$ , and which acts on the exponential factor as

$$-2\ell^2\partial \exp(-z\bar{z}/2\ell^2) = \bar{z} \exp(-z\bar{z}/2\ell^2) \quad , \quad (3.34)$$

thereby bringing down one factor of  $\bar{z}$  for each application of  $2\ell^2\partial$ . Thus, the action of an operator  $V(\mathbf{r})$  on the LLL wavefunction  $\psi(\mathbf{r}) = f(z) \exp(-z\bar{z}/4\ell^2)$  is tantamount to acting only on the holomorphic part  $f(z)$  with the operator<sup>1</sup>  $:V(2\ell^2\partial, z):$ . Thus, the Schrödinger equation in the LLL, dropping the constant  $\frac{1}{2}\hbar\omega_c$  zero point cyclotron energy term, is

$$:V(2\ell^2\partial, z): f(z) = E f(z) \quad . \quad (3.35)$$

As an example, consider the harmonic potential  $V(\mathbf{r}) = \frac{1}{2}K\mathbf{r}^2$ . Projected to the LLL, the eigenstates in this potential have holomorphic parts  $f(z)$  which satisfy

$$2\ell^2 \frac{\partial}{\partial z} [z f(z)] = E f(z) \quad . \quad (3.36)$$

Clearly the solutions are the angular momentum states, with  $f_m(z) = C_m z^m$ , where  $C_m$  is a normalization constant. The energy eigenvalues are then  $E_m = (m+1)K\ell^2$ .

A particularly important application for us will be that of the plane wave, for which

$$:\exp(-i\mathbf{k} \cdot \mathbf{r}): = \exp(-ik\ell^2\partial) \exp(-i\bar{k}z/2) \quad . \quad (3.37)$$

Note further that

$$:\exp(-i\mathbf{k} \cdot \mathbf{r}): f(z) = \exp(-\bar{k}k\ell^2/2) \exp(-i\bar{k}z/2) f(z - ik\ell^2) \quad . \quad (3.38)$$

Thus, the holomorphic coordinate within the function  $f(z)$  is displaced by  $-ik\ell^2$ .

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<sup>1</sup>S. M. Girvin and T. Jach, *Phys. Rev. B* **29**, 5617 (1984).

## 3.2 The Wigner Crystal

Let's first consider the interacting 2DEG in a field, but in the absence of disorder. The Hamiltonian is

$$H = \frac{1}{2m^*} \sum_{i=1}^N (\mathbf{p}_i + \frac{e}{c} \mathbf{A}_i)^2 + \sum_{i<j} v(\mathbf{r}_i - \mathbf{r}_j) \quad , \quad (3.39)$$

with  $v(r) = e^2/\epsilon r$  is the *three-dimensional* Coulomb interaction. We reiterate that while the electrons in are confined to a 2DEG, they interact via the three-dimensional  $1/r$  potential and not the two-dimensional  $\ln(1/r)$  form. This is because the field lines between charges in the 2DEG are themselves not confined to the 2DEG, but exist throughout the three-dimensional host heterostructure. The static dielectric constant in GaAs is  $\epsilon = 13.13$ .

In heterojunctions, the electron number  $N$  is fixed by the density of dopants. At fixed area  $A$ , the number of fluxoids in a finite area  $A$  under uniform  $\mathbf{B} = -B\hat{z}$  is  $N_\phi = BA/\phi_0$ , where the Dirac flux quantum is  $\phi_0 = hc/e = 4.137 \times 10^5 \text{ T} \cdot \text{\AA}^2$ . The Landau level filling fraction,  $\nu = N/N_\phi$ , may then be adjusted by varying the field strength  $B$ . In Si MOSFETs, the electron density is set by the gate voltage  $V_g$  and can be varied during an experiment, as can  $B$ . Thus there are two ways to change  $\nu$  in a MOSFET.

Recall that in the LLL, the kinetic energy is quenched, hence  $H_{\text{LLL}} = \Pi_0 H \Pi_0 = \frac{1}{2} N \hbar \omega_c + \tilde{V}$ , where

$$\tilde{V} = \Pi_0 \sum_{i<j} v(\mathbf{r}_i - \mathbf{r}_j) \Pi_0 = \sum_{i<j} v(\mathcal{R}_i - \mathcal{R}_j) \quad , \quad (3.40)$$

where  $\mathcal{R}_i = (\mathcal{X}_i, \mathcal{Y}_i)$  are the guiding center coordinates for the  $i^{\text{th}}$  particle<sup>2</sup>. Recall that

$$[\mathcal{X}_i, \mathcal{Y}_j] = -i\ell^2 \delta_{ij} \quad , \quad (3.41)$$

with  $[\mathcal{X}_i, \mathcal{X}_j] = [\mathcal{Y}_i, \mathcal{Y}_j] = 0$ . We may drop the constant  $\frac{1}{2} N \hbar \omega_c$  piece in  $H_{\text{LLL}}$  and take the LLL-projected Hamiltonian to be  $\tilde{V}$ . Projecting onto a single Landau level ignores Landau level mixing effects. We expect this approximation is justified provided the typical Coulomb energy scale  $e^2\sqrt{\pi n}/\epsilon$  is sufficiently smaller than the cyclotron energy gap  $\hbar\omega_c$ , or the Zeeman gap  $\zeta\hbar\omega_c$ , where  $\zeta = g^*m^*/2m_e$  (see §2.2.5). With  $n = \nu/2\pi\ell^2$  and  $\omega_c = \hbar/m^*\ell^2$ , this criterion, up to dimensionless factors of order unity, is given by

$$\sqrt{\nu} \ell \ll \sqrt{2} a_{\text{B}}^* \quad , \quad (3.42)$$

where  $a_{\text{B}}^* = \epsilon\hbar^2/m^*e^2$  is the effective Bohr radius, which is large in GaAs, with  $m^* = 0.067 m_e$  and  $\epsilon = 13$ , we obtain  $a_{\text{B}}^* = 104 \text{ \AA}$ , and with  $\ell = 257 \text{ \AA} \sqrt{B[T]}$ , our criterion then becomes  $\nu \ll 0.33 B[T]$ , which is reasonably satisfied within the LLL ( $\nu \leq 1$ ) for  $B \gtrsim 10 \text{ T}$ . In fact, there

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<sup>2</sup>Recall that there are some subtleties associated with the LLL projection, having to do with normal-ordering, as discussed in §1.3.2.

will always be some degree of LL mixing, and the issue is really whether the actual ground state  $|\Psi_0\rangle$  is adiabatically connected to some model or trial state that is conveniently expressed solely within the LLL, *i.e.* what *phase of matter* is present.

### 3.2.1 Classical Wigner crystal

At  $\nu = 1$  there is a unique state corresponding to a filled LLL, but for  $N < N_\phi$ , the number of possible many-body states is given by the size of the Slater determinant basis, which is

$$\Omega(N, N_\phi) = \binom{N_\phi}{N} \simeq e^{c(\nu)N_\phi} \quad , \quad (3.43)$$

where  $c(\nu) = -\nu \ln \nu - (1 - \nu) \ln(1 - \nu)$ . There are thus exponentially many states to consider, so we need some intuition or physical principle to help us choose among them. One thing we can do is throw up our hands and *ignore quantum mechanics*<sup>3</sup> and pretend that the components of  $\mathcal{R}_i$  commute. This is equivalent to considering the classical potential energy function

$$V(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_{i < j} v(\mathbf{r}_i - \mathbf{r}_j) \quad , \quad (3.44)$$

with  $v(r) = e^2/\epsilon r$ . To be more precise, we could place our  $N$  particles in a circular disk of radius  $\Lambda$ , along with a uniform neutralizing background. Without the background, the energy will diverge as  $N \rightarrow \infty$  with no thermodynamic limit (*i.e.* the energy will scale as  $N^2$  rather than as  $N$ ), but the neutralizing background, which is of course physical, fixes this problem<sup>4</sup>. What is the ground state? The simplest guess would be that it is a crystal. In some cases this can be proven mathematically, such as the case of ‘sticky disks’ where  $v(r) = +\infty$  for  $r < a$ ,  $v(r) = -1$  for  $r = a$ , and  $v(r) = 0$  for  $r > 0$ . For this case, the ground state is a triangular Bravais lattice<sup>5</sup>. Recall that the Abrikosov vortex lattice in a type-II  $s$ -wave superconductor, where the vortices interact by a screened repulsive logarithmic potential, is also triangular. That *weak crystallization*, meaning crystallization in a weakly first-order transition, should result in a triangular lattice in  $d = 2$  was argued by Alexander and McTague<sup>6</sup> based on a Landau theory of the transition. The argument is as follows. Let  $\varrho_G$  be the amplitude of the Fourier component of the density  $\varrho(\mathbf{r})$  with wavevector  $\mathbf{G}$ , which is a reciprocal lattice vector of the

<sup>3</sup>I am a trained professional. Students should not try this at home.

<sup>4</sup>Recall that in heterostructures, the neutralizing background is due to the dopants, which are typically recessed by several hundred Ångströms from the 2DEG.

<sup>5</sup>R. C. Heitmann and C. Radin, *J. Stat. Phys.* **22**, 281 (1980) and also C. Radin, *J. Stat. Phys.* **26**, 365 (1981), where the original proof was extended to ‘soft disks’ where there is an annulus over which the interaction potential varies linearly with distance between  $v(a) = -1$  and  $v(b) = 0$ .

<sup>6</sup>S. Alexander and J. McTague, *Phys. Rev. Lett.* **41**, 702 (1978). See also E. I. Kats, V. V. Lebedev, and A. R. Muranov, *Phys. Rep.* **228**, 1 (1993).

incipient crystalline phase. Then construct the free energy

$$F[\{\varrho_{\mathbf{G}}\}] = \frac{1}{2} \sum_{\mathbf{G}} \chi^{-1}(\mathbf{G}) |\varrho_{\mathbf{G}}|^2 - \frac{1}{3} B \sum_{\mathbf{G}_1} \sum_{\mathbf{G}_2} \sum_{\mathbf{G}_3} \varrho_{\mathbf{G}_1} \varrho_{\mathbf{G}_2} \varrho_{\mathbf{G}_3} \delta_{\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3, \mathbf{0}} + \frac{1}{4} C \sum_{\mathbf{G}_1} \sum_{\mathbf{G}_2} \sum_{\mathbf{G}_3} \sum_{\mathbf{G}_4} \varrho_{\mathbf{G}_1} \varrho_{\mathbf{G}_2} \varrho_{\mathbf{G}_3} \varrho_{\mathbf{G}_4} \delta_{\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{G}_4, \mathbf{0}} + \dots, \quad (3.45)$$

where

$$\chi^{-1}(\mathbf{k}) = r + b(\mathbf{k}^2 - G^2)^2 \quad (3.46)$$

is the inverse static susceptibility at wavevector  $\mathbf{k}$ , which for fixed  $r$  is minimized for  $|\mathbf{k}| = G$ . The quadratic term determines the magnitude of the preferred wavevectors at which condensation takes place at  $r = r_c = 0$ , but this energy is degenerate over the circle (or sphere in  $d = 3$ ) of radius  $G$ . For weak crystallization, then, the cubic term determines the crystal structure, and evidently prefers structures whose reciprocal lattices contain the maximum number of triangles, in order to satisfy the  $\mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3 = 0$  condition. In  $d = 2$  this prefers a reciprocal lattice which is triangular, hence the underlying direct Bravais lattice is also triangular (or honeycomb). In  $d = 3$ , this condition prefers the fcc structure among all regular lattices, and the corresponding direct lattice is thus bcc. It should be emphasized that the Alexander-McTague theory applies to the weak crystallization of a fluid, and really describes the formation of a charge density wave structure, rather than a Wigner crystal of point particles.

The energy per particle of  $d = 2$  crystalline lattices of charges interacting by the potential  $v(r) = e^2/\epsilon r$ , in the presence of a uniform neutralizing background, was computed by Bonsall and Maradudin (BM) using the Ewald summation method<sup>7</sup>. They obtained the general result,

$$u_{\text{WC}} = \frac{U_{\text{WC}}}{N} = -\frac{e^2}{\epsilon \sqrt{\Omega}} \left\{ 2 - \sum_{\mathbf{R}}' \phi_{-1/2}(\pi \mathbf{R}^2 / \Omega) \right\} \quad (3.47)$$

where the sum is over all nonzero direct lattice vectors  $\mathbf{R}$ ,  $\Omega$  is the unit cell area, and

$$\phi_n(z) = \int_1^{\infty} dt t^n e^{-zt} \quad (3.48)$$

is known as the Misra function. BM obtained the following results:

$$u_{\text{WC}} = -\frac{e^2}{\epsilon} \left( \frac{2\pi}{\Omega} \right)^{1/2} \times \begin{cases} 0.777990 & \text{(square)} \\ 0.782133 & \text{(triangular)} \end{cases} . \quad (3.49)$$

Thus, the triangular lattice configuration has lower energy per particle. Note that  $n\Omega = 1$  where  $n = \nu/2\pi\ell^2$  is the density, hence  $(2\pi/\Omega)^{1/2} = \sqrt{\nu}/\ell$ .

<sup>7</sup>L. Bonsall and A. A. Maradudin, *Phys. Rev. B* **15**, 1959 (1977).

### 3.2.2 Quantum Wigner crystal

How can we restore quantum mechanics, *i.e.* the noncommutativity of  $[\mathcal{X}_i, \mathcal{Y}_j] = -i\ell^2 \delta_{ij}$ ? Maki and Zotos<sup>8</sup> constructed a trial LLL Wigner crystal wavefunction, using the coherent states  $\varphi_{\mathbf{R}}(\mathbf{r})$  as a basis, where the set  $\{\mathbf{R}\}$  corresponds to a triangular lattice, *i.e.*  $\mathbf{R}_{mn} = m\mathbf{a}_1 + n\mathbf{a}_2$  with  $\mathbf{a}_{1,2} = \frac{1}{2}a(\hat{x} \mp \sqrt{3}\hat{y})$  and  $\Omega = n^{-1} = \frac{1}{2}\sqrt{3}a^2$ . Recall the form of the LLL coherent state wavefunction from §1.3.7,

$$\varphi_{\mathbf{R}}(\mathbf{r}) = \langle \mathbf{r} | \mathbf{R} \rangle = (2\pi\ell^2)^{-1/2} \exp(i\mathbf{R} \times \mathbf{r} \cdot \hat{z}/2\ell^2) \exp(-|\mathbf{r} - \mathbf{R}|^2/4\ell^2) \quad . \quad (3.50)$$

The Maki-Zotos wavefunction is then

$$|\Psi_{\text{MZ}}\rangle = N^{-1/2} \sum_{\sigma \in \mathcal{S}_N} \text{sgn}(\sigma) |\mathbf{R}_{\sigma(1)}, \dots, \mathbf{R}_{\sigma(N)}\rangle \quad . \quad (3.51)$$

Note that the MZ wavefunction is not normalized, due to the fact that the von Neumann lattice of coherent states is not an orthonormal basis:

$$\langle \Psi_{\text{MZ}} | \Psi_{\text{MZ}} \rangle = \sum_{\sigma \in \mathcal{S}_N} \prod_{j=1}^N \langle \mathbf{R}_j | \mathbf{R}_{\sigma(j)} \rangle \quad , \quad (3.52)$$

where

$$\langle \mathbf{R} | \mathbf{R}' \rangle = \exp(-i\mathbf{R} \times \mathbf{R}' \cdot \hat{z}/2\ell^2) \exp(-|\mathbf{R} - \mathbf{R}'|^2/4\ell^2) \quad . \quad (3.53)$$

Therefore, in computing the expectation value of any operator  $\mathcal{O}$  in the MZ state, one must compute

$$\langle \mathcal{O} \rangle_{\text{MZ}} = \frac{\langle \Psi_{\text{MZ}} | \mathcal{O} | \Psi_{\text{MZ}} \rangle}{\langle \Psi_{\text{MZ}} | \Psi_{\text{MZ}} \rangle} \quad . \quad (3.54)$$

For nearest neighbors, the overlap magnitude is  $|\langle \mathbf{R} | \mathbf{R}' \rangle| = \exp(-a^2/4\ell^2) = \exp(2\pi/\sqrt{3}\nu)$ , which says that exchange effects are negligible in the low density limit  $\nu \rightarrow 0$ .

Taking into account only electron-electron interactions, *i.e.* with no neutralizing background as of yet, the energy of the MZ Wigner crystal atate is

$$E = \frac{e^2}{\epsilon} \frac{\langle \Psi_{\text{MZ}} | \sum_{i<j} \frac{1}{r_{ij}} | \Psi_{\text{MZ}} \rangle}{\langle \Psi_{\text{MZ}} | \Psi_{\text{MZ}} \rangle} = \sum_{i<j} \tilde{V}_2(\mathbf{R}_i - \mathbf{R}_j) + \sum_{i<j<k} \tilde{V}_3(\mathbf{R}_i, \mathbf{R}_j, \mathbf{R}_k) + \dots \quad , \quad (3.55)$$

where

$$\begin{aligned} \tilde{V}_2(\mathbf{R}) &= \frac{e^2}{\epsilon} \frac{\langle \mathbf{0}, \mathbf{R} | \frac{1}{|\mathbf{r}-\mathbf{r}'|} | \mathbf{0}, \mathbf{R} \rangle - \langle \mathbf{0}, \mathbf{R} | \frac{1}{|\mathbf{r}-\mathbf{r}'|} | \mathbf{R}, \mathbf{0} \rangle}{\langle \mathbf{0}, \mathbf{R} | \mathbf{0}, \mathbf{R} \rangle - \langle \mathbf{0}, \mathbf{R} | \mathbf{R}, \mathbf{0} \rangle} \\ &= \frac{\sqrt{\pi} e^2}{4\epsilon\ell} \text{sech}(R^2/8\ell^2) I_0(R^2/8\ell^2) = \begin{cases} \sqrt{\pi} e^2/4\epsilon\ell & R \rightarrow 0 \\ e^2/\epsilon R & R \rightarrow \infty \end{cases} \quad . \quad (3.56) \\ &= \frac{e^2}{\epsilon R} \left\{ 1 + \frac{\ell^2}{R^2} + \frac{9\ell^4}{2R^4} + \frac{75\ell^6}{2R^6} + \dots \right\} \quad . \end{aligned}$$

<sup>8</sup>K. Maki and X. Zotos, *Phys. Rev. B* **28**, 4849 (1983).

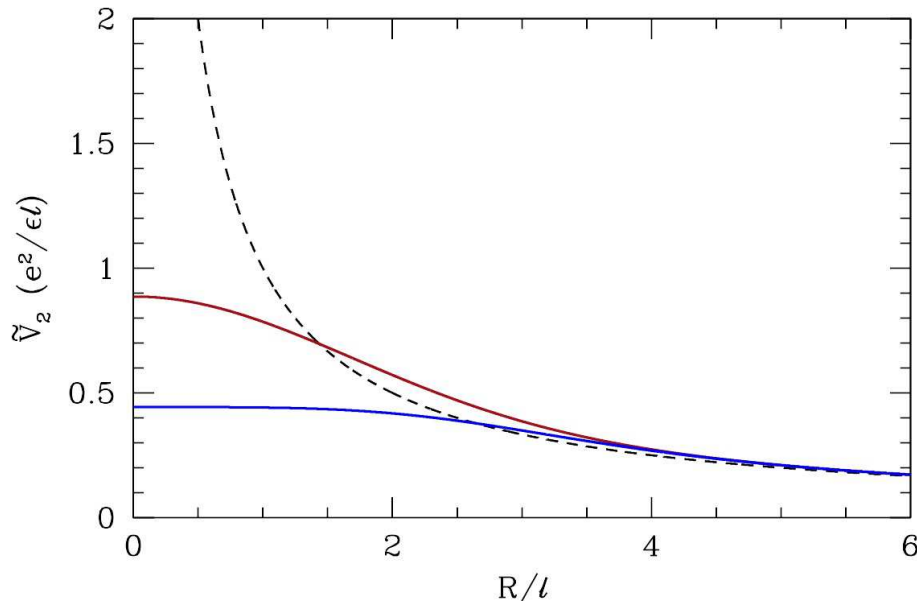


Figure 3.1: Bare (dashed, black), projected (red) and exchange-corrected (blue) Coulomb interaction in the LLL.

For details on the three-body term, see Maki and Zotos. Note that without the exchange correction, we would have

$$\begin{aligned} \tilde{V}'_2(\mathbf{R}) &= \frac{e^2}{\epsilon} \langle \mathbf{0}, \mathbf{R} | \frac{1}{|\mathbf{r} - \mathbf{r}'|} | \mathbf{0}, \mathbf{R} \rangle \\ &= \frac{\sqrt{\pi} e^2}{2\epsilon\ell} \exp(-R^2/8\ell^2) I_0(R^2/8\ell^2) = \begin{cases} \sqrt{\pi} e^2/2\epsilon\ell & R \rightarrow 0 \\ e^2/\epsilon R & R \rightarrow \infty \end{cases} . \end{aligned} \quad (3.57)$$

The consequences of projection and exchange correction are shown in Fig. 3.1.

The first term in the expansion of Eqn. 3.56 in powers of  $R^{-1}$  gives the classical energy, which diverges as  $N^2$  in the absence of a neutralizing background. With the background, this term gives us the Bonsall-Maradudin result  $u_{\text{WC}} = -0.782133 \sqrt{\nu} e^2/\epsilon\ell$  per particle. The remaining contributions, which are positive, are the quantum contributions to the correlation energy per particle. Note that the correlation energy of the filled Landau level, which we computed in Eqn. 3.31, is  $u_{\text{corr}}(\nu = 1) = -0.626657 e^2/\epsilon\ell$ , which is clearly greater than the classical WC energy at this density. This is due to zero point quantum fluctuations of the electron coordinates relative to their classical energy-minimizing locations. Maki and Zotos found that their correlation energy compared well with Hartree-Fock CDW calculations by Yoshioka and Lee<sup>9</sup>, with agreement to 1% throughout the regime  $\nu < \frac{1}{2}$ . Exchange effects were important to consider for  $\nu \in [0.4, 0.5]$  but accounted for less than a percent of the correlation energy at lower fillings. The salient feature here is that the correlation energy  $u_{\text{WC}}(\nu)$  is a *smooth* function of the filling  $\nu$ .

<sup>9</sup>D. Yoshioka and P. A. Lee, *Phys. Rev. B* **27**, 4986 (1983).

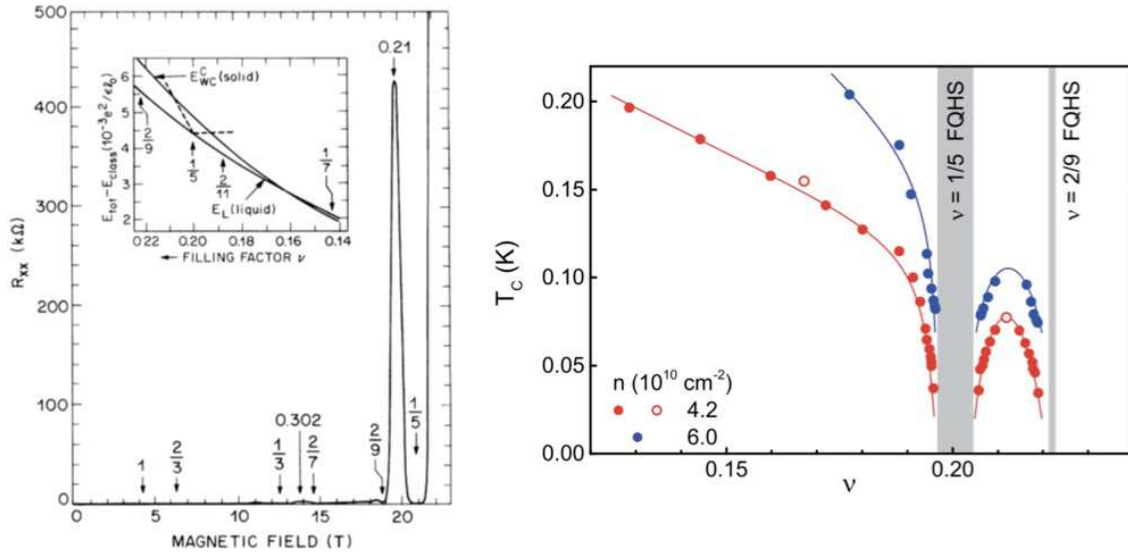


Figure 3.2: Evidence of Wigner crystal behavior for  $\nu \lesssim 0.2$ . Left: Data from H. W. Jiang *et al.*, *Phys. Rev. Lett.* **65**, 633 (1990). The longitudinal resistance  $R_{xx}$  exhibits pronounced peaks, dwarfing those in all FQH states, for  $\nu \approx 0.21$  and for  $\nu < 0.19$ , suggesting reentrant solid behavior. Right: Wigner crystal phase diagram inferred from capacitive measurements of effective screening by H. Deng *et al.*, *Phys. Rev. Lett.* **122**, 116601 (2019), showing reentrant solid behavior between  $\nu = \frac{1}{5}$  and  $\nu = \frac{2}{9}$ . The WC phase screens very poorly, and screening efficiency improves once the WC melts.

As we shall see, this is inconsistent with the phenomenology of the FQHE, which requires that the free energy  $F(\nu)$  have *cusps* when  $\nu$  corresponds to the filling at a FQH plateau. In addition, a Wigner crystal or charge density wave state breaks translational invariance, and is subject to pinning and the formation of Imry-Ma domains<sup>10</sup>. Observations of Wigner crystal phases of GaAs/AlGaAs heterojunction 2DEGs were first reported by E. Andrei *et al.*, *Phys. Rev. Lett.* **60**, 2765 (1988).

### 3.2.3 Magnetophonons in the Wigner crystal and in charged elastic media

Within the harmonic approximation, the energy of the deformed crystal, with  $\mathbf{R} \rightarrow \mathbf{R} + \mathbf{u}_R$ , is

$$\begin{aligned}
 U &= \frac{1}{2} \sum_{\mathbf{R} \neq \mathbf{R}'} \tilde{V}(\mathbf{R} + \mathbf{u}_R - \mathbf{R}' - \mathbf{u}_{R'}) \\
 &= U_0 + \frac{1}{2} \sum_{\mathbf{R} \neq \mathbf{R}'} (u_R^\alpha - u_{R'}^\alpha) (u_R^\beta - u_{R'}^\beta) \frac{\partial^2 \tilde{V}(\mathbf{R} - \mathbf{R}')}{\partial R^\alpha \partial R^\beta} + \dots,
 \end{aligned} \tag{3.58}$$

<sup>10</sup>H. Fukuyama and P. A. Lee, *Phys. Rev. B* **18**, 6245 (1978).



where each  $\mathbf{R}$  is a triangular lattice site. Now write

$$u_{\mathbf{R}}^{\alpha} = \frac{1}{\sqrt{N_c}} \sum_{\mathbf{k}} \hat{u}^{\alpha}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{R}} \quad (3.59)$$

where  $N_c = N$  is the number of unit cells<sup>11</sup>, in which case

$$U = U_0 + \frac{1}{2} \sum_{\mathbf{k}} \sum_{\alpha,\beta} \hat{\Phi}_{\alpha\beta}(\mathbf{k}) \hat{u}^{\alpha}(\mathbf{k}) \hat{u}^{\beta}(-\mathbf{k}) + \mathcal{O}(u^3) \quad , \quad (3.60)$$

with the dynamical matrix

$$\hat{\Phi}_{\alpha\beta}(\mathbf{k}) = \sum_{\mathbf{R}} (1 - \cos \mathbf{k} \cdot \mathbf{R}) \frac{\partial^2 \tilde{V}(\mathbf{R})}{\partial R^{\alpha} \partial R^{\beta}} \quad . \quad (3.61)$$

We now write  $\tilde{V}(\mathbf{R})$  as a Fourier integral, *viz.*

$$\tilde{V}(\mathbf{R}) = \int \frac{d^2 q}{(2\pi)^2} \hat{V}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{R}} \quad , \quad (3.62)$$

and make use of the Poisson summation formula,

$$\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} = \frac{(2\pi)^2}{\Omega} \sum_{\mathbf{G}} \delta(\mathbf{k} - \mathbf{G}) \quad , \quad (3.63)$$

where  $\Omega = 1/n$  is the area per unit cell. This gives

$$\hat{\Phi}_{\alpha\beta}(\mathbf{k}) = \frac{\nu}{2\pi\ell^2} \sum_{\mathbf{G}} \left[ (G_{\alpha} + k_{\alpha})(G_{\beta} + k_{\beta}) \hat{V}(\mathbf{G} + \mathbf{k}) - G_{\alpha} G_{\beta} \hat{V}(\mathbf{G}) \right] \quad (3.64)$$

Now we may quantize, writing

$$\hat{z}(\mathbf{k}) = \hat{u}_x(\mathbf{k}) + i\hat{u}_y(\mathbf{k}) = \sqrt{2} \ell b_{-\mathbf{k}}^{\dagger} \quad , \quad (3.65)$$

in the LLL (*cf.* Eqn. 1.48), to obtain the magnetophonon Hamiltonian,

$$H_{\text{MP}}^0 = \sum_{\mathbf{k}} \left[ \Omega_{\mathbf{k}} (b_{\mathbf{k}}^{\dagger} b_{\mathbf{k}} + b_{-\mathbf{k}}^{\dagger} b_{-\mathbf{k}}) + \Delta_{\mathbf{k}} b_{\mathbf{k}} b_{-\mathbf{k}} + \Delta_{\mathbf{k}}^* b_{\mathbf{k}}^{\dagger} b_{-\mathbf{k}}^{\dagger} \right] \quad , \quad (3.66)$$

where

$$\begin{aligned} \Omega_{\mathbf{k}} &= \frac{1}{2} \ell^2 \left[ \hat{\Phi}_{xx}(\mathbf{k}) + \hat{\Phi}_{yy}(\mathbf{k}) \right] \\ \Delta_{\mathbf{k}} &= \frac{1}{2} \ell^2 \left[ \hat{\Phi}_{xx}(\mathbf{k}) - \hat{\Phi}_{yy}(\mathbf{k}) + 2i\hat{\Phi}_{xy}(\mathbf{k}) \right] \quad . \end{aligned} \quad (3.67)$$

---

<sup>11</sup>There is one electron per unit cell, because the triangular lattice is a Bravais lattice.

Diagonalizing  $H_{\text{MP}}^0$  via a Bogoliubov transformation, we obtain the dispersion

$$\omega_{\mathbf{k}} = \sqrt{\Omega_{\mathbf{k}}^2 - |\Delta_{\mathbf{k}}|^2} = \ell \sqrt{\hat{\Phi}_{xx}(\mathbf{k}) \hat{\Phi}_{yy}(\mathbf{k}) - \hat{\Phi}_{xy}^2(\mathbf{k})} \quad . \quad (3.68)$$

Let  $\tilde{V}(\mathbf{k}) = (2\pi e^2 \ell / \epsilon) \hat{F}(k\ell)$ . Then

$$\hat{\Phi}_{\alpha\beta}(\mathbf{k}) = \frac{\nu e^2}{\epsilon \ell} \left( [\hat{F}(k\ell) + C_0] k_{\alpha} k_{\beta} + C_1 \mathbf{k}^2 \delta_{\alpha\beta} + \mathcal{O}(k^3) \right) \quad (3.69)$$

with

$$\begin{aligned} C_0 &= \sum'_{\mathbf{G}} \left[ \hat{F}(G\ell) + \frac{7}{8} (G\ell) \hat{F}'(G\ell) + \frac{1}{8} (G\ell)^2 \hat{F}''(G\ell) \right] \\ C_1 &= \sum'_{\mathbf{G}} \left[ \frac{3}{16} (G\ell) \hat{F}'(G\ell) + \frac{1}{16} (G\ell)^2 \hat{F}''(G\ell) \right] \quad . \end{aligned} \quad (3.70)$$

where the primes on the sums indicate that  $\mathbf{G} = 0$  is excluded. For the unprojected Coulomb potential  $v(r) = e^2/\epsilon r$ , the Fourier transform yields  $\hat{F}(k\ell) = 1/k\ell$  and the sums fail to converge. One must then reformulate the problem using the Ewald summation method. However in our case,  $\tilde{V}(\mathbf{r})$  is the LLL-projected and exchange-corrected potential of Eqn. 3.56. In this case  $\hat{F}(k\ell)$  behaves as  $1/k\ell$  in the infrared (*i.e.* as  $k \rightarrow 0$ ), but in the ultraviolet the short distance blunting of the  $1/r$  divergence from the LLL projection results in an exponential decay in  $k\ell$ , as in the case of the Yukawa potential. In this case, the sums for  $C_{0,1}$  converge nicely<sup>12</sup>. It is left as an exercise to the reader to verify the long wavelength dispersion,

$$\omega_{\mathbf{k}} = \frac{\nu e^2}{\epsilon} k^2 \sqrt{C_0 \hat{F}(k\ell) + C_1 (C_0 + C_1)} \quad . \quad (3.71)$$

As  $k \rightarrow 0$ , then, we have  $F(k\ell) = 1/k\ell$  dominates inside the radical, and  $\omega_{\mathbf{k}} \propto k^{3/2}$ . Note that if  $F(k\ell)$  were to approach a constant as  $k \rightarrow 0$ , corresponding to  $v(\mathbf{r}) \sim \delta(\mathbf{r})$ , we'd have  $\omega_{\mathbf{k}} \propto k^2$ . Conversely, if the potential were logarithmic, we'd obtain  $\omega_{\mathbf{k}} \propto k$ .

### Classical derivation

Consider an elastic medium with potential energy density

$$\mathcal{U}(\mathbf{x}) = \mu \text{Tr}(\varepsilon^2) + \frac{1}{2} \lambda (\text{Tr } \varepsilon)^2 \quad (3.72)$$

where  $\varepsilon(\mathbf{x})$  is the symmetric strain tensor, with components

$$\varepsilon_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial u^{\alpha}}{\partial x^{\beta}} + \frac{\partial u^{\beta}}{\partial x^{\alpha}} \right) \quad , \quad (3.73)$$

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<sup>12</sup>We assume both  $C_0$  and  $C_1$  are positive.

where  $u(x)$  is the local displacement field.. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} n_0 m (\dot{u}_x^2 + \dot{u}_y^2) + \frac{neB}{2c} (u_x \dot{u}_y - u_y \dot{u}_x) - \mathcal{U}(x) \quad , \quad (3.74)$$

where  $n_0$  is the number density,  $n_0 m$  is the mass density and  $(-n_0 e)$  is the charge density. Writing the action in terms of the Fourier modes  $u_{\mathbf{k}}^\alpha$ , we have

$$S = \int dt \sum_{\mathbf{k}} \left\{ \frac{1}{2} n_0 m \dot{u}_{\mathbf{k}}^\alpha \dot{u}_{-\mathbf{k}}^\alpha + \frac{n_0 eB}{2c} \epsilon_{\alpha\beta} u_{\mathbf{k}}^\alpha \dot{u}_{-\mathbf{k}}^\beta - \frac{1}{2} \left[ \mu (\delta^{\alpha\beta} - \hat{k}^\alpha \hat{k}^\beta) + (\lambda + 2\mu) \hat{k}^\alpha \hat{k}^\beta \right] \mathbf{k}^2 \right\} \quad . \quad (3.75)$$

We now express  $u_{\mathbf{k}}^\alpha$  in terms of longitudinal and transverse modes:

$$\mathbf{u}_{\mathbf{k}} = i \hat{\mathbf{k}} u_{\mathbf{k}}^\parallel + i \hat{\mathbf{z}} \times \hat{\mathbf{k}} u_{\mathbf{k}}^\perp \quad . \quad (3.76)$$

The factors of  $i$  ensure that  $\mathbf{u}_{\mathbf{k}}^* = \mathbf{u}_{-\mathbf{k}}$  if  $(u_{\mathbf{k}}^{\parallel/\perp})^* = u_{-\mathbf{k}}^{\parallel/\perp}$ , i.e. they are all Fourier components of real fields. Now we have  $L = T - U$  with

$$\begin{aligned} T &= \sum_{\mathbf{k}} \left\{ \frac{1}{2} n_0 m (\dot{u}_{\mathbf{k}}^\parallel \dot{u}_{-\mathbf{k}}^\parallel + \dot{u}_{\mathbf{k}}^\perp \dot{u}_{-\mathbf{k}}^\perp) + \frac{n_0 eB}{2c} (u_{\mathbf{k}}^\parallel \dot{u}_{-\mathbf{k}}^\perp - u_{\mathbf{k}}^\perp \dot{u}_{-\mathbf{k}}^\parallel) \right\} \\ U &= \sum_{\mathbf{k}} \left\{ \frac{1}{2} (\lambda + 2\mu) \mathbf{k}^2 |u_{\mathbf{k}}^\parallel|^2 + \frac{1}{2} \mu \mathbf{k}^2 |u_{\mathbf{k}}^\perp|^2 \right\} \quad . \end{aligned} \quad (3.77)$$

Now write the Lagrangian  $L = T - U$  and take the functional variation of the action  $S = \int dt L$  with respect to  $u_{-\mathbf{k}}^\parallel$  and with respect to  $u_{-\mathbf{k}}^\perp$  to get

$$\begin{aligned} \frac{\delta S}{\delta u_{-\mathbf{k}}^\parallel} = 0 &\Rightarrow n_0 m \ddot{u}_{\mathbf{k}}^\parallel - \frac{n_0 eB}{c} \dot{u}_{\mathbf{k}}^\perp = (\lambda + 2\mu) \mathbf{k}^2 u_{\mathbf{k}}^\parallel \\ \frac{\delta S}{\delta u_{-\mathbf{k}}^\perp} = 0 &\Rightarrow n_0 m \ddot{u}_{\mathbf{k}}^\perp + \frac{n_0 eB}{c} \dot{u}_{\mathbf{k}}^\parallel = \mu \mathbf{k}^2 u_{\mathbf{k}}^\perp \quad . \end{aligned} \quad (3.78)$$

In frequency space, this is equivalent to the system

$$\begin{pmatrix} \omega^2 - \omega_L^2(\mathbf{k}) & i \omega \omega_c \\ -i \omega \omega_c & \omega^2 - \omega_T^2(\mathbf{k}) \end{pmatrix} \begin{pmatrix} u_{\mathbf{k}}^\parallel \\ u_{\mathbf{k}}^\perp \end{pmatrix} = 0 \quad , \quad (3.79)$$

where  $\omega_c = eB/mc$ , and where

$$\omega_L(\mathbf{k}) = \left( \frac{\lambda + 2\mu}{n_0 m} \right)^{1/2} |\mathbf{k}| \quad , \quad \omega_T(\mathbf{k}) = \left( \frac{\mu}{n_0 m} \right)^{1/2} |\mathbf{k}| \quad (3.80)$$

are the long wavelength longitudinal and transverse phonon dispersions when  $B = 0$ . Setting the determinant to zero, we obtain the two normal modes,

$$\omega_{\pm}(\mathbf{k}) = \left[ \frac{1}{2} [\omega_c^2 + \omega_L^2(\mathbf{k}) + \omega_T^2(\mathbf{k})]^2 \pm \frac{1}{2} \sqrt{(\omega_c^2 + \omega_L^2(\mathbf{k}) + \omega_T^2(\mathbf{k}))^2 - 4 \omega_L^2(\mathbf{k}) \omega_T^2(\mathbf{k})} \right]^{1/2} \quad . \quad (3.81)$$

In the long wavelength ( $k \rightarrow 0$ ) limit, then,

$$\begin{aligned}\omega_+(\mathbf{k}) &= \omega_c + \frac{\omega_L^2(\mathbf{k}) + \omega_T^2(\mathbf{k})}{2\omega_c} + \dots \\ \omega_-(\mathbf{k}) &= \frac{\omega_L(\mathbf{k})\omega_T(\mathbf{k})}{\omega_c} + \dots\end{aligned}\quad (3.82)$$

Since both  $\omega_L(\mathbf{k})$  and  $\omega_T(\mathbf{k})$  vanish linearly with  $k$  in this limit, we find that the lower mode disperses as  $k^2$  and the upper mode is gapped with  $\omega_+(0) = \omega_c$ .

How do we add Coulomb interactions to this model? Note that  $\text{Tr } \varepsilon = \nabla \cdot \mathbf{u}$ , which is related to the local variation of the number density according to

$$n(\mathbf{x}) = n_0 (1 + \nabla \cdot \mathbf{u}) \quad , \quad (3.83)$$

*i.e.*  $\delta n(\mathbf{x}) = n(\mathbf{x}) - n_0 = n_0 \nabla \cdot \mathbf{u}$ . Thus

$$\Delta U = \frac{1}{2} \int d^2x \int d^2x' \delta n(\mathbf{x}) v(\mathbf{x} - \mathbf{x}') \delta n(\mathbf{x}') = \frac{n_0^2}{2} \sum_{\mathbf{k}} \hat{v}(\mathbf{k}) k^2 |u_{\mathbf{k}}^{\parallel}|^2 \quad , \quad (3.84)$$

and the Coulomb interaction  $\hat{v}(\mathbf{k}) = 2\pi e^2/\epsilon|\mathbf{k}|$  is accommodated by the replacement of the Lamé parameter  $\lambda$  with an effective Lamé parameter  $\lambda(\mathbf{k})$ , *viz.*

$$\lambda \rightarrow \lambda(\mathbf{k}) = \lambda + n_0^2 \hat{v}(\mathbf{k}) \quad . \quad (3.85)$$

We then have  $\omega_L(\mathbf{k}) = (2\pi n_0 e^2/\epsilon m)^{1/2} k^{1/2}$  at long wavelengths. This is the dispersion of the two-dimensional plasmon. Note that for  $\hat{v}(\mathbf{k}) \propto k^{-2}$ , as would be the case for a logarithmic potential, the 2D plasmon would be gapped, as is the plasmon in  $d = 3$  with  $1/r$  interactions, corresponding to  $\hat{v}_{3D}(\mathbf{k}) = 4\pi e^2/\epsilon k^2$ . The  $k^{1/2}$  longitudinal mode for  $B = 0$  in  $d = 2$  then entails  $\omega_-(\mathbf{k}) \propto k^{3/2}$  for a charged elastic medium in a uniform magnetic field. This is the famous  $k^{3/2}$  magnetophonon!

### 3.2.4 Imry-Ma argument: pinning by quenched disorder

In a crystalline phase there is long-ranged positional order and a breaking of the continuous symmetry of translation. It then behooves us to ask how such phases fare in the presence of quenched disorder, which in the case of QH systems is due to the random positions of dopant ions, each of which becomes a Coulomb impurity scatterer. The issue of how quenched randomness affects a system's attempt to order was taken up in a beautiful paper by Imry and Ma in 1975<sup>13</sup>. Quenched disorder in these systems is typically modeled as a local field. In systems with discrete symmetries, such as the Ising model, one would take  $V_{\text{dis}} = -\sum_r H_r \sigma_r$ . In

<sup>13</sup>J. Imry and S.-K. Ma, *Phys. Rev. Lett.* **35**, 1399 (1975).

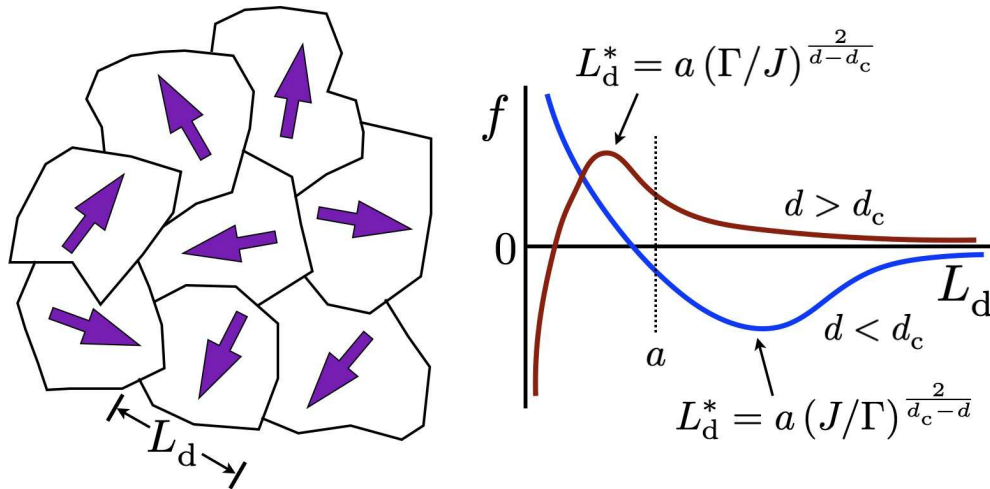


Figure 3.3: Adding disorder to systems which spontaneously break a discrete or continuous symmetry results in the formation of Imry-Ma domains of finite size  $L_d$  if  $d$  is less than the critical dimension  $d_c$ , where  $d_c = 2$  for discrete symmetry and  $d_c = 4$  for continuous symmetry. Left: Imry-Ma domains for a model with  $O(2)$  symmetry. Right: Energetics of domain formation. The region  $L_d < a$  is unphysical.

systems with continuous symmetries, such as the  $O(n)$  model,  $V_{\text{dis}} = -\sum_r \mathbf{H}_r \cdot \mathbf{S}_r$ . In charge density wave systems, if  $\Psi(\mathbf{r})$  is the order parameter which encodes the local amplitude  $|\Psi(\mathbf{r})|^2$  and phase  $\arg\Psi(\mathbf{r})$  of the local density variation  $\delta\rho(\mathbf{r})$  relative to the homogeneous liquid, we may write

$$V_{\text{dis}} = -\int d^2r \operatorname{Re} [H^*(\mathbf{r}) \Psi(\mathbf{r})] \quad , \quad (3.86)$$

where  $H(\mathbf{r}) \in \mathbb{C}$  is a complex number with a random amplitude and phase. Imry and Ma reasoned that such systems could try to lower their free energy by forming spatial domains in which the order parameter takes advantage of the local fluctuations in the random field. They presumed that such domains have a typical length scale  $L_d$ , which is determined by the following energy minimization argument.

There are two contributions to the energy of a given domain: bulk and surface terms from the disorder. The bulk energy is given by  $E_{\text{bulk}} = -\Gamma(L_d/a)^{d/2}$ , where  $a$  is an ultraviolet cutoff, typically set by an atomic lattice spacing, and where  $\Gamma = H_{\text{RMS}} = \langle |H(\mathbf{r})|^2 \rangle$  is the root mean square amplitude of the random field. This is the Central Limit Theorem at work: if the phase of the CDW is locked over a patch of linear dimension  $L_d$ , then adding  $L_d/a$  random fields gives us a contribution proportional to the square root of the number of such terms. The surface energy corresponds to the energy for creating a domain wall in the order parameter, which goes as

$$E_{\text{surf}} \propto \begin{cases} J(L_d/a)^{d-1} & \text{(discrete symmetry)} \\ J(L_d/a)^{d-2} & \text{(continuous symmetry)} \end{cases} \quad , \quad (3.87)$$

where  $J$  is the stiffness of the order parameter field arising from an energy density term  $J|\nabla\Psi|^2$ . For the discrete case, the width of the domain wall may be taken to be  $a$ , in which case the surface energy is proportional to the surface area. For continuous symmetry, the domain wall is described by a continuous twist in the order parameter over a distance  $L_d$  perpendicular to the DW interface. If we take  $\phi(x) = 2\pi x/L_d$  and assume  $|\Psi| \approx 1$  in the CDW, then

$$J \int_0^{L_d} dx (\partial_x \phi)^2 = \frac{4\pi^2 J}{L_d} \quad , \quad (3.88)$$

which introduces a factor of  $1/L_d$  relative to the discrete case. Thus the free energy density per unit cell volume  $a^d$  is

$$f(L_d) = \left(\frac{a}{L_d}\right)^d (E_{\text{bulk}} + E_{\text{surf}}) \approx J \left(\frac{a}{L_d}\right)^p - \Gamma \left(\frac{a}{L_d}\right)^{d/2} \quad , \quad (3.89)$$

where  $p = 1$  for discrete and  $p = 2$  for continuous symmetry of the order parameter. Extremizing, we find that there is an extremum at

$$\frac{L_d^*}{a} = \left(\frac{d_c}{d} \cdot \frac{J}{\Gamma}\right)^{\frac{2}{d_c-d}} \quad , \quad (3.90)$$

where  $d_c = 2p$  is  $d_c = 2$  (discrete) or  $d_c = 4$  (continuous). If  $d < d_c$ , the extremum is a local maximum. For weak disorder,  $\Gamma \ll J$ , and thus  $L_d \gg a$ . If  $d > d_c$ , the free energy attains a local maximum at  $L_d^*$ , but the sign of the exponent is reversed, and thus for weak disorder one has  $L_d^* \ll a$ . Since  $L_d$  cannot become smaller than the UV cutoff scale  $a$ , the entire region  $L_d < a$  is unphysical, and the apparent instability where  $f(L_d \rightarrow 0) \rightarrow -\infty$  is avoided. The minimum value then occurs at  $L_d^* = \infty$ , meaning that the LRO phase exists. The situation is summarized in Fig. 3.3.

Thus we conclude that a Wigner crystal phase in  $d = 2$  with true LRO cannot exist in the presence of quenched disorder. Nevertheless, as we have seen, the length scale for Imry-Ma domains may be quite large, and it therefore makes good sense to speak of *local* crystalline order. Not included in the above analysis is the condensation energy of the ordered phase itself, which is dominated by local effects<sup>14</sup>, and in assessing the stability of correlated liquid states, which we shall next discuss, it is generally sensible to compare to the energy density of the hypothetical pristine Wigner crystal.

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<sup>14</sup>Although, as we have seen, the ground state energy density of triangular *versus* square Wigner crystals is slight and long-ranged features of the potential may possibly determine the specific crystallographic symmetry of the ordered state.

### 3.3 The Principal Sequence of Laughlin States

Two weeks before the publication of Laughlin's theory, Yoshioka, Halperin, and Lee<sup>15</sup> presented results of numerical studies which suggested that the ground state of the (Coulomb) interacting 2DEG at high magnetic fields does not possess solid-like order, and that the actual ground state energy lies somewhat below the corresponding Hartree-Fock CDW values<sup>16</sup>. They furthermore concluded,

*We regard our data as supportive of the idea that the ground state is not crystalline, but a translationally invariant "liquid". We speculate that this liquid has commensuration energy at  $\nu = \frac{1}{3}$  (and possibly at other simple rational values), and that for a large but finite system, the ground state at  $\nu = \frac{1}{3}$  is threefold degenerate and separated by an energy gap from a variety of excited states. By going to a moving frame, it is then clear that at  $\nu = \frac{1}{3}$  a Hall current will flow without dissipation, even in the presence of impurities. At  $\nu$  close to  $\frac{1}{3}$ , we further suppose that the ground state, which is now highly degenerate, can be described as the  $\nu = \frac{1}{3}$  ground state plus an additional small density of quasi "particles" or "holes". This leads naturally to a downward cusp in the energy as a function of  $\nu$ . The Hall plateau at  $\sigma_{xy} = e^2/3h$  can then be explained if the quasiparticles are localized by impurities and thus do not contribute to the Hall current, which is simply carried by the underlying  $\nu = \frac{1}{3}$  state. Very recently, we have learned of a very original proposal by Laughlin of a wave function for a liquid state at  $\nu = 1/p$ , for  $p$  odd, which appears to have the requisite commensuration energy.*

One of the first good omens observed by Laughlin<sup>17</sup> was that his  $\nu = \frac{1}{3}$  fluid ground state weighed in at a lower energy than did even the best CDW estimates. This sat well with those who viewed solid-like order with great uneasiness in light of earlier investigations of CDW pinning by disorder, which would be inconsistent with the observed finite (and indeed quantized) Hall conductivity at  $\nu = \frac{1}{3}$ . By comparing energies of the Laughlin fluid and the best correlated CDW states, a crude one-parameter phase diagram emerged, predicting a transition between correlated fluid and Wigner crystal phases at  $\nu \approx \frac{1}{5}$  filling<sup>18</sup>.

An argument by Allan MacDonald<sup>19</sup> concludes that the FQH state must be *incompressible* in the absence of disorder in order to be consistent with experiment. Recall that the isothermal compressibility of a thermodynamic system is defined to be

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = \frac{1}{n^2} \left( \frac{\partial n}{\partial \mu} \right)_T . \quad (3.91)$$

<sup>15</sup>D. Yoshioka, B. I. Halperin, and P. A. Lee, *Phys. Rev. Lett.* **50**, 1219 (1983).

<sup>16</sup>The Maki-Zotos Wigner crystal describes the same state as the Hartree-Fock CDW and for  $\nu < \frac{1}{2}$  has almost the same energy.

<sup>17</sup>R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).

<sup>18</sup>As we have seen above (see Fig. 3.2, there is also experimental evidence for reentrant crystalline behavior.

<sup>19</sup>A. H. MacDonald, *Introduction to the Physics of the Quantum Hall Regime*, arXiv: cond-mat/9410047.

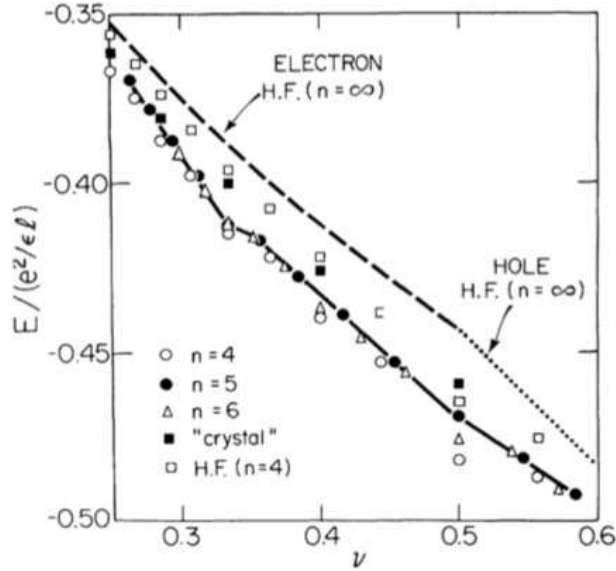


Figure 3.4: Energies for particle for the 2DEG as a function of  $\nu$ , from D. Yoshioka, B. I. Halperin, P. A. and Lee, *Phys. Rev. Lett.* **50**, 1219 (1983). The dashed and dotted lines show electron and hole Hartree-Fock crystal energies for the infinite system. Open circles, closed circles, and triangles show exact diagonalization results for  $N = 4, 5$ , and  $6$  electrons. Energies for the crystalline state with  $N = 4$  are shown with closed (exact diagonalization) and open (HF) squares. The solid line interpolating the  $N = 5$  data is a guide to the eye.

It follows that when  $\kappa = 0$ , *i.e.* when the system is incompressible, the chemical potential  $\mu$  is a discontinuous function of the density  $n$ . Consider now a quantum Hall droplet in which there is a current density  $j(r)$ . The magnetization of the droplet is given by

$$\mathbf{M} = \frac{1}{2c} \int d^2r \mathbf{r} \times \mathbf{j}(r) \quad \Rightarrow \quad \delta \mathbf{M} = \frac{1}{2c} \int d^2r \mathbf{r} \times \delta \mathbf{j}(r) \quad . \quad (3.92)$$

Now let us imagine changing the chemical potential  $\mu$  by an amount  $\delta\mu$ . If  $\mu = \varepsilon_F$  lies within a mobility gap, the only change in the current distribution can take place along the edge, which is some closed curve  $\mathbf{R}(s)$ . The parameterization is unimportant, but to be concrete we may take  $s$  to be the length along the curve. The differential change  $\delta j(r)$  in current density is then

$$\delta \mathbf{j}(r) = \delta I \int d\mathbf{R} \delta(\mathbf{r} - \mathbf{R}) \quad , \quad (3.93)$$

where  $\delta I$  is the additional edge current. This entails the relation

$$\delta \mathbf{M} = \frac{\delta I}{2c} \oint \mathbf{R} \times d\mathbf{R} = \frac{A}{c} \delta I \hat{z} \quad , \quad (3.94)$$

where  $A$  is the enclosed area. Thus,

$$\delta I = \frac{c}{A} \delta M = \frac{c}{A} \left( \frac{\partial M}{\partial \mu} \right)_B \delta \mu = \frac{c}{A} \left( \frac{\partial N}{\partial B} \right)_\mu \delta \mu \quad , \quad (3.95)$$



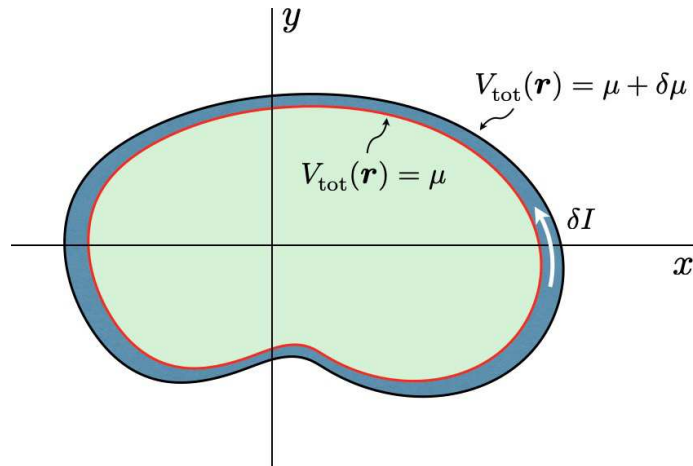


Figure 3.5: An expanding QH droplet, with  $\delta j(\mathbf{r}) = \delta I \int d\mathbf{R} \delta(\mathbf{r} - \mathbf{R})$ . The single electron potential  $V_{\text{tot}}(\mathbf{r})$  is the sum of the random background and confining potentials plus Coulomb contributions from all the other electrons.

where we have invoked a Maxwell relation. This establishes the result

$$\frac{\delta I}{\delta \mu} = c \left( \frac{\partial n}{\partial B} \right)_{\mu} . \quad (3.96)$$

Therefore, if  $n(B)$  depends on  $B$ , i.e. if  $(\partial n / \partial B)_{\mu} \neq 0$ , and if  $\mu$  lies within a spectral gap (or, more generally, a mobility gap) such that no internal currents are generated by an increase in chemical potential  $\delta \mu$ , then there must be gapless edge excitations.

As acknowledged by MacDonald, there are a couple of weak points to this argument. For example, it would seem that the Hall voltage  $V_H$  should have to be small in magnitude compared with  $\hbar \omega_c / e$ , yet in experiments good quantization is observed even if the former is hundreds of times larger than the latter. In addition, it may not be that all the transport current flows at the edges of the system<sup>20</sup>. A more realistic approach to boundary conditions in the QHE was considered by Niu and Thouless<sup>21</sup>.

### 3.3.1 Laughlin's excellent idea

Recall that all  $N$ -electron LLL states may be written in the form

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = P(z_1, \dots, z_N) \exp\left(-\frac{1}{4\ell^2} \sum_{j=1}^N |z_j|^2\right) , \quad (3.97)$$

<sup>20</sup>C. Wexler and D. J. Thouless, *Phys. Rev. B* **49**, 4815 (1994).

<sup>21</sup>Q. Niu and D. J. Thouless, *Phys. Rev. B* **35**, 2188 (1987).

where  $P(Z)$  is a multinomial function which is odd under exchange, *i.e.*

$$P(Z_\sigma) = \text{sgn}(\sigma)P(Z) \quad , \quad (3.98)$$

where  $Z_\sigma = \{z_{\sigma(1)}, \dots, z_{\sigma(N)}\}$ , and where, restricting the single particle states to the angular momentum basis, with  $m \in \{0, \dots, N_\phi - 1\}$ , the highest degree in each holomorphic coordinate  $z_j$  is  $z_j^{N_\phi - 1}$ . Furthermore, if  $\Psi$  is a state of definite total angular momentum  $J = L_z$ , then  $F(Z)$  must be homogeneous, with  $F(\lambda Z) = \lambda^{\text{deg}(F)} F(Z)$  where  $J = \text{deg}(F)$ . The filling fraction is then given by  $\nu = N/N_\phi$ . We've already encountered the example of the Vandermonde determinant,

$$V(Z) = \det(z_j^k) = \prod_{i>j} (z_i - z_j) \quad , \quad (3.99)$$

for which  $J = \frac{1}{2}N(N-1)$  and  $N_\phi = N$ .

Laughlin (1983) proposed the sequence of FQHE states

$$\Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i>j} (z_i - z_j)^q \exp\left(-\frac{1}{4\ell^2} \sum_{j=1}^N |z_j|^2\right) \quad , \quad (3.100)$$

*i.e.*  $F_q(Z) = [V(Z)]^q$ . Since  $V(Z)$  is completely antisymmetric, we have

$$F_q(Z_\sigma) = (\text{sgn}(\sigma))^q F_q(Z) \quad , \quad (3.101)$$

hence  $\Psi_q$  corresponds to a fermionic wavefunction provided  $q$  is an odd integer. The total electronic angular momentum is  $J = \frac{1}{2}qN(N-1)$  and the highest individual degree in any  $z_j$  is  $N_\phi - 1 = q(N-1)$ . Thus  $\nu = N/N_\phi = q^{-1}$  in the thermodynamic limit. Note that for bosons, the Laughlin wavefunctions have  $m$  even.

### 3.3.2 Plasma analogy

The Laughlin states are a generalization of the Bijl-Jastrow pair product form,  $\Psi_{\text{BJ}} = \prod_{i<j} f(r_{ij})$ . Consider, for example, the extended form,

$$\Psi_{\text{GBJ}}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_i \exp\left\{-\frac{1}{2}u(r_i)\right\} \prod_{i>j} \exp\left\{-\frac{1}{2}v(r_{ij})\right\} \prod_{i>j>k} \exp\left\{-\frac{1}{2}w(r_{ij}, r_{jk})\right\} \quad , \quad (3.102)$$

where  $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ . Then the  $N$ -particle probability density is

$$|\Psi_{\text{GBJ}}|^2 = \exp\left\{-\beta\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N)\right\} \quad (3.103)$$

where

$$\beta\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \sum_i u(r_i) + \sum_{i>j} v(r_{ij}) + \sum_{i>j>k} w(r_{ij}, r_{jk}) \quad . \quad (3.104)$$

Here we assume  $u$ ,  $v$ , and  $w$  are all real. Thus the many-particle probability density is equivalent to the Boltzmann weight of a *classical* problem *in the same number of dimensions*, with one-body, two-body, three-body, *etc.* potentials.

For the Laughlin wavefunction, we have  $|\Psi_q|^2 = \exp(-\beta\Phi)$ , with  $\beta = 1/q$  and

$$\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) = -2q^2 \sum_{i>j} \ln |\mathbf{r}_i - \mathbf{r}_j| + \frac{q}{2\ell^2} \sum_i \mathbf{r}_i^2 \quad . \quad (3.105)$$

This is the classical *two-dimensional one-component plasma*, or 2DOCP, consisting of  $N$  point charges, each of strength  $\theta = q\sqrt{2}$ , interacting by the potential  $v(\mathbf{r}) = \theta^2\phi(\mathbf{r}) = 2q^2\phi(\mathbf{r})$ , with  $\phi(\mathbf{r}) = -\ln r$ , and subject to the background potential  $u(\mathbf{r}) = q\mathbf{r}^2/2\ell^2$ , all at temperature  $k_B T = q$ . Note that  $\nabla^2\phi(\mathbf{r}) = -2\pi\delta(\mathbf{r})$ , hence  $\nabla^2u(\mathbf{r}) = 2q/\ell^2$ , corresponding to the interaction of a charge  $q\sqrt{2}$  with a uniform background of charge density  $\rho = -1/\sqrt{2}\pi\ell^2$ . To minimize the Coulomb energy, the  $N$  point charges form a disk of number density  $n = 1/2\pi q\ell^2$ , so that total charge neutrality holds, *i.e.*  $nq\sqrt{2} + \rho = 0$ . The radius  $R$  of this disk is then given by the condition  $\pi R^2 n = N$ , hence  $R = \sqrt{2qN}\ell$ .

### 3.3.3 The 2DOCP

Properties of the 2DOCP are discussed in a review article by J. M. Caillol *et al.*<sup>22</sup> To fix the problem precisely, consider a classical system of particles each of charge  $e$ , interacting via a potential  $v(r) = -e^2 \ln(r/d)$ , where  $d$  is a length scale. Clearly  $d$  is irrelevant as it enters the energy additively and thus sets the location of the zero of energy. Consider  $N$  such particles in a disk of radius  $R$ . The mean particle number density is thus  $n = N/\pi R^2$ ; then  $a \equiv (\pi n)^{-1/2}$  is called the *ion disk radius*. Let the disk be filled with a uniform neutralizing background of charge density  $(-en)$ . Taking into account particle-particle, particle-background, and background-background interactions, the energy is then

$$H(\mathbf{r}_1, \dots, \mathbf{r}_N) = -e^2 \sum_{j<k}^N \ln \left( \frac{|\mathbf{r}_j - \mathbf{r}_k|}{d} \right) + \frac{1}{2} N e^2 \sum_{i=1}^N \left( \frac{r_i}{R} \right)^2 + \frac{1}{2} N^2 e^2 \left[ \ln \left( \frac{R}{d} \right) - \frac{3}{4} \right] \quad (3.106)$$

The partition function is

$$Z = \int_0^R d^2 r_1 \cdots \int_0^R d^2 r_N e^{-\beta H(\mathbf{r}_1, \dots, \mathbf{r}_N)} \equiv e^{-N\beta f} \quad , \quad (3.107)$$

where  $f = -N^{-1}k_B T \ln Z$  is the free energy per particle. Defining  $\mathbf{x}_i \equiv \mathbf{r}_i/R$ ,

$$\beta f = \left(1 - \frac{1}{4}\Gamma\right) \ln(\pi n) - \frac{1}{2}\Gamma \ln d - \frac{3}{8}N\Gamma + \frac{1}{4}\Gamma \ln N - \frac{1}{N} \ln W(N, \Gamma) \quad , \quad (3.108)$$

<sup>22</sup>J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen, *J. Stat. Phys.* **28**, 325 (1982).

where  $\Gamma \equiv \beta e^2$  is the dimensionless *plasma parameter* and

$$W(N, \Gamma) = \int_0^1 d^2 x_1 \cdots \int_0^1 d^2 x_N \prod_{j < k} |\mathbf{x}_j - \mathbf{x}_k|^\Gamma \prod_{i=1}^N e^{-N\Gamma x_i^2/2} . \quad (3.109)$$

From the thermodynamic relation

$$df|_N = -s dT + pn^{-2} dn , \quad (3.110)$$

we have the equation of state

$$p = (1 - \frac{1}{4}\Gamma) nk_B T . \quad (3.111)$$

The isothermal compressibility is then  $\kappa_T = n^{-1}(\partial n / \partial p)_T = 1/p$ , and we define the dimensionless isothermal compressibility  $\chi_T \equiv nk_B T \kappa_T = (1 - \frac{1}{4}\Gamma)^{-1}$ .

The equilibrium properties of the plasma are dependent solely on  $\Gamma$ . When  $\Gamma$  is small, the plasma is said to be weakly coupled. For  $\Gamma > \Gamma_c \approx 140$ , the 2DOCP crystallizes. Much of the physics of the 2DOCP is reflected in the behavior of the pair distribution function,

$$n g(\mathbf{r}) = \frac{1}{N} \left\langle \sum_{i \neq j}^N \delta(\mathbf{r} + \mathbf{r}_j - \mathbf{r}_i) \right\rangle \quad (3.112)$$

and the associated pair correlation function  $h(\mathbf{r}) = g(\mathbf{r}) - 1$ . The static structure factor, for example, is given by

$$\hat{s}(\mathbf{k}) = 1 + n \int d^2 r h(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} = 1 + n \hat{h}(\mathbf{k}) . \quad (3.113)$$

The Fourier transform of the *direct correlation function*,  $\hat{c}(\mathbf{k})$ , is defined by

$$\hat{c}(\mathbf{k}) \equiv \frac{\hat{h}(\mathbf{k})}{1 + n \hat{h}(\mathbf{k})} \iff \hat{h}(\mathbf{k}) = \frac{\hat{c}(\mathbf{k})}{1 - n \hat{c}(\mathbf{k})} , \quad (3.114)$$

which is equivalent to the relation

$$h(\mathbf{r}) = c(\mathbf{r}) + n \int d^3 r' h(\mathbf{r} - \mathbf{r}') c(\mathbf{r}') . \quad (3.115)$$

This is known as the Ornstein-Zernike equation. Physically it says that the correlation between a particle at position  $\mathbf{0}$  and a particle at position  $\mathbf{r}$  can be written as a sum of the direct correlator  $c(\mathbf{r})$  plus an term arising from the indirect effect of the particle at  $\mathbf{0}$  with a third particle at  $\mathbf{r}'$  which affects that at  $\mathbf{r}$  both directly and indirectly.

The asymptotic long wavelength behavior of  $\hat{c}(\mathbf{k})$  is believed to correspond to a weak coupling limit, in which case

$$\hat{c}(\mathbf{k}) \longrightarrow -\beta \hat{v}(\mathbf{k}) = -Q^2/nk^2 , \quad (3.116)$$

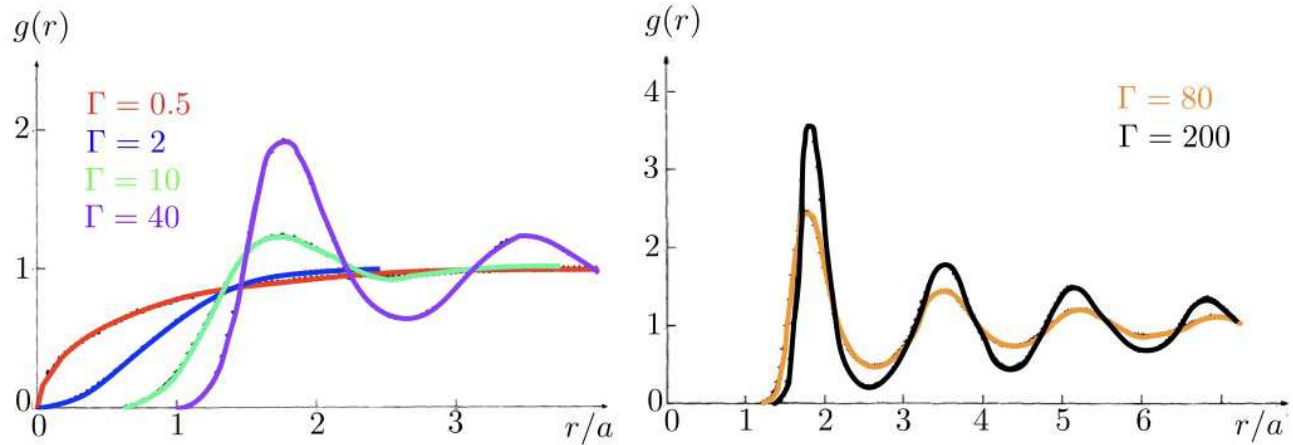


Figure 3.6: Pair distribution function  $g(r)$  for the 2DOCP. Monte Carlo results from Figs. 1 and 2 of Caillol *et al.* (1982).

where  $Q = (2\pi n\beta e^2)^{1/2} = (2\pi n\Gamma)^{1/2}$  is the Debye wavevector. Separating out this singular term, one writes  $\hat{c}(\mathbf{k}) = -Q^2/nk^2 + \hat{c}^R(\mathbf{k})$  where  $\hat{c}^R(\mathbf{k}) = \hat{c}^R(0) + \mathcal{O}(k^2)$ . The regular part  $\hat{c}^R(\mathbf{k})$  is related to the dimensionless isothermal compressibility:

$$\lim_{k \rightarrow 0} \hat{c}^R(\mathbf{k}) = \hat{c}^R(0) = n^{-1}(1 - \chi_T^{-1}) \quad . \quad (3.117)$$

The above results entail the expansion

$$n \hat{h}(\mathbf{k}) = -1 + \frac{k^2}{Q^2} + \frac{k^4}{\chi_T Q^4} + \mathcal{O}(k^6) \quad . \quad (3.118)$$

In real space, then, we have the following sum rules on moments of the pair correlation function:

$$n \int d^2r h(\mathbf{r}) = n \hat{h}(\mathbf{k})|_{k=0} = -1 \quad , \quad (3.119)$$

known as the *charge neutrality* sum rule,

$$n \int d^2r r^2 h(\mathbf{r}) = -n \nabla_{\mathbf{k}}^2 \hat{h}(\mathbf{k})|_{k=0} = -\frac{4}{Q^2} \quad , \quad (3.120)$$

known as the *perfect screening* sum rule, and

$$n \int d^2r r^4 h(\mathbf{r}) = n (\nabla_{\mathbf{k}}^2)^2 \hat{h}(\mathbf{k})|_{k=0} = -\frac{64}{\chi_T Q^4} \quad , \quad (3.121)$$

known as the *compressibility* sum rule.

From Monte Carlo studies, we know that the 2DOCP crystallizes at  $\Gamma \simeq 140$ . At this point, the structure factor  $s(\mathbf{k})$  exhibits Bragg peaks. Since the Coulomb potential is long-ranged, the system can evade the usual Hohenberg-Mermin-Wagner restrictions which forbid broken continuous translational symmetry at finite temperature in  $d \leq 2$  dimensions.

### 3.3.4 Laughlin vs. Wigner crystal

The correspondence between the Laughlin wavefunction  $|\Psi_q|^2 = \exp(-\beta H)$  and the 2DOCP Hamiltonian  $H$  is then:

$$\beta = \frac{1}{q} \quad , \quad e^2 = 2q^2 \quad \Gamma = 2q \quad , \quad n = \frac{1}{2\pi q \ell^2} \quad . \quad (3.122)$$

The sum rules provide information on the long-wavelength behavior of the structure factor in the Laughlin states, *viz.*

$$\hat{s}(\mathbf{k}) = \frac{1}{2} k^2 \ell^2 + \frac{1}{8} (q-1) k^4 \ell^4 + \mathcal{O}(k^6) \quad . \quad (3.123)$$

Crystallization of the 2DOCP (into a triangular structure) at  $\Gamma = 140$  means that the Laughlin wavefunction has triangular crystalline order for  $q > 70$ . However, recall that the Laughlin state is an *Ansatz* wavefunction. It isn't even a proper variational state, since the only free parameter  $q$  is discrete and is fixed by the filling, with  $\nu = q^{-1}$ .

The actual 2DEG in the LLL crystallizes well before the filling fraction gets as low as  $\frac{1}{70}$ . Upon taking the thermodynamic limit and properly including the effects of the uniform neutralizing background, the energy per particle is given by

$$u = \frac{U}{N} = \frac{n}{2} \int d^2r v(\mathbf{r}) [g(\mathbf{r}) - 1] \quad , \quad (3.124)$$

where  $v(r) = e^2/\epsilon r$ . The pair distribution function in the Laughlin states has been evaluated using the so-called hypernetted chain approximation (Laughlin, 1983) and by Monte Carlo methods<sup>23</sup>. At  $\nu = \frac{1}{3}$ , these calculations yield energies  $u_L^{\text{HNC}}(q=3) = -0.4156 \pm 0.0012$  and  $u_L^{\text{MC}}(q=3) = -0.410 \pm 0.001$ , respectively (units of  $e^2/\epsilon\ell$ ). Exact diagonalization studies by Haldane and Rezayi<sup>24</sup> extrapolated finite size results on the sphere for  $N \leq 7$  to  $N = \infty$  and obtained  $u_L^{\text{ED}}(q=3) = -0.415 \pm 0.005$ . In contrast, the energy for the Wigner crystal, as computed by Yoshioka, Halperin, and Lee, or by Maki and Zotos, is about  $u^{\text{WC}}(\nu = \frac{1}{3}) \approx -0.38$ , which is much higher. In order to give the Wigner crystal a fighting chance, Lam and Girvin<sup>25</sup> investigated the correlated Wigner crystal wavefunction,

$$\Psi_{\text{CWC}}(\mathbf{r}_1, \dots, \mathbf{r}_N) = \exp\left(\frac{1}{2} \sum_{i,j} (z_i - R_i) B_{ij} (z_j - R_j)\right) \prod_l \varphi_{R_l}(\mathbf{r}_l) \quad , \quad (3.125)$$

where the prefactor is a holomorphic function in  $\{z_1, \dots, z_N\}$  and  $B_{ij} = B(\mathbf{R}_i - \mathbf{R}_j)$  is a correlating matrix. As the effects of antisymmetrization below  $\nu = 0.4$  were found by Maki and

<sup>23</sup>See D. Levesque, J. J. Weis, and A. H. MacDonald, *Phys. Rev. B* **30**, 1056 (1984) and R. Morf and B. I. Halperin, *Phys. Rev. B* **32**, 2221 (1986).

<sup>24</sup>F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **54**, 237 (1985).

<sup>25</sup>P. K. Lam and S. M. Girvin, *Phys. Rev.* **30**, 473 (1984).

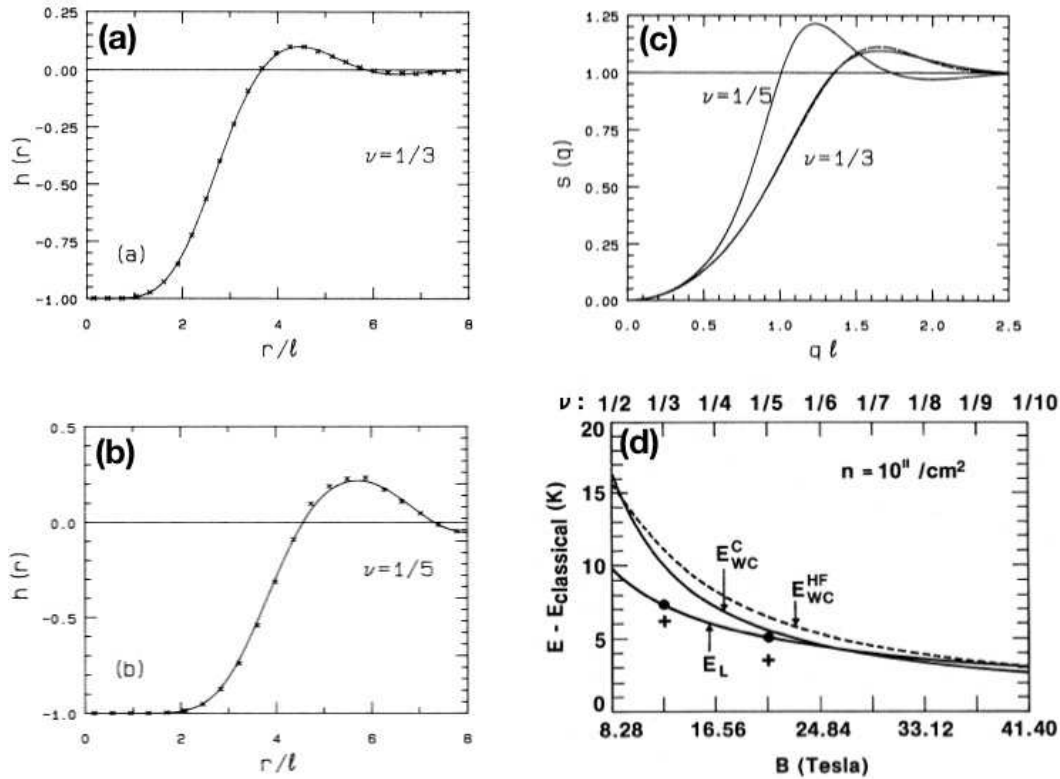


Figure 3.7: Pair correlation function (a,b) and structure factor (c) for  $q = 3$  and  $q = 5$  Laughlin states. From S. M. Girvin, A. M. MacDonald, and P. M. Platzman, *Phys. Rev. B* **33**, 2481 (1986). Panel (d) shows a comparison of the (interpolated Laughlin state energy with that of the correlated Wigner crystal, indicating a transition to the WC state at  $\nu^{-1} \approx 6.5$ . From P. K. Lam and S. M. Girvin, *Phys. Rev.* **30**, 473 (1984).

Zotos to be insignificant, no antisymmetrization was imposed. The matrix  $B_{ij}$  was then determined variationally by minimizing the Coulomb energy in this state. Within the harmonic approximation, the lattice Fourier transform  $\hat{B}(\mathbf{k}) = \sum_{\mathbf{R}} B(\mathbf{R}) e^{-i\mathbf{k}\cdot\mathbf{R}}$  is given by

$$\hat{B}(\mathbf{k}) = \frac{\omega_L(\mathbf{k}) - \omega_T(\mathbf{k})}{\omega_L(\mathbf{k}) + \omega_T(\mathbf{k})}, \quad (3.126)$$

where  $\omega_{L,T}(\mathbf{k})$  are the longitudinal and transverse magnetophonon frequencies. Such an optimized WC state significantly lowers the correlation energy, to  $u_{\text{cwc}}(\nu = \frac{1}{3}) = -0.3948 \pm 0.0005$ , which is still about 2.7% higher than that of the  $q = 3$  Laughlin state. The Laughlin state is remarkably robust in terms of its Coulomb energy with respect to variational tweaking, and the extrapolated differences between the Coulomb energy in the Laughlin state and that from exact diagonalization differ by about 0.05%<sup>26</sup>.

<sup>26</sup>See M. Fremling, J. Fulsebakke, N. Moran, and J. K. Slingerland, *Phys. Rev. B*, **93**, 235149 (2016).

### 3.3.5 Haldane pseudopotentials

It was realized by Haldane<sup>27</sup> and by Trugman and Kivelson<sup>28</sup> that the Laughlin state  $\Psi_q$  is the *exact* ground state for sufficiently short-ranged interaction potentials<sup>29</sup>. Whenever the interaction potential  $v(\mathbf{r})$  is central, *i.e.* a function of  $r = |\mathbf{r}|$  alone, its Fourier transform,

$$\hat{v}(\mathbf{k}) = \int d^2r v(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} = 2\pi \int_0^\infty dr r v(r) J_0(kr) \quad , \quad (3.127)$$

may be expanded as a power series in  $k^2$ , *i.e.*  $\hat{v}(\mathbf{k}) = \sum_{j=0}^\infty A_j (-k^2 \ell^2)^j$ . Thus in real space, we may write  $v(\mathbf{r})$  as an expansion in powers of the Laplacian acting on a delta-function:

$$v(\mathbf{r}) = \sum_{j=0}^\infty A_j (\ell^2 \nabla^2)^j \delta(\mathbf{r}) \quad . \quad (3.128)$$

For a totally antisymmetric wavefunction  $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$ , the  $j = 0$  term will not contribute to the total energy.

The interaction energy per particle is given by<sup>30</sup>

$$\frac{E_{\text{int}}}{N} = \langle \Psi | H_{\text{int}} | \Psi \rangle = \frac{1}{2} n \int d^2r v(\mathbf{r}) g(\mathbf{r}) = \frac{1}{2} n \sum_{j=0}^\infty A_j (\ell^2 \nabla^2)^j g(\mathbf{r}) \Big|_{r=0} \quad , \quad (3.129)$$

where  $g(\mathbf{r})$  is the pair distribution function. In the Laughlin state  $\Psi_q$ , the  $r \rightarrow 0$  behavior of  $g(\mathbf{r})$  is given by

$$g(\mathbf{r}) = c_q (r/\ell)^{2q} + c_{q+1} (r/\ell)^{2(q+1)} + \dots \quad , \quad (3.130)$$

where the  $\{c_j\}$  are constant coefficients. Thus all terms in the potential with coefficients  $A_j$  with  $j < q$  will make no contribution to  $E_{\text{int}}$ . Suppose now that  $A_{1,2} > 0$  but  $A_3 = A_4 = \dots = 0$ . If a wavefunction  $\Psi = P(Z) \prod_j e^{-|z_j|^2/4\ell^2}$  is to satisfy  $\langle \Psi | H_{\text{int}} | \Psi \rangle = 0$ , it is clear that it must vanish at least as fast as  $(z_i - z_j)^3$  as the pair separation tends to zero, and hence  $P(Z)$  must contain at least 3 factors of  $V(Z)$ , where  $V(Z)$  is the Vandermonde determinant. On the other hand, for homogeneous states, the filling factor is given by

$$\nu = \frac{N(N-1)}{2J} \quad , \quad (3.131)$$

<sup>27</sup>F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).

<sup>28</sup>S. A. Trugman and S. Kivelson, *Phys. Rev. B* **31**, 5280 (1985).

<sup>29</sup>On the sphere, the Laughlin state is nondegenerate. On the torus, the degeneracy is  $q$ , and on a Riemann surface of genus  $g$ , the degeneracy is  $q^g$ . See X. G. Wen and Q. Niu, *Phys. Rev. B* **41**, 9377 (1990). On the plane, with a confining potential, there will be a continuum of gapless edge excitations.

<sup>30</sup>This energy includes only particle-particle interactions and does not include the effects of any neutralizing background.



where  $J$  is the total angular momentum. Since  $\deg(V) = \frac{1}{2}N(N-1)$ , we conclude that the holomorphic part of  $\Psi$  is  $P(Z) = C[V(Z)]^q$ , where  $C$  is a constant. Thus the Laughlin state with  $q = 3$  is the *only* homogeneous state at  $\nu = \frac{1}{3}$  which has zero energy. Similarly, when  $A_{1,2,3,4} > 0$  and  $A_5 = A_6 = \dots = 0$ , the  $q = 5$  Laughlin state is the sole  $\nu = \frac{1}{5}$  state which lies at zero energy<sup>31</sup>.

There is a convenient parameterization (Haldane 1983) of the pair interaction in terms of relative coordinate ‘pseudopotentials’. Writing

$$v(\mathbf{r}_i - \mathbf{r}_j) = \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot(\mathbf{r}_i - \mathbf{r}_j)} \quad , \quad (3.132)$$

and invoking the separation into cyclotron and guiding center ladder operators<sup>32</sup>  $z = \sqrt{2}\ell(a + b^\dagger)$ , we have that the LLL-projected interaction is

$$\begin{aligned} \Pi_0 v(\mathbf{r}_i - \mathbf{r}_j) \Pi_0 &= \int \frac{d^2k}{(2\pi)^2} \hat{v}(\mathbf{k}) \exp\left[\frac{i\ell k}{\sqrt{2}}(b_i - b_j)\right] \exp\left[\frac{i\ell \bar{k}}{\sqrt{2}}(b_i^\dagger - b_j^\dagger)\right] \\ &= \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \int \frac{d^2k}{(2\pi)^2} \hat{v}(\mathbf{k}) (-k^2 \ell^2)^n (J_{ij}^-)^n (J_{ij}^+)^n \quad . \end{aligned} \quad (3.133)$$

The operators  $J_{ij}^\pm$  raise and lower the *relative angular momentum* of the pair  $(ij)$ :

$$J_{ij}^+ = \frac{1}{\sqrt{2}}(b_i^\dagger - b_j^\dagger) \quad , \quad J_{ij}^- = \frac{1}{\sqrt{2}}(b_i - b_j) \quad , \quad (3.134)$$

with

$$J_{ij} = \Pi_0 \left( \frac{1}{2}(\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{p}_i - \mathbf{p}_j) \cdot \hat{\mathbf{z}} \right) \Pi_0 = J_{ij}^+ J_{ij}^- \quad . \quad (3.135)$$

The relative angular momentum raising and lowering operators satisfy the algebra

$$[J_{ij}^-, J_{ij}^+] = \frac{1}{2}(\delta_{ik} + \delta_{jl} - \delta_{il} - \delta_{jk}) \quad . \quad (3.136)$$

If we define the projector  $P_s(ij)$  to be the projector of the pair  $(ij)$  onto the relative angular momentum  $J_{ij} = s$  subspace, then

$$\begin{aligned} \Pi_0 v(\mathbf{r}_i - \mathbf{r}_j) \Pi_0 &= \sum_{s=0}^{\infty} V_s P_s(ij) \\ V_s &= \int \frac{d^2k}{(2\pi)^2} \hat{v}(\mathbf{k}) L_s(\mathbf{k}^2 \ell^2) e^{-\mathbf{k}^2 \ell^2} \quad , \end{aligned} \quad (3.137)$$

where  $L_s(x)$  is the Laguerre polynomial. This provides us with another way to expand the real space potential, *i.e.* in terms of the pseudopotential amplitudes  $\{V_s\}$ :

$$v(\mathbf{r}) = 4\pi\ell^2 \sum_{s=0}^{\infty} V_s L_s(-\ell^2 \nabla^2) \delta(\mathbf{r}) \quad . \quad (3.138)$$

<sup>31</sup>Other than center-of-mass degeneracies on Riemann surfaces of genus  $g > 0$ .

<sup>32</sup>Recall we have been working in the symmetric gauge since §1.3.7.

which may be compared with Eqn. 3.128. One then finds

$$A_j = \frac{4\pi\ell^2}{j!} \sum_{s=j}^{\infty} \binom{s}{s-j} V_s \quad . \quad (3.139)$$

If  $V_s = 0$  for all  $s > s_0$ , then the power series expansion of Eqn. 3.128 is simply a rearrangement of the pseudopotential expansion, up to the same leading order in powers of  $\nabla^2$ . Again, only odd pseudopotentials contribute to the energy for a fermionic system.

For the Coulomb interaction  $v(r) = e^2/\epsilon r$ , one finds

$$V_s^{\text{COUL}} = \frac{1}{4^s} \binom{2s}{s} \cdot \frac{\sqrt{\pi}e^2}{2\epsilon\ell} \quad . \quad (3.140)$$

As  $s \rightarrow \infty$ , we have  $V_s \simeq e^2/2\sqrt{s}\epsilon\ell \propto s^{-1/2}$ , corresponding to the fact that the single particle state with angular momentum  $s$  encloses an area  $\pi r^2 = 2\pi s\ell^2$ . Thus  $V_1^{\text{COUL}} = \sqrt{\pi}e^2/4\epsilon\ell$  and  $V_3^{\text{COUL}} = \frac{5}{8} V_1^{\text{COUL}}$ ,  $V_5^{\text{COUL}} = \frac{63}{238} V_1^{\text{COUL}}$ , etc. If we define

$$V_s(\lambda) = (1 - \lambda)V_1^{\text{COUL}} \delta_{s,1} + \lambda V_s^{\text{COUL}} \quad , \quad (3.141)$$

which interpolates between the truncated pseudopotential  $V_s(0) = V_1^{\text{COUL}} \delta_{s,1}$  at  $\lambda = 0$  and the full Coulomb  $V_s(1) = V_s^{\text{COUL}}$  at  $\lambda = 1$ . One can then ask whether the gap at  $\lambda = 0$  remains finite for all  $\lambda \in [0, 1]$ , in which case no phase boundaries are crossed as one evolves from the truncated pure  $V_1$  model to the full Coulomb interaction. Alternatively, Haldane<sup>33</sup> considered the potential given by

$$V_s = (V_1 - V_1^{\text{COUL}})\delta_{s,1} + V_s^{\text{COUL}} \quad (3.142)$$

as a function of  $V_1$ , which takes the Coulomb interaction and replaces the  $s = 1$  pseudopotential component  $V_1^{\text{COUL}}$  with the parameter  $V_1$ . Results for  $N = 6$  particles in a spherical geometry are shown in Fig. 3.8. As the  $V_1$  pseudopotential is decreased from its Coulomb value of  $V_1^{\text{COUL}} = 0.4781$  on the sphere<sup>34</sup>, the bulk gap is found to collapse at  $V_3 \approx 0.37$ , heralding a second order phase transition to a compressible phase.

To understand better the spectrum of relative angular momentum in the Laughlin states, define the complex center-of-mass and relative coordinates for a select pair ( $i = 1, j = 2$ ) as  $W \equiv \frac{1}{2}(z_1 + z_2)$  and  $w \equiv z_2 - z_1$ , so that  $z_{1,2} = W \mp \frac{1}{2}w$ . Then

$$V(z_1, \dots, z_N) = w \prod_{j=3}^N \left[ (W - z_j)^2 - \frac{1}{4}w^2 \right] V(z_3, \dots, z_N) \quad . \quad (3.143)$$

The spectrum of relative angular momentum states for the pair (1, 2) can be gleaned by identifying terms homogeneous in the relative coordinate  $w$ , *i.e.* terms proportional to  $w^l$  for some  $l$ .

<sup>33</sup>F. D. M. Haldane in *The Quantum Hall Effect*, R. E. Prange and S. M. Girvin, eds. (Springer, 1987).

<sup>34</sup>On the sphere, the Coulomb interaction is proportional to the inverse chord length, yielding  $V_1^{\text{COUL}} = 0.4781$  and  $V_3^{\text{COUL}} = 0.3092$ , rather than  $V_1^{\text{COUL}} = 0.4431$  and  $V_3^{\text{COUL}} = 0.2769$  as obtained on the plane.

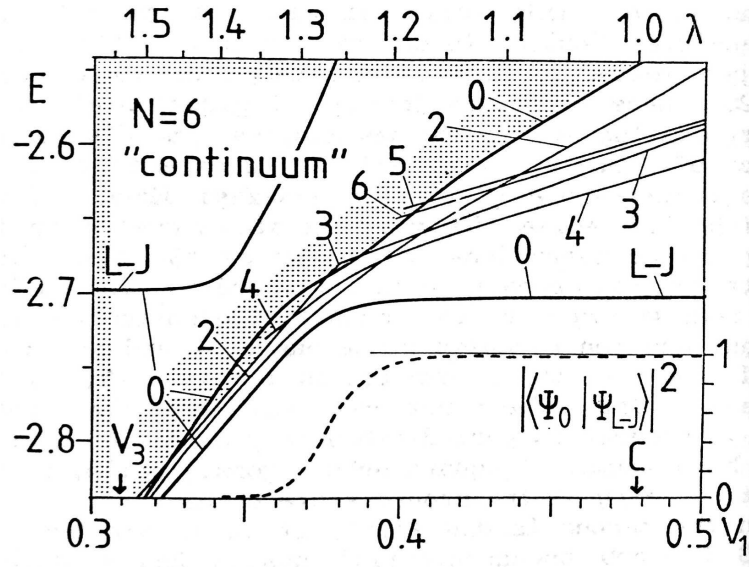


Figure 3.8: Effect of varying the  $V_1$  pseudopotential on low-lying energy states. Excited energy levels are labeled by their total angular momentum  $L$  on the sphere ( $N = 6$ ). The Laughlin-Jastrow (L-J) state has  $L = 0$ . The overlap of the ground state with the Laughlin state is also shown. Arrows on the  $V_1$  axis indicate values of  $V_1^{\text{COUL}}$  and  $V_3^{\text{COUL}}$  (units of  $e^2/\epsilon\ell$ ). For  $V_1 > 0.37$  the system is gapped and incompressible. Below this value the system is gapless and hence compressible. From Haldane (1987).

Clearly we have contributions from  $l \in \{1, 3, 5, \dots, 2N - 3\}$ . Note that only odd  $l$  terms enter the spectrum. When we raise  $V(Z)$  to the power  $q$ , we obtain

$$l \in \{q, q + 2, q + 4, \dots, 2q(N - 2) + 1\} \quad . \quad (3.144)$$

Thus, in the Laughlin state  $\Psi_q$ , the spectrum of relative coordinate angular momenta is all odd integers  $l$  starting at  $l_{\min} = q$ , and terminating at the cutoff  $l_{\max} = 2q(N - 1) + 1$ .

Given an arbitrary many-body state, the pair distribution function  $g(r)$  may be expanded in powers of  $r^2$  for small  $r$ , viz.

$$g(r) = \sum_{p=0}^{\infty} c_p (r^2/\ell^2)^p \quad . \quad (3.145)$$

If the holomorphic factor  $P(Z)$  contains  $q$  factors of the Vandermonde determinant  $V(Z)$ , then  $c_p = 0$  for  $p = 0, \dots, q - 1$ . Exact diagonalization studies by Yoshioka<sup>35</sup> for small fermionic ( $N \leq 8$ ) and bosonic ( $N \leq 7$ ) systems in a toroidal geometry examined the behavior of the coefficients  $c_p$  as a function of filling fraction. Results are shown in Fig. 3.9. One sees that for fermionic states with  $\nu < \frac{1}{3}$ , the coefficients  $c_{1,2}$  are both exceedingly small, suggesting that the cube of the Vandermonde determinant approximately divides the holomorphic part

<sup>35</sup>D. Yoshioka, *Phys. Rev. B* **29**, 6822 (1984).

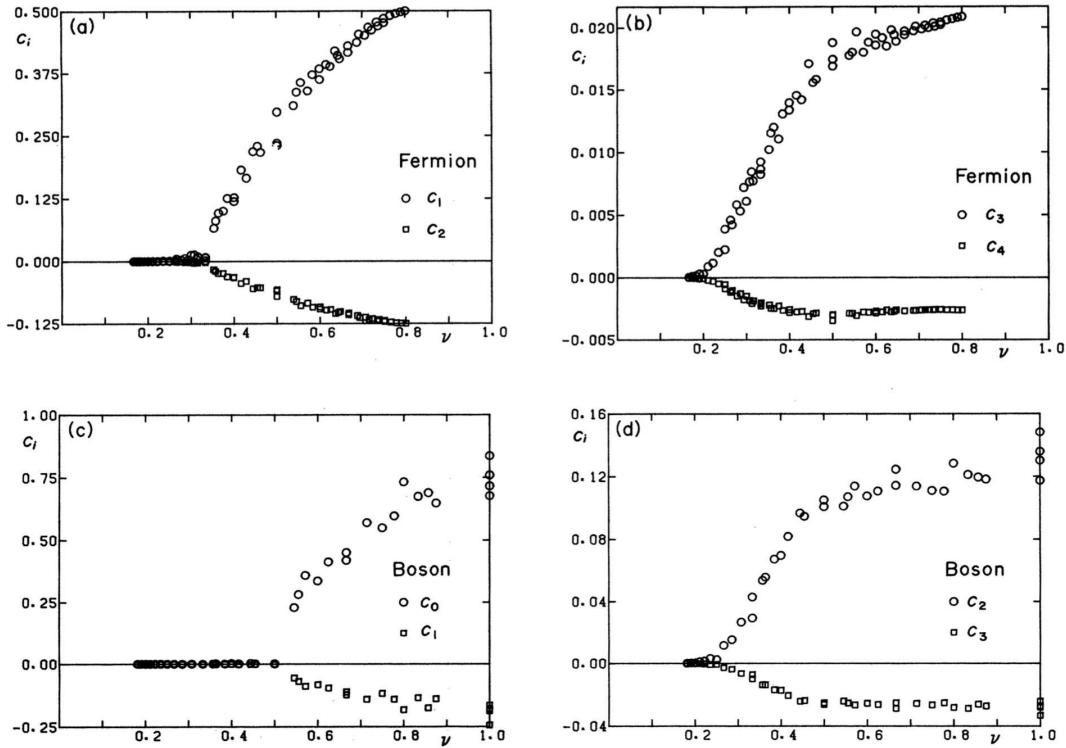


Figure 3.9: Pair distribution function coefficients in exact finite  $N$  ground states *versus* filling fraction. (a) Coefficients  $c_1$  (circles) and  $c_2$  (squares) for fermionic states. (b) Coefficients  $c_3$  (circles) and  $c_4$  (squares) for fermionic states. (c) Coefficients  $c_0$  (circles) and  $c_1$  (squares) for bosonic states. (d) Coefficients  $c_2$  (circles) and  $c_3$  (squares) for bosonic states. From D. Yoshioka, *Phys. Rev. B* **29**, 6822 (1984).

$P(Z)$  of the exact wavefunction. Similar results are found *vis-a-vis*  $c_{3,4}$  when  $\nu < \frac{1}{5}$ . Yoshioka's numerics establish that for Coulomb systems with  $\nu \leq q^{-1}$ , for both bosons and fermions, the holomorphic part  $P(Z)$  of the ground state wavefunction is almost perfectly divided by the  $q^{\text{th}}$  power of the Vandermonde determinant.

### 3.3.6 Quasiparticles

Laughlin also proposed wavefunctions for localized charged excitations, called *quasiholes* and *quasielectrons*. The quasihole wavefunctions are the easiest to understand and are of the form

$$\Psi_q^{\text{QH}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\xi}) = \prod_{l=1}^N (z_l - \xi) \prod_{j>k}^N (z_j - z_k)^q \prod_{i=1}^N \exp(-z_i \bar{z}_i / 4\ell^2) \quad , \quad (3.146)$$

where  $\xi$  is the complexified quasihole position  $\xi = \xi_x + i\xi_y$ . Laughlin argues that such a state should be gauge-equivalent to an eigenstate of a many-electron system in which  $\Psi_q$  is a ground

state, since it results from adiabatic flux threading by  $\phi_0 = hc/e$  parallel to the applied field at the point  $\mathbf{r} = \boldsymbol{\xi}$ . The associated plasma Hamiltonian is then

$$\Phi_q(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\xi}) = -m \ln |\Psi_q|^2 = -2q^2 \sum_{j>k} \ln |\mathbf{r}_j - \mathbf{r}_k| + \frac{q}{2\ell^2} \sum_i \mathbf{r}_i^2 - 2q \sum_l \ln |\mathbf{r}_l - \boldsymbol{\xi}| \quad . \quad (3.147)$$

The last term corresponds to a charge  $\theta^* = \sqrt{2}$  object at  $\boldsymbol{\xi}$  interacting via the logarithmic potential  $v(\mathbf{r}) = -\theta\theta^* \ln |\mathbf{r}_i - \boldsymbol{\xi}|$  with charge  $e^* = q\sqrt{2}$  particles at each  $\mathbf{r}_i$ . Since any charged impurities are completely screened in the plasma phase of the 2DOCP, there will be a deficit of  $\theta^*/\theta = q^{-1}$  particles localized about the position  $\boldsymbol{\xi}$  on the scale of the Debye length, which is  $\lambda_D = 1/Q = a/\sqrt{2\Gamma}$  where  $a = (\pi n)^{-1/2} = \sqrt{2q}\ell$  is the ion disk radius of the 2DOCP and  $\Gamma = 2q$  is the plasma parameter. Thus  $\lambda_D = \ell/\sqrt{2}$  is the screening length.

The quasielectron wavefunctions are somewhat more complicated. Adiabatically inserting flux  $\phi_0$  parallel to the applied field has the effect of  $|m\rangle \rightarrow |m+1\rangle$  on the single particle angular momentum basis states with origin at  $\boldsymbol{\xi}$ . If we adiabatically insert flux  $\phi_0$  antiparallel to the applied field, it stands to reason that  $|m\rangle \rightarrow |m-1\rangle$ , in which case whither  $|m=0\rangle$ ? Laughlin's proposed quasielectron wavefunction is given by

$$\Psi_q^{\text{QE}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\eta}) = \prod_{i=1}^N \exp(-z_i \bar{z}_i / 4\ell^2) \prod_{l=1}^N \left( 2\ell^2 \frac{\partial}{\partial z_l} - \bar{\eta} \right) \prod_{j>k}^N (z_j - z_k)^q \quad , \quad (3.148)$$

where  $\bar{\eta} = \eta_x - i\eta_y$ . This form is inspired by the Girvin-Jach substitution  $\bar{z} \rightarrow 2\ell^2\partial$ , and results in a localized defect of increased number density  $q^{-1}$  within  $\lambda_D$  of the quasielectron at  $\boldsymbol{\eta}$ . Note that dividing  $\Psi_q$  by the product  $\prod_j (z_j - \eta)$  results in a nonanalyticity and the resulting state is no longer in the LLL. Morf and Halperin (1986) calculated the quasihole and quasielectron energies at  $\nu = \frac{1}{3}$  via Monte Carlo, obtaining

$$\tilde{\epsilon}_{\text{MH}}^{\text{QH}} = (0.0268 \pm 0.0033) \frac{e^2}{\epsilon\ell} \quad , \quad \tilde{\epsilon}_{\text{MH}}^{\text{QP}} = (0.073 \pm 0.008) \frac{e^2}{\epsilon\ell} \quad . \quad (3.149)$$

It should be stressed that these are so-called "proper quasiparticle energies", representing the change in energy of the system at fixed  $N$  and  $R$  when the total magnetic flux is changed by a single Dirac quantum. The gap  $E_g \equiv \tilde{\epsilon}^{\text{QH}} + \tilde{\epsilon}^{\text{QP}} = (0.099 \pm 0.009) e^2/\epsilon\ell$  compares well with computations of Haldane and Rezayi (1985), who obtained  $E_g = (0.105 \pm 0.005) e^2/\epsilon\ell$ , extrapolated from finite size exact diagonalization results.

A few words on how the sausage is made: Morf and Halperin worked with the Laughlin ground state and quasihole/quasielectron wavefunctions, evaluating the Coulomb energy for the system on the plane. Recall that the radius  $R$  of a Laughlin droplet at filling  $\nu = q^{-1}$  is given by  $R = \sqrt{2qN}\ell$ . When a quasihole is created at the origin at fixed  $N$ , a small bubble is blown in the droplet, as each single particle angular momentum state is effectively shifted from  $|m\rangle$  to  $|m+1\rangle$  (with some consequential changes in normalization<sup>36</sup>), hence the outer radius of the

<sup>36</sup>Note that since  $\varphi_m(\mathbf{r}) = (2\pi\ell^2 m!)^{-1/2} (z/\sqrt{2}\ell)^m \exp(-|z|^2/4\ell^2)$  that  $z\varphi_m(\mathbf{r}) = \sqrt{2(m+1)}\ell \varphi_{m+1}(\mathbf{r})$ , and the  $j$ -dependence means that this multiplication is not precisely equivalent to adiabatically increasing the  $j$  quantum number.

droplet increases slightly. Since the largest single particle angular momentum is  $m = q(N - 1)$ , in the state with a single quasihole at the origin the outer droplet radius shifts according to

$$R = \sqrt{2q(N - 1)} \ell \quad \rightarrow \quad R = \sqrt{2[q(N - 1) + 1]} \ell \quad . \quad (3.150)$$

In order to keep the droplet radius fixed at the original value of  $R = \sqrt{2q(N - 1)} \ell$ , Morf and Halperin adjust the magnetic field, increasing it by a factor  $1 + [q(N - 1)]^{-1}$ , which has the effect of decreasing  $\ell$  by just the right amount to prevent the outer radius of the droplet from shifting. Similarly for the quasielectron states, the field is reduced by a factor  $1 - [q(N - 1)]^{-1}$ .

MacDonald and Girvin<sup>37</sup> proposed the trial quasiparticle states

$$\begin{aligned} |\Psi_q^{\text{QH}}\rangle &= \hat{U} |\Psi_q\rangle \\ |\Psi_q^{\text{QE}}\rangle &= (1 - \nu)^{-1/2} \hat{D} |\Psi_q\rangle \quad , \end{aligned} \quad (3.151)$$

where  $\hat{U} |m\rangle = |m + 1\rangle$  and  $\hat{D} |m\rangle = (1 - \delta_{m,0}) |m - 1\rangle$  raise and lower the single particle angular momentum quantum number of each electron, respectively<sup>38</sup>. Their quasiparticle energies were somewhat larger than those of Morf and Halperin:

$$\tilde{\varepsilon}_{\text{GM}}^{\text{QH}} = (0.0287 \pm 0.001) \frac{e^2}{\epsilon \ell} \quad , \quad \tilde{\varepsilon}_{\text{GM}}^{\text{QP}} = (0.085 \pm 0.002) \frac{e^2}{\epsilon \ell} \quad . \quad (3.152)$$

The energy gap  $E_g$  is related to the discontinuity in the chemical potential,

$$\Delta\mu = \left. \frac{\partial E}{\partial N} \right|_{\nu^+} - \left. \frac{\partial E}{\partial N} \right|_{\nu^-} = \frac{\tilde{\varepsilon}^{\text{QE}} + \tilde{\varepsilon}^{\text{QH}}}{|e^*/e|} = qE_g \quad , \quad (3.153)$$

where  $e^* = \pm e/q$ . A pristine system should exhibit thermally activated resistivity according to  $\rho_{xx}(T) \propto \exp(-E_g/2k_B T)$ .

### Adiabatic calculation of fractional quasiparticle charge and statistics

Consider a Hamiltonian  $H(\lambda)$  where  $\lambda = \{\lambda_1, \dots, \lambda_K\}$  are a set of parameters. Recall the definition of the geometric (Berry) connection

$$\mathcal{A}(\lambda) = i \langle \Psi(\lambda) | \nabla_\lambda | \Psi(\lambda) \rangle \quad (3.154)$$

and the geometric phase

$$\gamma(\mathcal{C}) = \oint_{\mathcal{C}} d\lambda \cdot \mathcal{A}(\lambda) \quad . \quad (3.155)$$

<sup>37</sup>A. H. MacDonald and S. M. Girvin, *Phys. Rev. B* **33**, 4414 (1986).

<sup>38</sup>The  $(1 - \nu)^{-1/2}$  factor in the quasielectron wavefunction is necessary to preserve normalization.

Recall also the Laughlin quasihole and quasielectron wavefunctions,

$$\begin{aligned}\Psi_q^{\text{QH}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\xi}) &= \mathcal{M}(\boldsymbol{\xi}) \prod_{l=1}^N (z_l - \xi) \prod_{j>k}^N (z_j - z_k)^q \prod_{i=1}^N \exp(-z_i \bar{z}_i / 4\ell^2) \\ \Psi_q^{\text{QE}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\eta}) &= \mathcal{N}(\boldsymbol{\eta}) \prod_{i=1}^N \exp(-z_i \bar{z}_i / 4\ell^2) \prod_{l=1}^N \left( 2\ell^2 \frac{\partial}{\partial z_l} - \bar{\eta} \right) \prod_{j>k}^N (z_j - z_k)^q \quad ,\end{aligned}\tag{3.156}$$

where  $\mathcal{M}(\boldsymbol{\xi})$  and  $\mathcal{N}(\boldsymbol{\eta})$  are normalization constants, which without loss of generality may be assumed to be real functions of their arguments. Treating the quasihole and quasielectron coordinates as adiabatic parameters, we may compute the geometric phase accrued as they each traverse a closed loop in two-dimensional space. Taking the differential of the quasihole wavefunction, we have

$$d\Psi^{\text{QH}}(\boldsymbol{\xi}) = \left[ d \ln \mathcal{M}(\boldsymbol{\xi}) + \sum_{i=1}^N d \ln(z_i - \xi) \right] \Psi^{\text{QH}}(\boldsymbol{\xi}) \quad ,\tag{3.157}$$

where for notational simplicity we write  $\Psi_q^{\text{QH}}(\mathbf{r}_1, \dots, \mathbf{r}_N; \boldsymbol{\xi}) \equiv \Psi^{\text{QH}}[\boldsymbol{\xi}]$ . With  $\boldsymbol{\lambda} = \boldsymbol{\lambda}(t)$  along the path  $\mathcal{C}$ , we compute the differential  $d\gamma = \dot{\gamma} dt$ , and find

$$d\gamma^{\text{QH}} = i d \ln \mathcal{M}(\boldsymbol{\xi}) + i \langle \Psi^{\text{QH}}(\boldsymbol{\xi}) | \sum_{i=1}^N d \ln(z_i - \xi) | \Psi^{\text{QH}}(\boldsymbol{\xi}) \rangle \quad .\tag{3.158}$$

The number density in the quasihole state is

$$n_{\boldsymbol{\xi}}^{\text{QH}}(\mathbf{r}) = \langle \Psi^{\text{QH}}(\boldsymbol{\xi}) | \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) | \Psi^{\text{QH}}(\boldsymbol{\xi}) \rangle \quad .\tag{3.159}$$

Since the normalization  $\mathcal{M}(\boldsymbol{\xi})$  is a real single-valued function of its argument, it cannot contribute to the integral for  $\gamma(\mathcal{C})$ , since it is the same at the initial and end points of any closed path. We now write  $n_{\boldsymbol{\xi}}^{\text{QH}}(\mathbf{r}) = n + \delta n_{\boldsymbol{\xi}}(\mathbf{r})$ , where  $n = \nu / 2\pi\ell^2$  is the density in the Laughlin ground state with  $\nu = q^{-1}$  and  $\delta n_{\boldsymbol{\xi}}(\mathbf{r})$  is concentrated about the location  $\boldsymbol{\xi}$  of the quasihole defect. From the plasma analogy, we expect that in the thermodynamic limit that  $\delta n_{\boldsymbol{\xi}}(\mathbf{r})$  should be a function of  $|\mathbf{r} - \boldsymbol{\xi}|$  decaying on the scale of the Debye screening length  $\ell/\sqrt{2}$ . Therefore we have

$$\gamma^{\text{QH}}(\mathcal{C}) = i \oint_{|\boldsymbol{\xi}|=R} d\boldsymbol{\xi} \int d^2r \frac{n + \delta n_{\boldsymbol{\xi}}^{\text{QH}}(\mathbf{r})}{\boldsymbol{\xi} - z}\tag{3.160}$$

Integrating  $\boldsymbol{\xi}(t)$  over a circle of radius  $R$ , we have

$$\oint_{|\boldsymbol{\xi}|=R} \frac{d\boldsymbol{\xi}}{\boldsymbol{\xi} - z} = 2\pi i \Theta(R - |z|) \quad ,\tag{3.161}$$

where  $\Theta(x)$  is a step function. Thus the background density term in  $n_{\xi}^{\text{QH}}(\mathbf{r})$  yields a contribution to the total Berry phase of

$$\gamma_0^{\text{QH}}(\mathcal{C}) = i \int d^2r \, 2\pi i n \Theta(R - r) = -2\pi \langle N \rangle_{\mathcal{C}} = -2\pi\nu \Phi(\mathcal{C})/\phi_0 \quad , \quad (3.162)$$

where  $\Phi(\mathcal{C}) = \pi R^2 B = \phi_0 R^2/2\ell^2$  is the total magnetic flux enclosed by the loop  $\mathcal{C}$ . This is consistent with a Bohm-Aharonov phase of a charge  $e_{\text{QH}}^* = \nu e$  quasihole. The  $\delta n_{\xi}^{\text{QH}}(\mathbf{r})$  term integrates to zero if  $\delta n_{\xi}^{\text{QH}}(\mathbf{r}) = \delta n(|\mathbf{r} - \xi|)$  is a rotationally symmetric function in the difference  $\mathbf{r} - \xi$ . This will be exponentially accurate if  $\xi$  lies in the bulk of the Laughlin droplet, far from the edge. Thus the full Berry phase is  $\gamma^{\text{QH}}(\mathcal{C}) = -2\pi\nu \Phi(\mathcal{C})/\phi_0$ .

To determine the statistics of the quasihole, we consider the state with two quasiholes:

$$\Psi^{\text{QH}}(\xi, \xi') = \mathcal{M}(\xi, \xi') \prod_{l=1}^N (z_l - \xi)(z_l - \xi') \prod_{j>k}^N (z_j - z_k)^q \prod_{i=1}^N \exp(-z_i \bar{z}_i/4\ell^2) \quad . \quad (3.163)$$

We now carry out the same adiabatic calculation of the Berry phase by taking  $\xi$  around a circle of radius  $R$ , with  $\xi'$  held fixed. If  $\xi'$  lies outside the circle  $|\xi| = R$  by a distance  $d$  which is greater than just a few magnetic lengths, then the above analysis is unchanged, and the phase is  $\gamma^{\text{QH}}(\mathcal{C}) = -2\pi\nu \Phi(\mathcal{C})/\phi_0$ . If, on the other hand,  $\xi'$  lies inside the loop, then there is a deficit in  $\langle N \rangle_{\mathcal{C}}$  of  $(-\nu)$ , and the accrued phase is  $\gamma'(\mathcal{C}) = \gamma^{\text{QH}}(\mathcal{C}) + 2\pi\nu$ , which says that when one quasihole winds around another, the wavefunction accumulates an extra *statistical phase*  $\Delta\gamma(\mathcal{C}) = 2\pi\nu$ . For exchange, one need only traverse half a circle, *i.e.* the relative angle of  $\xi$  and  $\xi'$  changes by  $\pi$ , leading to the *statistical angle*  $\theta = \pi\nu$  for exchange of quasiholes. At  $\nu = 1$ , the statistical angle is  $\theta_{\text{QH}} = \pi$ , corresponding to Fermi statistics, but for  $\nu = q^{-1}$ , the statistical angle is  $\theta_{\text{QH}} = \pi/q$ , corresponding to *fractional statistics*.

Fractional statistics for particles in  $d = 2$  was first discussed by Leinaas and Myrheim<sup>39</sup>. An interpretation of particles obeying fractional statistics in terms of charge-flux composites was first discussed by Wilczek<sup>40</sup>, who called such particles *anyons*, because they could exhibit any type of exchange statistics. It was Halperin who first suggested that FQH quasiparticles obey fractional statistics, and argued that condensation of gases of these anyonic quasiparticles gave rise to a hierarchy of new FQH states at fillings  $\nu = p/q$  with  $p \neq 1$  and thus outside the principal Laughlin sequence of states. The adiabatic calculation of quasiparticle charge was first carried out by Arovas, Schrieffer, and Wilczek<sup>41</sup> (ASW).

The adiabatic calculation for quasielectrons is a tricky affair. The adiabatic argument of ASW analyzed

$$d\gamma^{\text{QE}} = i d \ln \mathcal{N}(\boldsymbol{\eta}) + i \langle \Psi^{\text{QE}}(\boldsymbol{\eta}) | e^{-G(Z)} \left( \sum_{i=1}^N d \ln (2\ell^2 \partial_i - \bar{\eta}) \right) e^{G(Z)} | \Psi^{\text{QE}}(\boldsymbol{\eta}) \rangle \quad , \quad (3.164)$$

<sup>39</sup>J. M. Leinaas and J. Myrheim, *Il Nuovo Cimento* **37**, 1, 1977.

<sup>40</sup>F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982).

<sup>41</sup>D. P. Arovas, J. R. Schrieffer, and F. Wilczek, *Phys. Rev. Lett.* **53**, 722 (1984).



where  $G(Z) = \frac{1}{4\ell^2} \sum_{j=1}^N |z_j|^2$ , appealing to the Girvin-Jach replacement  $2\ell^2\partial_i \leftrightarrow \bar{z}_i$ . This would appear to give

$$\gamma^{\text{QE}}(\mathcal{C}) = i \oint_{|\eta|=R} d\bar{\eta} \int d^2r \frac{n + \delta n_{\eta}^{\text{QE}}(\mathbf{r})}{\bar{\eta} - \bar{z}} \quad . \quad (3.165)$$

The  $\bar{\eta}$  integral is taken clockwise around the path  $\mathcal{C}$ , hence this calculation yields the opposite value of the charge, *i.e.*  $e_{\text{QE}}^* = -\nu e$  for the quasielectron. Since the quasielectron thus represents an *increase* in electron density at  $\mathbf{r} = \boldsymbol{\eta}$ , there are two cancelling factors of  $(-1)$  in the calculation, and the statistical angle for quasielectrons is also  $\theta_{\text{QE}} = \pi/q$ . Numerical calculations using Laughlin wavefunctions at  $\nu = \frac{1}{3}$  for up to 200 particles were effected by Kjønberg and Myrheim<sup>42</sup>, who found good convergence for the adiabatic quasihole charge and statistics, but surprising poor convergence for the quasielectron values (especially the quasielectron statistical angle). These authors also found that boundary effects complicate the glib application of the Girvin-Jach replacement in the ASW calculations for the quasielectron.

The adiabatic method for determining effective quasiparticle charge and statistics is based on the adiabatic effective Lagrangian prescription of Moody, Shapere, and Wilczek<sup>43</sup>, which goes as follows. Let  $\Psi[\boldsymbol{\lambda}]$  be an adiabatic wavefunction and  $\boldsymbol{\lambda}$  the adiabatic parameters. The adiabatic Lagrangian  $L(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}})$  is then given by

$$L(\boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}) = - \langle \Psi[\boldsymbol{\lambda}] | \left\{ i \frac{d}{dt} + H(\boldsymbol{\lambda}) \right\} | \Psi[\boldsymbol{\lambda}] \rangle = -\mathcal{A}(\boldsymbol{\lambda}) \cdot \frac{d\boldsymbol{\lambda}}{dt} - E(\boldsymbol{\lambda}) \quad , \quad (3.166)$$

where  $\mathcal{A}(\boldsymbol{\lambda})$  is the Berry connection from §1.7.1. In the Born-Oppenheimer approach, where the nuclear coordinates  $\{\mathbf{R}_1, \dots, \mathbf{R}_{N_{\text{nuc}}}\}$  are regarded as adiabatically varying so far as the electrons are concerned, the total effective Lagrangian is

$$L(\{\mathbf{R}_j\}, \{\dot{\mathbf{R}}_j\}) = \sum_{i=1}^{N_{\text{nuc}}} \frac{1}{2} M \dot{\mathbf{R}}_i^2 - \sum_{i=1}^{N_{\text{nuc}}} \mathcal{A}_i(\{\mathbf{R}_j\}) \cdot \frac{d\mathbf{R}_i}{dt} - E_{\text{elec}}(\{\mathbf{R}_j\}) - E_{\text{nuc}}(\{\mathbf{R}_j\}) \quad . \quad (3.167)$$

### 3.3.7 Excitons

An exciton is a neutral entity formed from a quasielectron-quasihole pair bound by their mutually attractive Coulomb force. A callow description consists of charges  $\pm e^* = \pm e/q$  located at positions  $\mathbf{r}_{1,2}$ . One then defines center-of-mass and relative coordinates  $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$  and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ . Within the LLL, these objects have no inertial mass. If we assign each a mass  $m$ , the Lagrangian of the exciton system in the symmetric gauge becomes

$$L = m\dot{\mathbf{R}}^2 + \frac{1}{4}m\dot{\mathbf{r}}^2 - \frac{e^*B}{2c} \hat{\mathbf{z}} \cdot (\mathbf{R} \times \dot{\mathbf{r}} + \mathbf{r} \times \dot{\mathbf{R}}) - v(\mathbf{r}) \quad , \quad (3.168)$$

<sup>42</sup>H. Kjønberg and J. Myrheim, *Int. Jour. Mod. Phys. A* **14**, 537, 1999.

<sup>43</sup>J. Moody, A. Shapere, and F. Wilczek, *Adiabatic Effective Lagrangians*, in *Geometric Phases in Physics* F. Wilczek and A. Shapere, eds. (World Scientific, 1989).

where  $v(\mathbf{r}) = -e^{*2}/\epsilon r$  is the Coulomb interaction<sup>44</sup>. Thus

$$\begin{aligned} \mathbf{P} &= \frac{\partial L}{\partial \dot{\mathbf{R}}} = 2m\dot{\mathbf{R}} + \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{r} \\ \mathbf{p} &= \frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{1}{2}m\dot{\mathbf{r}} - \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{R} \quad . \end{aligned} \quad (3.169)$$

The equations of motion are

$$\begin{aligned} 2m\ddot{\mathbf{R}} + \frac{e^*B}{c} \hat{\mathbf{z}} \times \dot{\mathbf{r}} &= 0 \\ \frac{1}{2}m\ddot{\mathbf{r}} - \frac{e^*B}{c} \hat{\mathbf{z}} \times \dot{\mathbf{R}} &= -\nabla v(\mathbf{r}) \quad , \end{aligned} \quad (3.170)$$

and therefore

$$\frac{d}{dt} \left( 2m\dot{\mathbf{R}} + \frac{e^*B}{c} \hat{\mathbf{z}} \times \mathbf{r} \right) = 0 \quad , \quad (3.171)$$

Averaging over the fast cyclotron motion, we obtain

$$\langle \dot{\mathbf{R}} \rangle = -\frac{c}{e^*B} \hat{\mathbf{z}} \times \nabla \tilde{v}(\mathbf{r}) \quad (3.172)$$

where  $\tilde{v}(\mathbf{r})$  is an averaged potential as in §1.1.4 and  $\langle \dot{\mathbf{r}} \rangle = 0$ . Thus, the component charges of the exciton maintain their relative separation  $\mathbf{r}$  and drift with speed  $c v'(\mathbf{r})/e^*B$  in a direction perpendicular to  $\mathbf{r}$ .

The Hamiltonian of the exciton system is

$$H = \frac{1}{4m} \left( \mathbf{P} - \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{r} \right)^2 + \frac{1}{m} \left( \mathbf{p} - \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{R} \right)^2 + v(\mathbf{r}) \quad . \quad (3.173)$$

where  $\mathbf{P}$  and  $\mathbf{p}$  are the CM and relative coordinate canonical momenta, respectively. One may now define

$$\mathbf{G} = \sqrt{2} \left( \mathbf{p} - \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{R} \right) \quad , \quad \mathbf{A} = \frac{1}{\sqrt{2}} \left( \frac{c}{e^*B} \hat{\mathbf{z}} \times \mathbf{P} + \frac{1}{2} \mathbf{r} \right) \quad (3.174)$$

and

$$\mathbf{g} = \mathbf{P} + \frac{e^*B}{2c} \hat{\mathbf{z}} \times \mathbf{r} \quad , \quad \boldsymbol{\lambda} = -\frac{c}{e^*B} \hat{\mathbf{z}} \times \mathbf{p} + \frac{1}{2} \mathbf{R} \quad , \quad (3.175)$$

which satisfy

$$[G_\alpha, A_\beta] = -i\hbar\delta_{\alpha\beta} \quad , \quad [g_\alpha, \lambda_\beta] = -i\hbar\delta_{\alpha\beta} \quad , \quad (3.176)$$

which no other nonzero commutators. Then

$$H = \frac{\mathbf{G}^2}{2m} + \frac{1}{2}m\omega_c^2 \mathbf{A}^2 + v \left( \sqrt{2} \mathbf{A} + \frac{c}{e^*B} \mathbf{g} \times \hat{\mathbf{z}} \right) \quad , \quad (3.177)$$

<sup>44</sup>If  $r \lesssim \ell$ , the LLL projection blunts the  $1/r$  divergence of the Coulomb interaction, as computed in §3.2.2.

where  $\omega_c = e^*B/mc$ . Thus  $[H, \mathbf{g}] = 0$  and we may specify the momentum  $\mathbf{g}$ . Note that  $\mathbf{g} = 2m\dot{\mathbf{R}} + \frac{e^*B}{c} \hat{\mathbf{z}} \times \mathbf{r}$  was classically conserved. Recall that in charged systems at most one component of the total momentum could be fixed, e.g. in the Landau strip basis. Because the exciton is neutral, there is a conserved momentum. If we project the  $\mathbf{G}$  and  $\mathbf{A}$  degrees of freedom onto their lowest harmonic oscillator state, then we have  $\langle \mathbf{A} \rangle = 0$  and  $\langle \mathbf{A}^2 \rangle = \hbar c/e^*B$ , and for large values of  $g$  we have that the energy is

$$\Delta_{\text{EX}}(\mathbf{g}) = \tilde{\varepsilon}^{\text{QE}} + \tilde{\varepsilon}^{\text{QH}} + v \left( \frac{c}{e^*B} \mathbf{g} \times \hat{\mathbf{z}} \right) . \quad (3.178)$$

Note  $c/e^*B = q\ell^2/\hbar$  at filling  $\nu = q^{-1}$ .

### 3.3.8 Collective excitations

One might expect that there exist excited states of the FQH ground states corresponding to long-wavelength density oscillations, *i.e.* phonons. Whenever the ground state  $\Psi$  is of uniform density, it can often be argued on general grounds that such excitations are adequately represented by the *Ansatz*

$$|\Phi_{\mathbf{k}}\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N e^{i\mathbf{k}\cdot\mathbf{r}_i} |\Psi\rangle = N^{-1/2} \rho_{\mathbf{k}} |\Psi\rangle , \quad (3.179)$$

where

$$\rho_{\mathbf{k}} = \int d^2r e^{i\mathbf{k}\cdot\mathbf{r}} \overbrace{\sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i)}^{n(\mathbf{r})} = \sum_{i=1}^N e^{i\mathbf{k}\cdot\mathbf{r}_i} \quad (3.180)$$

is the Fourier transform of the density. For  $\mathbf{k} \rightarrow 0$ , the state  $|\Phi_{\mathbf{k}}\rangle$  will feature the same short-ranged correlations which favor the ground state  $|\Psi\rangle$ , yet  $\langle \Psi | \Phi_{\mathbf{k}} \rangle = (2\pi)^2 N^{-1/2} n \delta(\mathbf{k})$  which vanishes for  $\mathbf{k} \neq 0$ . Thus,  $|\Phi_{\mathbf{k}}\rangle$  serves as a trial state whose expected energy is a rigorous upper bound to the exact lowest excitation energy at wavevector  $\mathbf{k}$ .

The excitation energy of the state  $|\Phi_{\mathbf{k}}\rangle$  is given by

$$\Delta(\mathbf{k}) = \frac{\langle \Phi_{\mathbf{k}} | H - E_0 | \Phi_{\mathbf{k}} \rangle}{\langle \Phi_{\mathbf{k}} | \Phi_{\mathbf{k}} \rangle} = \frac{f(\mathbf{k})}{s(\mathbf{k})} , \quad (3.181)$$

where

$$\begin{aligned} s(\mathbf{k}) &= \frac{1}{N} \langle \Psi | \rho_{\mathbf{k}}^\dagger \rho_{\mathbf{k}} | \Psi \rangle \\ f(\mathbf{k}) &= \frac{1}{2N} \langle \Psi | [\rho_{\mathbf{k}}^\dagger, [H, \rho_{\mathbf{k}}]] | \Psi \rangle . \end{aligned} \quad (3.182)$$

The quantity  $s(\mathbf{k})$  is the static structure factor, and  $f(\mathbf{k})$  is known as the oscillator strength. When the Hamiltonian is of the form

$$H = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} + \sum_{j<k} v(\mathbf{r}_j - \mathbf{r}_k) \quad , \quad (3.183)$$

the oscillator strength is given by

$$f(\mathbf{k}) = \frac{\hbar^2 \mathbf{k}^2}{2m} \quad , \quad (3.184)$$

independent of the potential  $v$ . This is known as the  $f$ -sum rule. It is also valid in a uniform magnetic field in which case  $\mathbf{p}_i$  is replaced by  $\boldsymbol{\pi}_i = \mathbf{p}_i + \frac{e}{c} \mathbf{A}(\mathbf{r}_i)$  for electrons. The static structure factor is

$$s(\mathbf{k}) = 1 + n \int d^2r [g(\mathbf{r}) - 1] e^{-i\mathbf{k}\cdot\mathbf{r}} + (2\pi)^2 n \delta(\mathbf{k}) \quad . \quad (3.185)$$

where the pair distribution function  $g(\mathbf{r})$  is defined in Eqn. 3.112. Note that the last term above is not included in the definition of in Eqn. 3.113. This is because our definition in this section includes the diagonal  $i = j$  term in what is the sum in Eqn. 3.112. This is a matter of convention and has no consequences for the following developments.

This approximation was originally employed by Feynman in deducing the phonon-roton spectrum of superfluid  $^4\text{He}$ . It is really quite remarkable, for it allows one to represent a collective mode excitation spectrum solely in terms of static correlations in the ground state. In  $^4\text{He}$ ,  $s(\mathbf{k})$  rises linearly<sup>45</sup> at small  $k$  and peaks at  $k \equiv k_R \propto n^{1/2}$ , where  $n$  is the ground state number density. As a result,  $\Delta(\mathbf{k})$  exhibits a local minimum at  $k_R$ , called the *roton minimum*.

The dynamic structure factor (dsf)  $S(\mathbf{k}, \omega)$  is given by

$$S(\mathbf{k}, \omega) = \frac{1}{N} \sum_j |\langle \Psi_j | \rho_{\mathbf{k}} | \Psi_0 \rangle|^2 \delta(\omega - \omega_j) \quad (3.186)$$

where  $\hbar\omega_j \equiv E_j - E_0$ . Here  $|\Psi_0\rangle$  is the ground state and the sum is over the entire many-body spectrum of states  $|\Psi_j\rangle$ . We then have

$$\begin{aligned} s(\mathbf{k}) &= \int_0^\infty d\omega S(\mathbf{k}, \omega) \\ f(\mathbf{k}) &= \int_0^\infty d\omega S(\mathbf{k}, \omega) \hbar\omega \quad , \end{aligned} \quad (3.187)$$

and he have that  $\Delta(\mathbf{k})$  is the first moment of the dsf. The expression for  $\Delta(\mathbf{k})$  is therefore exact if  $S(\mathbf{k}, \omega)$  as a function of  $\omega$  for each  $\mathbf{k}$  has no variance, *i.e.* if  $S(\mathbf{k}, \omega) = S_{\text{SMA}}(\mathbf{k}, \omega)$ , where

$$S_{\text{SMA}}(\mathbf{k}, \omega) = s(\mathbf{k}) \delta(\omega - \hbar^{-1} \Delta_{\text{SMA}}(\mathbf{k})) \quad . \quad (3.188)$$

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<sup>45</sup>See the Appendix §3.7.

Thus,  $\Delta(\mathbf{k})$  is exact if a *single mode* saturates all the oscillator strength at wavevector  $\mathbf{k}$ . For this reason, this procedure is known as the *single mode approximation* or SMA. Precisely, *the SMA energy*  $\Delta_{\text{SMA}}(\mathbf{k})$  *is the exact first moment of the dynamic structure factor at wavevector*  $\mathbf{k}$ .

If one naïvely applies the SMA to the Laughlin ground state, a disappointing result is found. Invoking the result of Eqn. 3.123 for  $\mathbf{k} \neq 0$ , and with the  $f$ -sum rule fixing  $f(\mathbf{k})$ , we obtain

$$\Delta_{\text{SMA}}(\mathbf{k}) = \hbar\omega_c \left[ 1 - \frac{1}{4}(q-1)k^2\ell^2 + \mathcal{O}(k^4\ell^4) \right] , \quad (3.189)$$

*i.e.* the excitation energy as  $\mathbf{k} \rightarrow 0$  is independent of filling fraction and the interaction potential, and lies outside the lowest Landau level, at energy  $\hbar\omega_c$ . The problem here is that the density operator  $\rho_{\mathbf{k}}$  creates a mixture of inter-LL and intra-LL excitations, and for small  $\mathbf{k}$  almost all of the oscillator strength is saturated by the inter-LL piece. This is known as *Kohn's theorem*<sup>46</sup>, which says that the cyclotron resonance mode in systems without disorder saturates the oscillator strength up to terms of order  $k^2$  as  $k \rightarrow 0$ . For any interaction  $v_{ij} = v(\mathbf{r}_i - \mathbf{r}_j)$ , the  $N$ -electron Hamiltonian

$$H = \sum_{i=1}^N \frac{\pi_i^2}{2m^*} + \sum_{i<j} v(\mathbf{r}_i - \mathbf{r}_j) \quad (3.190)$$

satisfies  $[H, \Pi^\alpha] = i\hbar\omega_c \epsilon_{\alpha\beta} \Pi^\beta$ , with  $\Pi^\alpha = \sum_i \pi_i^\alpha$ . In particular, note that  $[V, \Pi^\alpha] = 0$ . Thus,  $\Pi = \Pi^x + i\Pi^y$  is an eigenoperator of  $H$ , satisfying  $[H, \Pi] = \hbar\omega_c \Pi$ . This means that if  $|\Psi_0\rangle$  is the exact ground state of  $H$ , with  $H|\Psi_0\rangle = E_0|\Psi_0\rangle$ , then defining

$$|\Psi_1\rangle \equiv \frac{\Pi|\Psi_0\rangle}{\langle\Psi_0|\Pi^\dagger\Pi|\Psi_0\rangle^{1/2}} , \quad (3.191)$$

we have  $H|\Psi_1\rangle = (E_0 + \hbar\omega_c)|\Psi_1\rangle$ , *i.e.*  $|\Psi_1\rangle$  is an exact excited state with excitation energy  $\hbar\omega_c$  above the ground state, *independent of the interaction potential*. This is the Kohn mode.

To get at the intra-LL collective mode, Girvin, MacDonald, and Platzman<sup>47</sup> realized that what was needed is to project the density operator  $\rho_{\mathbf{k}}$  onto the LLL. This is easily accomplished:

$$\bar{\rho}_{\mathbf{k}} = \Pi_0 \sum_{i=1}^N e^{ik\bar{z}_i/2} e^{i\bar{k}z_i/2} \Pi_0 = \sum_{i=1}^N e^{ik\ell b_i/\sqrt{2}} e^{i\bar{k}\ell b_i^\dagger/\sqrt{2}} , \quad (3.192)$$

where we work in the symmetric gauge and invoke Eqn. 1.48, which expresses the complex coordinate  $z$  in terms of the cyclotron and guiding center ladder operators:  $z = \sqrt{2}\ell(a + b^\dagger)$ . Recall also the definition of the single particle magnetic translation operator,

$$t(\mathbf{d}) = e^{(db - \bar{d}b^\dagger)/\sqrt{2}\ell} . \quad (3.193)$$

<sup>46</sup>W. Kohn, *Phys. Rev.* **123**, 142 (1961).

<sup>47</sup>S. M Girvin, A. H. MacDonald, and P. M. Platzman, *Phys. Rev. B* **33**, 2381 (1986).

Thus, we may write

$$\bar{\rho}_{\mathbf{k}} = e^{-\mathbf{k}^2 \ell^2 / 4} \sum_{i=1}^N t_i(\ell^2 \hat{\mathbf{z}} \times \mathbf{k}) \quad , \quad (3.194)$$

where  $t_i(\mathbf{d})$  is the MTO for the  $i^{\text{th}}$  particle. The MTOs satisfy the relations

$$t(\mathbf{a}) t(\mathbf{b}) = e^{i\hat{\mathbf{z}} \cdot \mathbf{a} \times \mathbf{b} / 2\ell^2} t(\mathbf{a} + \mathbf{b}) = e^{i\hat{\mathbf{z}} \cdot \mathbf{a} \times \mathbf{b} / \ell^2} t(\mathbf{b}) t(\mathbf{a}) \quad . \quad (3.195)$$

Thus the projected density operators  $\bar{\rho}_{\mathbf{k}}$  form a closed Lie algebra, *viz.*

$$[\bar{\rho}_{\mathbf{k}}, \bar{\rho}_{\mathbf{k}'}] = (e^{\bar{k}k' \ell^2 / 2} - e^{k\bar{k}' \ell^2 / 2}) \bar{\rho}_{\mathbf{k}+\mathbf{k}'} \quad , \quad (3.196)$$

which is known as the Girvin-MacDonald-Platzman (GMP) algebra<sup>48</sup>. Another useful result may be derived based on these relations:

$$\Pi_0 \rho_{\mathbf{k}} \rho_{\mathbf{k}'} \Pi_0 = \bar{\rho}_{\mathbf{k}} \bar{\rho}_{\mathbf{k}'} + (1 - e^{\bar{k}k' \ell^2 / 2}) \bar{\rho}_{\mathbf{k}+\mathbf{k}'} \quad (3.197)$$

The LLL-projected Hamiltonian, after dropping the constant  $\frac{1}{2} N \hbar \omega_c$  cyclotron energy per particle, may be written

$$\begin{aligned} \bar{H} &= \Pi_0 \sum_{i < j} v(\mathbf{r}_i - \mathbf{r}_j) \Pi_0 = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \hat{v}(\mathbf{k}) \Pi_0 \rho_{\mathbf{k}}^\dagger \rho_{\mathbf{k}} \Pi_0 \\ &= \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \hat{v}(\mathbf{k}) \left( \bar{\rho}_{\mathbf{k}}^\dagger \bar{\rho}_{\mathbf{k}} - N e^{-\mathbf{k}^2 \ell^2 / 2} \right) \quad . \end{aligned} \quad (3.198)$$

Applying a projected version of the SMA, then, GMP defined the state

$$|\Phi_{\mathbf{k}}\rangle = \frac{\bar{\rho}_{\mathbf{k}} |\Psi_0\rangle}{\langle \Psi_0 | \bar{\rho}_{\mathbf{k}}^\dagger \bar{\rho}_{\mathbf{k}} | \Psi_0 \rangle^{1/2}} \quad , \quad (3.199)$$

where  $|\Psi_0\rangle$  is the exact ground state. Treating  $|\Psi_{\mathbf{k}}\rangle$  as a trial state, we have

$$\Delta_{\text{SMA}}(\mathbf{k}) = \langle \Phi_{\mathbf{k}} | \bar{H} - E_0 | \Phi_{\mathbf{k}} \rangle = \frac{\bar{f}(\mathbf{k})}{\bar{s}(\mathbf{k})} \quad , \quad (3.200)$$

where

$$\begin{aligned} s(\mathbf{k}) &= \frac{1}{N} \langle \Psi_0 | \bar{\rho}_{\mathbf{k}}^\dagger \bar{\rho}_{\mathbf{k}} | \Psi_0 \rangle \\ f(\mathbf{k}) &= \frac{1}{2N} \langle \Psi_0 | [\bar{\rho}_{\mathbf{k}}^\dagger, [H, \bar{\rho}_{\mathbf{k}}]] | \Psi_0 \rangle \quad . \end{aligned} \quad (3.201)$$

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<sup>48</sup>Note that  $\rho_{\mathbf{k}}^\dagger = \rho_{-\mathbf{k}}$  and  $\bar{\rho}_{\mathbf{k}}^\dagger = \bar{\rho}_{-\mathbf{k}}$ , and also that  $\rho_{\mathbf{0}} = \bar{\rho}_{\mathbf{0}} = 1$ .

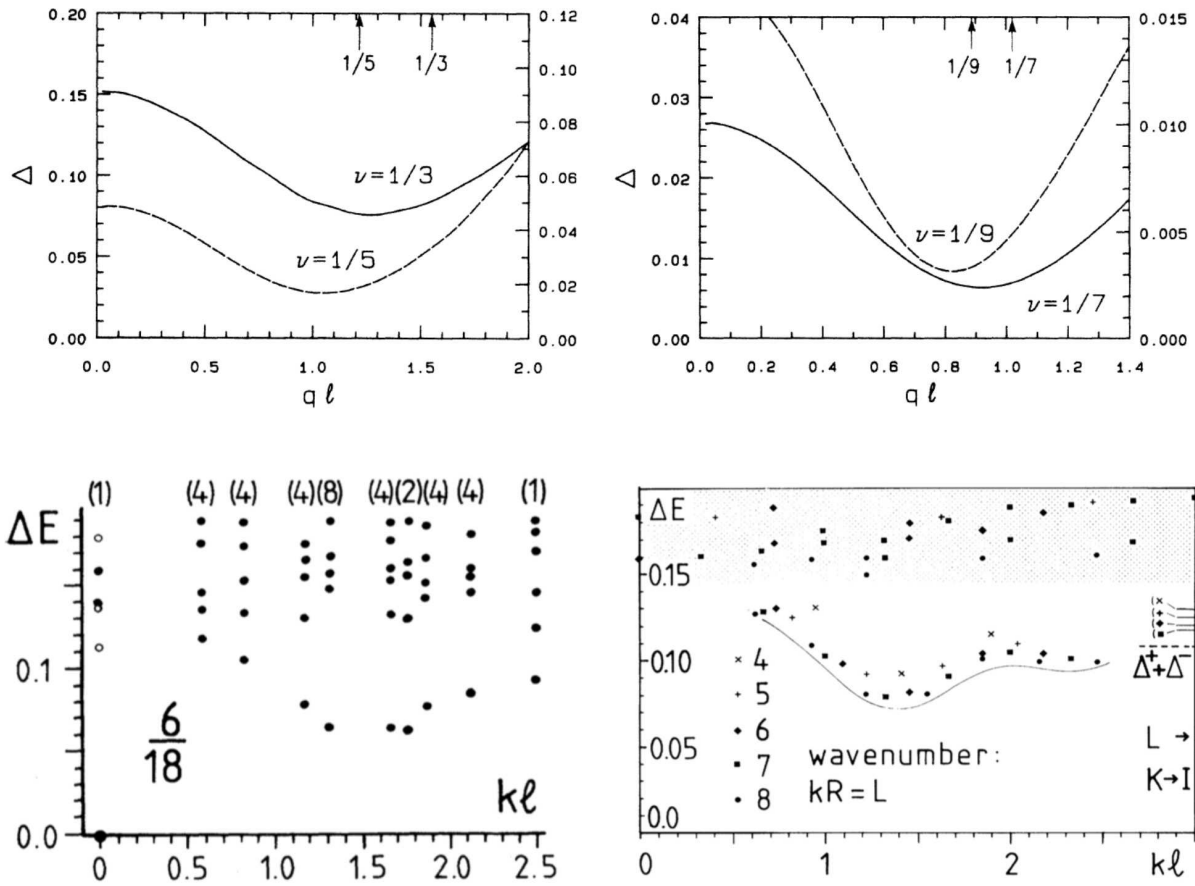


Figure 3.10: The theoretical and numerical dispersions for the magnetophonon-magnetoroton branch. Upper panels: Theoretical predictions of the collective mode dispersion at  $\nu = \frac{1}{3}$ ,  $\frac{1}{5}$ ,  $\frac{1}{7}$ , and  $\frac{1}{9}$  (units of  $e^2/\epsilon\ell$ ) from S. M. Girvin, A. M. MacDonald, and P. M. Platzman, *Phys. Rev. B* **33**, 2481 (1986). The arrows point to locations of the first reciprocal lattice vector of the corresponding Wigner crystal. Lower panels: Numerical computations of the excitation spectra for the Coulomb system at  $\nu = \frac{1}{3}$  on the torus (left) and sphere (right), showing a clear  $k \rightarrow 0$  gap and magnetoroton dip. From F. D. M. Haldane, *Phys. Rev. Lett.* **55**, 2095 (1985) and F. D. M. Haldane and E. H. Rezayi, *Phys. Rev. Lett.* **54**, 237 (1985).

are the projected structure factor and oscillator strength, which are given by

$$\begin{aligned} \bar{s}(\mathbf{k}) &= s(\mathbf{k}) - 1 + e^{-k^2\ell^2/2} \\ \bar{f}(\mathbf{k}) &= \int \frac{d^2p}{(2\pi)^2} \hat{v}(\mathbf{p}) \left( 1 - \cos(\ell^2 \hat{z} \cdot \mathbf{k} \times \mathbf{p}) \right) \left[ \bar{s}(\mathbf{k} + \mathbf{p}) e^{\ell^2 \mathbf{k} \cdot \mathbf{p}} - \bar{s}(\mathbf{p}) e^{-k^2\ell^2/2} \right] \end{aligned} \quad (3.202)$$

Appealing to Eqn. 3.123 and assuming this result, valid for the Laughlin states  $|\Psi_q\rangle$ , is also valid for the exact ground state  $|\Psi_0\rangle$ , we find

$$\bar{s}(\mathbf{k}) = \frac{1}{8}(q-1)k^4\ell^4 + \mathcal{O}(k^6\ell^6) \quad (3.203)$$

Note that the projected structure factor vanishes as  $k^4$  in the long wavelength limit, in contrast to  $s(\mathbf{k})$  itself, which vanishes as  $k^2$ . Now consider the lengthy expression for  $\bar{f}(\mathbf{k})$  in the limit  $k \rightarrow 0$ . The first factor in round brackets is proportional to  $k^2$ , The second factor in round brackets clearly vanishes when  $\mathbf{k} = 0$ , but the linear term in  $\mathbf{k}$  must vanish after integrating over  $\mathbf{p}$  whenever  $v(\mathbf{p})$  and  $\bar{s}(\mathbf{p})$  are isotropic. Thus the second factor in round brackets also vanishes as  $k^4$  in the long wavelength limit. We conclude that the projected SMA results in the prediction of a *gap* in the collective mode spectrum at  $\mathbf{k} = 0$ . Note that when  $q = 1$ , the projected structure factor vanishes to all orders in  $k$ , because the SMA wavefunction itself vanishes (more on this below)!

Both the structure factor  $s(\mathbf{k})$  and its projection  $\bar{s}(\mathbf{k})$  exhibit a peak at the wavevector  $k^* \approx \pi/a$  where  $\pi a^2 n = 1$  with  $n = \nu/2\pi\ell^2$ . Thus  $k^*\ell \approx \pi\sqrt{\nu/2}$ . Although we don't have much intuition about the behavior of  $\bar{f}(\mathbf{k})$ , a natural guess is that the SMA energy  $\Delta_{\text{SMA}}(\mathbf{k}) = \bar{f}(\mathbf{k})/\bar{s}(\mathbf{k})$  should exhibit a dip in the vicinity of  $k^*$ . Indeed this is what was found by GMP, whose results compared quite well with previous numerical studies of the excitation spectrum at  $\nu = \frac{1}{3}$  by Haldane and by Haldane and Rezayi (1985). The  $k \approx 0$  portion of this excitation branch is called the *magnetophonon*, and the  $k \approx k^*$  portion the *magnetoroton*, using terminology borrowed from the study of superfluid  $^4\text{He}$ . A comparison of theoretical predictions and numerical computations for this elementary excitation is shown in Fig. 3.10. Note that over a wide region centered at  $k = k^*$ , the magnetoroton appears as a true elementary excitation, *i.e.* as an isolated mode which contributes a  $\delta$ -function to the dynamical structure factor  $S(\mathbf{k}, \omega)$ . While other states at  $k \approx k^*$  are present at higher energies, they have much weaker oscillator strength, whence the accuracy of the SMA in this application.

GMP also examined the regime  $k\ell \gg 1$ , and found

$$\lim_{k \rightarrow \infty} \Delta_{\text{SMA}}(\mathbf{k}) = \frac{2\varepsilon_{\text{COH}}(\nu)}{1-\nu} \quad , \quad (3.204)$$

where  $\varepsilon_{\text{COH}}(\nu) = \varepsilon(\nu) - \nu\varepsilon(1)$  is the cohesive energy per particle<sup>49</sup>.

Note that while the SMA presumes that the underlying state  $|\Psi_0\rangle$  is the exact ground state, GMP used the Laughlin states  $|\Psi_q\rangle$  in its place. The agreement between the theoretical and numerically computed elementary excitation dispersion  $\Delta_{\text{SMA}}(\mathbf{k})$  in the vicinity of its minimum is due to the accurate short-distance correlations encoded in the Laughlin state, and to the fact that the magnetoroton appears as a sharp collective mode in the excitation spectrum, and that this feature is not destroyed as one proceeds from the truncated pseudopotential Hamiltonian, for which the Laughlin state is exact, to the exact Coulomb ground state.

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<sup>49</sup>The energy per particle is defined to be  $\varepsilon(\nu) = E(\nu)/N$ .



### Charge susceptibility in the SMA

Suppose we add a dynamical perturbation to the Hamiltonian which couples to the density, described by

$$\bar{H}' = - \int d^2r \bar{\rho}(\mathbf{r}) U(\mathbf{r}, t) \quad . \quad (3.205)$$

Then from linear response theory,

$$\langle \delta \bar{\rho}(\mathbf{r}, t) \rangle = \int_{-\infty}^{\infty} dt' \int d^2r' \chi(\mathbf{r} - \mathbf{r}', t - t') U(\mathbf{r}', t') \quad , \quad (3.206)$$

where  $\chi(\mathbf{r} - \mathbf{r}', t - t')$  is the *dynamical susceptibility*, whose Fourier transform is

$$\begin{aligned} \chi(\mathbf{k}, \omega) &= \sum_j |\langle \Psi_0 | \bar{\rho}_{\mathbf{k}} | j \rangle|^2 \left\{ \frac{1}{\omega + \omega_j + i\epsilon} - \frac{1}{\omega - \omega_j + i\epsilon} \right\} \\ &= \int_{-\infty}^{\infty} d\nu S(\mathbf{k}, \nu) \frac{2\nu}{\nu^2 - (\omega + i\epsilon)^2} \quad . \end{aligned} \quad (3.207)$$

where  $\epsilon = 0^+$  is a positive infinitesimal. Thus,

$$\delta \hat{\rho}(\mathbf{k}, \omega) = \chi(\mathbf{k}, \omega) \hat{U}(\mathbf{k}, \omega) \quad . \quad (3.208)$$

Replacing  $S(\mathbf{k}, \nu)$  in Eqn. 3.207 with the SMA result  $\bar{s}(\mathbf{k}) \delta(\hbar\nu - \Delta_{\text{SMA}}(\mathbf{k}))$ , we obtain

$$\chi_{\text{SMA}}(\mathbf{k}, \omega) = \frac{2\bar{s}(\mathbf{k}) \Delta_{\text{SMA}}(\mathbf{k})}{\Delta_{\text{SMA}}^2(\mathbf{k}) - (\hbar\omega + i\epsilon)^2} \quad . \quad (3.209)$$

For static properties, at zero frequency, then, we have

$$\chi_{\text{SMA}}(\mathbf{k}) = \frac{2\bar{s}(\mathbf{k})}{\Delta_{\text{SMA}}(\mathbf{k})} = \frac{2[\bar{s}(\mathbf{k})]^2}{f(\mathbf{k})} \quad . \quad (3.210)$$

Note that  $\chi_{\text{SMA}}(\mathbf{k}) \sim k^4$  in the limit  $k \rightarrow 0$ . As an application, consider the response to an impurity potential  $U(\mathbf{r})$ . The resulting induced number density is  $\langle \delta \hat{\rho}(\mathbf{k}) \rangle = \chi_{\text{SMA}}(\mathbf{k}) \hat{U}(\mathbf{k})$ . The total induced charge is obtained by taking the limit  $\mathbf{k} \rightarrow 0$ ; this vanishes provided  $\hat{U}(\mathbf{k})$  does not diverge as  $k^{-\sigma}$  with  $\sigma \geq 4$ , where perturbation theory surely fails. The change in Coulomb energy of the electron system resulting from the perturbation is given by

$$\delta E_{\text{COUL}} = -\frac{1}{2}n \text{Re} \int \frac{d^2k}{(2\pi)^2} \hat{v}(-\mathbf{k}) \langle \delta \hat{\rho}(\mathbf{k}) \rangle = -\frac{1}{2}n \text{Re} \int \frac{d^2k}{(2\pi)^2} \chi_{\text{SMA}}(\mathbf{k}) \hat{v}(-\mathbf{k}) \hat{U}(\mathbf{k}) \quad (3.211)$$

For a Coulomb impurity of charge  $+Ze$ , we have  $\hat{U}(\mathbf{k}) = +Z\hat{v}(\mathbf{k})$  and the energy shift within the SMA is found to be

$$\delta E_{\text{COUL}} = -\frac{1}{2}Zn \int \frac{d^2k}{(2\pi)^2} \chi_{\text{SMA}}(\mathbf{k}) \left( \frac{2\pi e^2}{\epsilon k} \right)^2 . \quad (3.212)$$

GMP report that evaluation of the above formula yields  $\delta E_{\text{COUL}} = -1.15 Ze^2/\epsilon\ell$  at  $\nu = \frac{1}{3}$  (with  $n = \nu/2\pi\ell^2$  as always), in substantial agreement with numerical calculations from exact diagonalization studies<sup>50</sup>.

### Orthogonality of SMA states of different wavevector

For the translationally invariant Bijl-Feynman wavefunction for  ${}^4\text{He}$ , the density operator  $\rho_{\mathbf{k}}$  changes the many-body momentum of the ground state. That is, if

$$T(\mathbf{d}) = \prod_{i=1}^N t_{0,i}(\mathbf{d}) = \prod_{i=1}^N e^{i\mathbf{p}_i \cdot \mathbf{d}/\hbar} \quad (3.213)$$

is the many-body translation operator, which implements  $\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{d}$  for all  $i \in \{1, \dots, N\}$ , then if the ground state is a state of definite momentum  $\mathbf{P}_0$ , then  $T(\mathbf{d})|\Psi_0\rangle = e^{i\mathbf{P}_0 \cdot \mathbf{d}/\hbar}|\Psi_0\rangle$  for any translation  $\mathbf{d}$ . It is then easy to prove that the state  $|\Psi_{\mathbf{k}}\rangle = \rho_{\mathbf{k}}|\Psi_0\rangle$  is a state of momentum  $\mathbf{P}_1 = \mathbf{P}_0 + \hbar\mathbf{k}$ . This establishes that  $\langle\Psi_{\mathbf{k}}|\Psi_{\mathbf{k}'}\rangle = 0$  for  $\mathbf{k} \neq \mathbf{k}'$ .

The situation is more complicated in the presence of a magnetic field, where the individual MTOs satisfy the algebra of Eqn. 3.215. If we place  $N$  electrons in a periodic (toroidal) geometry spanned by vectors  $\mathbf{L}_{1,2}$  such that  $\hat{z} \cdot \mathbf{L}_1 \times \mathbf{L}_2 = 2\pi\ell^2 N_\phi$ , then the many-body MTOs,

$$T(\mathbf{d}) = \prod_{i=1}^N t_i(\mathbf{d}) = \prod_{i=1}^N \exp\left(\frac{d b_i - \bar{d} b_i^\dagger}{\sqrt{2}\ell}\right) \quad (3.214)$$

satisfy the many-body version of Eqn. 3.215,

$$T(\mathbf{a})T(\mathbf{b}) = e^{iN\hat{z} \cdot \mathbf{a} \times \mathbf{b}/2\ell^2} T(\mathbf{a} + \mathbf{b}) = e^{iN\hat{z} \cdot \mathbf{a} \times \mathbf{b}/\ell^2} T(\mathbf{b})T(\mathbf{a}) . \quad (3.215)$$

Then  $T(\mathbf{L}_1)T(\mathbf{L}_2) = e^{2\pi i N N_\phi} T(\mathbf{L}_2)T(\mathbf{L}_1) = T(\mathbf{L}_2)T(\mathbf{L}_1)$  and therefore we can separately specify for each  $a \in \{1, 2\}$  that  $T(\mathbf{L}_a)|\Psi\rangle = e^{i\Theta_a}|\Psi\rangle$  for all states  $|\Psi\rangle$  in our many-body Hilbert space. One can now check that

$$T(\mathbf{L}_a) t_i(\ell^2 \hat{z} \times \mathbf{k}) T^\dagger(\mathbf{L}_a) = e^{i\mathbf{k} \cdot \mathbf{L}_a} t_i(\ell^2 \hat{z} \times \mathbf{k}) \quad (3.216)$$

<sup>50</sup>F. C. Zhang, V. Z. Vulovic, Y. Guo, and S. Das Sarma, *Phys. Rev. B* **32**, 6920 (1985) studied systems of up to  $N = 4$  electrons in a toroidal geometry. The loss of translational invariance due to the impurity makes larger systems difficult to study numerically.

and therefore provided  $e^{ik \cdot L_a} = 1$  all our SMA states lie in the same Hilbert space as specified by the eigenvalues of the unitaries  $T(L_{1,2})$ . This establishes the proper quantization of the SMA wavevectors  $\mathbf{k}$  in a periodic geometry, but it does not prove that SMA states of different wavevector are orthogonal.

We will delve more deeply into the many-body MTO algebra below. For now, let us establish the desired result by appealing to the thermodynamic limit for states on the plane. From Eqn. 3.197 we have, for any many-body state  $|\Psi\rangle$  purely within the LLL,

$$\langle \Psi | \bar{\rho}_{\mathbf{k}}^\dagger \bar{\rho}_{\mathbf{k}'} | \Psi \rangle = \langle \Psi | \rho_{\mathbf{k}}^\dagger \rho_{\mathbf{k}'} | \Psi \rangle - (1 - e^{-\bar{k}\bar{k}'\ell^2/2}) \langle \Psi | \rho_{\mathbf{k}'-\mathbf{k}} | \Psi \rangle \quad . \quad (3.217)$$

Now

$$\begin{aligned} \langle \Psi | \rho_{\mathbf{k}}^\dagger \rho_{\mathbf{k}'} | \Psi \rangle &= \int d^2r \int d^2r' n_2(\mathbf{r}, \mathbf{r}') e^{-i\mathbf{k} \cdot \mathbf{r}} e^{i\mathbf{k}' \cdot \mathbf{r}'} + \int d^2r n_1(\mathbf{r}) e^{i(\mathbf{k}'-\mathbf{k}) \cdot \mathbf{r}} \\ \langle \Psi | \rho_{\mathbf{k}'-\mathbf{k}} | \Psi \rangle &= \int d^2r n_1(\mathbf{r}) e^{i(\mathbf{k}'-\mathbf{k}) \cdot \mathbf{r}} \quad , \end{aligned} \quad (3.218)$$

where

$$n_j(\mathbf{r}_1, \dots, \mathbf{r}_j) = \frac{N!}{(N-j)!} \int d^2x_{j+1} \cdots \int d^2x_N |\Psi(\mathbf{r}_1, \dots, \mathbf{r}_j, \mathbf{x}_{j+1}, \dots, \mathbf{x}_N)|^2 \quad (3.219)$$

are the diagonal elements of the  $j$ -particle density matrix. The  $j$ -particle distribution function is then defined as the ratio

$$g_j(\mathbf{r}_1, \dots, \mathbf{r}_j) \equiv \frac{n_j(\mathbf{r}_1, \dots, \mathbf{r}_j)}{n_1(\mathbf{r}_1) \cdots n_1(\mathbf{r}_j)} \quad . \quad (3.220)$$

In the thermodynamic limit, we may write  $n_1(\mathbf{r}) = n$  and  $n_2(\mathbf{r}, \mathbf{r}') = n^2 g(\mathbf{r} - \mathbf{r}')$ , in which case

$$\langle \Psi | \bar{\rho}_{\mathbf{k}}^\dagger \bar{\rho}_{\mathbf{k}'} | \Psi \rangle = (2\pi)^2 (n^2 \hat{h}(\mathbf{k}) + n e^{-\mathbf{k}^2 \ell^2/2}) \delta(\mathbf{k} - \mathbf{k}') + (2\pi)^4 n^2 \delta(\mathbf{k}) \delta(\mathbf{k}') \quad , \quad (3.221)$$

where  $\hat{h}(\mathbf{k})$  is the Fourier transform of the pair correlation function. Note that in a system of finite area  $A$  we may replace  $(2\pi)^2 \delta(\mathbf{k} - \mathbf{k}') \rightarrow A \delta_{\mathbf{k}\mathbf{k}'}$ . We thus have established the result

$$\langle \Phi_{\mathbf{k}} | \Phi_{\mathbf{k}'} \rangle = A \bar{s}(\mathbf{k}) \delta_{\mathbf{k}\mathbf{k}'} + N^2 \delta_{\mathbf{k},\mathbf{0}} \delta_{\mathbf{k}',\mathbf{0}} \quad (3.222)$$

where  $|\Phi_{\mathbf{k}}\rangle = \bar{\rho}_{\mathbf{k}} |\Psi_0\rangle$ , and where  $\langle \Psi_0 | \Psi_0 \rangle = 1$  is assumed.

### Final remarks on the SMA

In Eqn. 3.38 we saw how projecting a plane wave  $e^{-i\mathbf{k} \cdot \mathbf{r}}$  onto the LLL yields the result

$$\Pi_0 e^{-i\mathbf{k} \cdot \mathbf{r}} f(z) e^{-r^2/4\ell^2} = e^{-\mathbf{k}^2 \ell^2/2} e^{-i\bar{k}z/2} f(z - i\bar{k}\ell^2) e^{-r^2/4\ell^2} \quad . \quad (3.223)$$

The generalization to a many-body wavefunction  $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = P(z_1, \dots, z_N) \prod_{i=1}^N e^{-r_i^2/4\ell^2}$  is

$$\begin{aligned} \Pi_0 \left( \prod_{i=1}^N e^{-\bar{z}_i z_i/4\ell^2} \right) \sum_{i=1}^N e^{-i\mathbf{k}\cdot\mathbf{r}_i} P(z_1, \dots, z_N) \\ = \left( \prod_{i=1}^N e^{-\bar{z}_i z_i/4\ell^2} \right) \underbrace{e^{-\mathbf{k}^2 \ell^2/2} \sum_{j=1}^N P(z_1, \dots, z_{j-1}, z_j - ik\ell^2, z_{j+1}, \dots, z_N) e^{-i\bar{k}z_j/2}}_{\equiv P_{\mathbf{k}}(z_1, \dots, z_N)} \end{aligned} \quad (3.224)$$

Note that on the RHS, we have each  $z_j$  in  $P_{\mathbf{k}}(Z)$  is in turn translated by  $-ik\ell^2$ . This messes with the zeroes of the wavefunction. In the Laughlin state, where  $P(Z) = [V(Z)]^q$ , the zeros of  $P(Z)$  as a function of one of the holomorphic coordinates (say  $z_1$ ) lie at the positions of all the other particles. In the SMA wavefunction, some of these zeros have shifted off the particle positions. However, since  $\rho_{\mathbf{k}}$  is still symmetric under interchange of particle labels, we still have that  $P_{\mathbf{k}}(Z_{\sigma}) = \text{sgn}(\sigma) P_{\mathbf{k}}(Z)$ . So at least  $(N-1)$  of the  $q(N-1)$  zeros lie on the positions of the other particles. The remaining  $(q-1)(N-1)$  zeros are shifted, and this results in an increase in energy, as is quite clear if we adopt a model truncated pseudopotential Hamiltonian.

What happens when  $q = 1$  and  $P(Z) = V(Z)$ ? Since  $V(Z)$  is the only polynomial in our Hilbert space when  $\nu = 1$ , it must be that  $\bar{\rho}_{\mathbf{k}}$  annihilates the filled LL whenever  $\mathbf{k} \neq 0$ . This is not so easy to show, however, and requires an appeal to the thermodynamic limit. Invoking the results of §3.1.2, we first obtain the second quantized form of the projected density operator in the angular momentum basis:

$$\bar{\rho}_{\mathbf{k}} = \sum_{m, m'} \langle m | e^{-i\mathbf{k}\cdot\mathbf{r}} | m' \rangle c_m^\dagger c_{m'} \quad , \quad (3.225)$$

where the matrix element is

$$I_{m, m'}(\mathbf{k}) \equiv \langle m | e^{-i\mathbf{k}\cdot\mathbf{r}} | m' \rangle = \int d^2r \varphi_m^*(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \varphi_{m'}(\mathbf{r}) \quad , \quad (3.226)$$

where  $\varphi_m(\mathbf{r}) = (2\pi\ell^2 m!)^{-1/2} (z/\sqrt{2}\ell)^m \exp(-z\bar{z}/4\ell^2)$ . We can use

$$I_{0,0}(\mathbf{k}) = \frac{1}{2\pi\ell^2} \int d^2r e^{-ik\bar{z}/2} e^{-i\bar{k}z/2} e^{-\bar{z}z/2\ell^2} = e^{-\bar{k}k\ell^2/2} \quad (3.227)$$

as a generating function, for which it is easily seen that

$$I_{m, m'}(\mathbf{k}) = \frac{1}{\sqrt{m! m'}} \left( \frac{i\sqrt{2}}{\ell} \frac{\partial}{\partial k} \right)^m \left( \frac{i\sqrt{2}}{\ell} \frac{\partial}{\partial \bar{k}} \right)^{m'} e^{-\bar{k}k\ell^2/2} = \frac{i^{m-m'}}{\sqrt{m! m'}} \left( \frac{\partial}{\partial w} \right)^m (w^{m'} e^{-\bar{w}w}) \quad (3.228)$$

with  $w = k\ell/\sqrt{2}$ ,  $\bar{w} = \bar{k}\ell/\sqrt{2}$ , and  $j = m' - m$ . Thus,

$$I_{m, m+j}(\mathbf{k}) = (-i)^j \left( \frac{m!}{(m+j)!} \right)^{1/2} \left( \frac{k\ell}{\sqrt{2}} \right)^m L_m^{(j)}(\mathbf{k}^2 \ell^2/2) e^{-\mathbf{k}^2 \ell^2/2} \quad , \quad (3.229)$$

where  $L_m^{(j)}(x)$  is a generalized Laguerre polynomial. Note  $I_{m+j,m}(\mathbf{k}) = I_{m,m+j}^*(-\mathbf{k})$ .

Now consider the action of the projected density operator  $\bar{\rho}_{\mathbf{k}} = \sum_{m,m'} I_{m,m'}(\mathbf{k}) c_m^\dagger c_{m'}$  on the filled Landau level  $|\Psi_1\rangle = \prod_{m=0}^{N_\phi-1} c_m^\dagger |0\rangle$ . In the thermodynamic limit, we have  $N_\phi \rightarrow \infty$ . Since every  $m$  state is occupied at  $\nu = 1$ , the off-diagonal terms in  $\bar{\rho}_{\mathbf{k}}$  with  $m \neq m'$  must annihilate  $|\Psi_1\rangle$ . This leaves only the diagonal elements, for which

$$I_{m,m}(\mathbf{k}) = L_m(\mathbf{k}^2 \ell^2 / 2) e^{-\mathbf{k}^2 \ell^2 / 2} \quad . \quad (3.230)$$

Thus,

$$\bar{\rho}_{\mathbf{k}} |\Psi_1\rangle = \sum_{m=0}^{\infty} I_{m,m}(\mathbf{k}) |\Psi_1\rangle = e^{-\mathbf{k}^2 \ell^2 / 2} \sum_{m=0}^{\infty} L_m(\mathbf{k}^2 \ell^2 / 2) |\Psi_1\rangle \quad . \quad (3.231)$$

But the generating function of  $L_n(x)$  yields<sup>51</sup>

$$\sum_{m=0}^{\infty} t^m L_m(x) = \frac{1}{1-t} \exp\left(-\frac{tx}{1-t}\right) \quad , \quad (3.232)$$

and thus taking  $t \uparrow 1$  we encounter an essential singularity and provided  $\mathbf{k} \neq 0$  the sum is zero! When  $\mathbf{k} = 0$ , from  $L_m(0) = 1$  and cutting off the  $m$  sum at  $N_\phi$ , we have  $\bar{\rho}_0 |\Psi_1\rangle = N_\phi |\Psi_1\rangle$ , which is also correct.

## 3.4 The Hierarchy

FQH plateaus have been observed at a number of odd-denominator rational fractions, including the principal Laughlin states at  $\nu = \frac{1}{3}$  and  $\nu = \frac{1}{5}$ , their particle-hole conjugates at  $\nu = \frac{2}{3}$  and  $\nu = \frac{4}{5}$ , and at a number of other fillings:  $\nu = \frac{2}{5}, \frac{3}{5}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{4}{9}, \frac{5}{11}, \frac{6}{13}$ , etc. Other plateaus have been observed at  $\nu = \frac{4}{3}, \frac{7}{5}, \frac{10}{7}$ , i.e. in higher Landau levels. Fractions  $\nu = p/q$  with  $p \notin \{1, q-1\}$  are understood in terms of a hierarchical scheme originally due to Haldane and to Halperin, and a very successful set of wavefunctions due to Jain known as the composite fermion construction.

### 3.4.1 Particle-hole conjugation

Consider an arbitrary  $N$ -electron wavefunction within the LLL,

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = P(z_1, \dots, z_N) \exp\left(-\frac{1}{4\ell^2} \sum_{i=1}^N |z_i|^2\right) \quad , \quad (3.233)$$

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<sup>51</sup>See Gradshteyn and Ryzhik 8.975.1.

how does one construct its particle-hole conjugate? Here we presume that  $P(Z)$  is a multinomial function of its arguments. We start with the filled Landau level with  $N_\phi$  total states, whose holomorphic component is the Vandermonde determinant  $V(z_1, \dots, z_{N_\phi})$ , viz.

$$\Psi_1(\mathbf{r}_1, \dots, \mathbf{r}_{N_\phi}) = V(z_1, \dots, z_{N_\phi}) \exp\left(-\frac{1}{4\ell^2} \sum_{i=1}^{N_\phi} |z_i|^2\right) . \quad (3.234)$$

The unnormalized particle-hole conjugate wavefunction of the  $N$ -particle state in Eqn. 3.233 is constructed by taking its 'image' in the filled LL:

$$\Psi^C(\mathbf{r}_{N+1}, \dots, \mathbf{r}_{N_\phi}) = \int d^2r_1 \cdots \int d^2r_N \Psi_1(\mathbf{r}_1, \dots, \mathbf{r}_{N_\phi}) \overline{\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)} , \quad (3.235)$$

where the bar denotes complex conjugation. This means that the holomorphic component of the  $M$ -particle wavefunction  $\Psi^C(\mathbf{r}_1, \dots, \mathbf{r}_M)$  is

$$P^C(z_1, \dots, z_M) = \int d^2y_1 \cdots \int d^2y_N V(y_1, \dots, y_N, z_1, \dots, z_M) \overline{P(y_1, \dots, y_N)} \exp\left(-\frac{1}{2\ell^2} \sum_{i=1}^N |y_i|^2\right) \quad (3.236)$$

where  $M + N = N_\phi$ . Clearly  $P^C(Z)$  is totally antisymmetric in  $Z = \{z_1, \dots, z_M\}$ . It is useful to introduce the notation  $P[N] \equiv P(z_1, \dots, z_N)$  to explicitly denote the number of holomorphic coordinates of  $P$ . We then have

$$\deg P^C[M] = \frac{1}{2}N_\phi(N_\phi - 1) - \deg P[N] , \quad (3.237)$$

and therefore if  $\deg P[N] = N(N - 1)/2\nu$ , with  $N = \nu N_\phi$ , then

$$\deg P^C[M] = \frac{N}{2\nu} \left( \frac{N}{\nu} - 1 \right) - \frac{N}{2\nu} (N - 1) = \frac{M(M - 1)}{2(1 - \nu)} , \quad (3.238)$$

where  $M = N_\phi - N = (\nu^{-1} - 1)N$ . This of course confirms that the filling is

$$\nu^C = \frac{M(M - 1)}{2 \deg P^C[M]} = 1 - \nu . \quad (3.239)$$

### 3.4.2 Particle-hole symmetry

Consider the LLL-projected Coulomb Hamiltonian in the presence of a neutralizing background of density  $\nu n_0$ , where  $n_0 = 1/2\pi\ell^2$ :

$$H_\nu = \frac{1}{2} \int d^2r \int d^2r' v(\mathbf{r} - \mathbf{r}') \left\{ \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) - 2\nu n_0 \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) + \nu^2 n_0^2 \right\} . \quad (3.240)$$

Recall that the electron field operator is

$$\psi(\mathbf{r}) = \sum_{\alpha} \varphi_{\alpha}(\mathbf{r}) c_{\alpha} \quad , \quad (3.241)$$

expressed in terms of the angular momentum basis (or any orthonormal basis complete in the LLL). Thus  $\{\psi(\mathbf{r}), \psi^{\dagger}(\mathbf{r}')\} = n_0 G(\mathbf{r}, \mathbf{r}')$  where  $n_0 = 1/2\pi\ell^2$  and

$$G(\mathbf{r}, \mathbf{r}') = e^{z\bar{z}'/2\ell^2} e^{-z\bar{z}/4\ell^2} e^{-z'\bar{z}'/4\ell^2} = e^{i\hat{z}\cdot\mathbf{r}\times\mathbf{r}'/2\ell^2} e^{-(\mathbf{r}-\mathbf{r}')^2/4\ell^2} = [G(\mathbf{r}', \mathbf{r})]^* \quad . \quad (3.242)$$

The anticommutation relations yield the following result,

$$\begin{aligned} \psi^{\dagger}(\mathbf{r}) \psi^{\dagger}(\mathbf{r}') \psi(\mathbf{r}') \psi(\mathbf{r}) &= \psi(\mathbf{r}) \psi(\mathbf{r}') \psi^{\dagger}(\mathbf{r}') \psi^{\dagger}(\mathbf{r}) + n_0 \psi^{\dagger}(\mathbf{r}) \psi(\mathbf{r}) - n_0 \psi(\mathbf{r}') \psi^{\dagger}(\mathbf{r}') \\ &+ n_0 G(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \psi^{\dagger}(\mathbf{r}) - n_0 G(\mathbf{r}', \mathbf{r}) \psi^{\dagger}(\mathbf{r}') \psi(\mathbf{r}) \quad , \end{aligned} \quad (3.243)$$

which establishes

$$H_{\nu}[\psi, \psi^{\dagger}] = H_{1-\nu}[\tilde{\psi}, \tilde{\psi}^{\dagger}] + \frac{1}{2} n_0 \int d^2r \int d^2r' v(\mathbf{r} - \mathbf{r}') \left\{ G(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}) \psi^{\dagger}(\mathbf{r}') - G(\mathbf{r}', \mathbf{r}) \psi^{\dagger}(\mathbf{r}) \psi(\mathbf{r}') \right\} \quad (3.244)$$

where  $\tilde{\psi}(\mathbf{r}) = \psi^{\dagger}(\mathbf{r})$  and  $\tilde{\psi}^{\dagger}(\mathbf{r}) = \psi(\mathbf{r})$  is a particle-hole canonical transformation. Note however that  $\{\tilde{\psi}(\mathbf{r}), \tilde{\psi}^{\dagger}(\mathbf{r}')\} = n_0 G(\mathbf{r}', \mathbf{r})$ . Of course this is because particles and holes have opposite electric charge! We can now show that in any state  $|\Psi_{\nu}\rangle$  in which  $\langle \Psi_{\nu} | c_m^{\dagger} c_n | \Psi_{\nu} \rangle = \nu \delta_{mn}$ , or equivalently  $\langle \Psi_{\nu} | \psi^{\dagger}(\mathbf{r}') \psi(\mathbf{r}) | \Psi_{\nu} \rangle = \nu G(\mathbf{r}, \mathbf{r}')$ , that

$$\langle \Psi_{\nu} | H_{\nu}[\psi, \psi^{\dagger}] | \Psi_{\nu} \rangle + N_{\phi} \nu^2 \sqrt{\frac{\pi}{8}} \frac{e^2}{\epsilon\ell} = \langle \tilde{\Psi}_{1-\nu} | H_{1-\nu}[\tilde{\psi}, \tilde{\psi}^{\dagger}] | \tilde{\Psi}_{1-\nu} \rangle + N_{\phi} (1 - \nu)^2 \sqrt{\frac{\pi}{8}} \frac{e^2}{\epsilon\ell} \quad , \quad (3.245)$$

where  $|\tilde{\Psi}_{1-\nu}\rangle \equiv |\Psi_{\nu}\rangle$ . This establishes that if there is a cusp in the total energy at filling fraction  $\nu$ , there will also be a cusp at filling fraction  $1 - \nu$ .

### 3.4.3 Hierarchical construction of FQH wavefunctions

Haldane<sup>52</sup> was the first to suggest that Laughlin's quasiparticles could themselves condense into a higher order FQH state, which, in turn would itself have quasiparticle excitations which could condense, *ad infinitum*. Energetics would then determine how far along this chain one can proceed and still have stable condensates<sup>53</sup>. In Haldane's analysis, the quasiparticles carry bosonic statistics. Subsequently Halperin<sup>54</sup> devised a hierarchy based on fractional quasiparticle statistics, which we discussed in §3.3.6. Laughlin<sup>55</sup> proposed explicit hierarchical wavefunctions for the  $\nu = \frac{2}{5}$  and  $\frac{2}{7}$  states, and argued that the quasiparticles obeyed fermionic statistics.

<sup>52</sup>F. D. M. Haldane, *Phys. Rev. Lett.* **51**, 605 (1983).

<sup>53</sup>One competing phase would be a Wigner crystal of quasiparticles, for example.

<sup>54</sup>B. I. Halperin, *Phys. Rev. Lett.* **52**, 1982 (1984).

<sup>55</sup>R. B. Laughlin, *Surf. Sci.* **142**, 163 (1984).

Consistent with the Chern-Simons Ginzburg-Landau field theory of the FQHE, which we shall discuss in §CSGL below, I believe that a proper understanding of quasiparticle statistics in the FQHE entails that they are anyons, but the matter of their exchange statistics will not enter the following discussion.

To elicit hierarchical wavefunctions, we will follow the scheme of MacDonald, Aers, and Dharma-wardana<sup>56</sup>, which relies heavily on the particle-hole conjugation formalism discussed in §3.4.1. In particular, recall the relation in Eqn. 3.236 between the  $N$ -particle holomorphic factor  $P(z_1, \dots, z_N)$  and its particle-hole conjugate  $P^C(z_1, \dots, z_M)$ . If at level  $t - 1$  of the hierarchy one has the holomorphic polynomial  $P_{t-1}(Z)$  corresponding to a filling  $\nu_{t-1}$ , we construct a new polynomial  $P_t(Z)$  in one of two ways. The first way is to write

$$P_t(Z) = P_{t-1}^C(Z) [V(Z)]^{2p_t} \quad (3.246)$$

with  $p_t$  a nonnegative integer. Note here that  $P_{t-1}^C(Z) = P^C[N]$  is a function of the  $N$  holomorphic coordinates  $\{z_1, \dots, z_N\}$ . Therefore, from Eqn. 3.238,

$$\begin{aligned} \deg P_t[N] &= \frac{N(N-1)}{2\nu_t} = \deg P_{t-1}^C[N] + 2p_t \deg V[N] \\ &= \frac{N(N-1)}{2(1-\nu_{t-1})} + p_t N(N-1) \quad , \end{aligned} \quad (3.247)$$

from which we obtain

$$\nu_t^{-1} = 2p_t + \frac{1}{1-\nu_{t-1}} = 2p_t + 1 + \frac{1}{\nu_{t-1}^{-1} - 1} \quad . \quad (3.248)$$

Starting at level  $i = 0$  of the hierarchy with the polynomial  $P_0(Z) \equiv 1$ , which has  $\deg P_0[N] = 0$ , corresponding to a filling  $\nu_0 = 0$ , the above formula then gives  $\nu_1 = 1/(2p_1 + 1)$ , which is one of the principal Laughlin states<sup>57</sup>. If we iterate the formula once more, with  $p_2 = 0$ , we obtain the particle-hole conjugate of the state at level  $i = 1$ , with filling  $\nu_2 = 2p_1/(2p_1 + 1)$ .

The second iterative construction is given by<sup>58</sup>

$$P_t(Z) = V(Z) \left( \frac{P_{t-1}^C(Z)}{V(Z)} \right)^\dagger [V(Z)]^{2p_t} \quad , \quad (3.249)$$

where  $z_i^\dagger \equiv 2\ell^2 \partial_t$ . Note

$$\deg \left( \frac{P_{t-1}^C[N]}{V[N]} \right) = \frac{N(N-1)}{2(1-\nu_{t-1})} - \frac{N(N-1)}{2} = \frac{N(N-1)}{2(\nu_{t-1}^{-1} - 1)} \quad , \quad (3.250)$$

<sup>56</sup>A. H. MacDonald, G. C. Aers, and M. W. C. Dharma-wardana, *Phys. Rev. B* **31**, 5529 (1985).

<sup>57</sup>While the initial polynomial  $P_0(Z) = 1$  is not antisymmetric, this is not an issue. All the polynomials at level  $i > 0$  in the hierarchy are completely antisymmetric. We can think of  $P_0(Z)$  as the limit as  $\nu \rightarrow 0$  of a proper fermionic state.

<sup>58</sup>I think that another possible choice here would be to take  $P_t(Z) = [P_{t-1}^C(Z)]^\dagger [V(Z)]^{2p_t}$ . This should yield the same fractions, but in with a different labeling.



and therefore

$$\deg P_t[Z] = \frac{N(N-1)}{2\nu_t} = \frac{1}{2}N(N-1) + p_t N(N-1) - \frac{N(N-1)}{2(\nu_{t-1}^{-1} - 1)} \quad , \quad (3.251)$$

and thus

$$\nu_t^{-1} = 2p_t + 1 - \frac{1}{\nu_{t-1}^{-1} - 1} \quad . \quad (3.252)$$

We can express both steps of the hierarchy by the formula

$$\nu_t^{-1} = 2p_t + 1 + \frac{\alpha_{t-1}}{\nu_{t-1}^{-1} - 1} \quad , \quad (3.253)$$

where  $\alpha_{t-1} \equiv +1$  if Eqn. 3.246 is used and  $\alpha_{t-1} \equiv -1$  if Eqn. 3.249 is used. If we iterate the formulae  $t$  times, we obtain

$$\nu_t \equiv [p_t, \alpha_{t-1} p_{t-1}, \dots, \alpha_0 p_0] = \frac{1}{1 + 2p_t + \frac{\alpha_{t-1}}{2p_{t-1} + \frac{\alpha_{t-2}}{\dots + \frac{\alpha_1}{2p_1 + \frac{\alpha_0}{2p_0}}}}} \quad (3.254)$$

It is convenient to write  $\alpha p = \bar{p}$  for  $\alpha = -1$ . One then finds

$$\begin{array}{ll} \frac{1}{3} = [1] & \frac{2}{3} = [0, 1] \\ \frac{1}{5} = [2] & \frac{4}{5} = [0, 2] \\ \frac{2}{5} = [1, \bar{1}] & \frac{3}{5} = [0, 1, \bar{1}] \\ \frac{2}{7} = [1, 1] & \frac{5}{7} = [0, 1, 1] \\ \frac{3}{7} = [1, \bar{1}, \bar{1}] & \frac{4}{7} = [0, 1, \bar{1}, \bar{1}] \\ \frac{4}{9} = [1, \bar{1}, \bar{1}, \bar{1}] & \frac{5}{9} = [0, 1, \bar{1}, \bar{1}, \bar{1}] \\ \frac{3}{11} = [1, 1, \bar{1}] & \frac{8}{11} = [0, 1, 1, \bar{1}] \quad . \end{array}$$

As is clear from the above, particle-hole conjugation means

$$\nu_t^C = 1 - \nu_t = [0, p_t, \alpha_{t-1} p_{t-1}, \dots, \alpha_0 p_0] \quad . \quad (3.255)$$

Note that the same fraction may be represented by more than one sequence. For example,

$$\frac{4}{7} = [0, 1, \bar{1}, \bar{1}] = [1, 1, \bar{1}, \bar{1}, \bar{1}] \quad , \quad (3.256)$$

which is a rather awkward aspect to this procedure. More seriously, one has to proceed rather deep into the hierarchy to arrive at several observed FQH fractions, such as  $\nu = \frac{4}{9} = [1, \bar{1}, \bar{1}, \bar{1}]$ .

### 3.4.4 Composite fermions

The composite fermion (CF) approach, pioneered by J. K. Jain<sup>59</sup>, starts with the  $\nu = r$  Slater determinant state of  $r$  filled LLs. In this state we have  $N/N_\phi = r$  in the thermodynamic limit. We denote this state as  $\Phi_r$  if  $B = -B\hat{z}$  as has been our convention, and as  $\Phi_{-r}$  if  $B = +B\hat{z}$ . Now consider the flux attachment operation where  $2p$  flux quanta are attached to each particle, yielding a *composite fermion*. This introduces  $2p$  additional zeros whenever any two particles coincide, and is accomplished by multiplying the wavefunction  $\Phi_{\pm r}$  for  $r$  filled LLs by the  $(2p)^{\text{th}}$  power of the Vandermonde determinant, yielding the  $N$ -particle wavefunction  $\Psi_{\pm r,p}[N] = V^{2p}[N] \Phi_{\pm r}[N]$ . The total flux per particle is then

$$\nu^{-1} = \frac{N_\phi}{N} = \pm \frac{1}{r} + 2p \quad \Rightarrow \quad \nu_{\pm r,p} = \frac{r}{2rp \pm 1} \quad . \quad (3.257)$$

Clearly this state will contain contributions from single-particle wavefunctions in the first  $n$  Landau levels, and if a *bona fide* LLL many-body wavefunction is desired, one should then *project* onto the  $n = 0$  LL, *viz.*

$$\tilde{\Psi}_{\pm r,p}[N] \equiv \Pi_0 \Psi_{\pm r,p}[N] = \Pi_0 V^{2p}[N] \Phi_{\pm r}[N] \quad . \quad (3.258)$$

Note that the projector  $\Pi_0$  commutes with the total particle number  $N$ . Note also that the projector is applied *after* the state  $\Phi_{\pm r}[N]$  is multiplied by  $V^{2p}[N]$ , since otherwise it is annihilated, *i.e.*  $\Pi_0 \Phi_{\pm r}[N] = 0$  for  $n > 1$ .

| $\nu_{\pm r,p}$ | $r = \bar{1}$ | $r = 1$ | $r = \bar{2}$ | $r = 2$ | $r = \bar{3}$ | $r = 3$ | $r = \bar{4}$ | $r = 4$ | $r = \bar{5}$ | $r = 5$ |
|-----------------|---------------|---------|---------------|---------|---------------|---------|---------------|---------|---------------|---------|
| $p = 0$         | $\bar{1}$     | 1       | $\bar{2}$     | 2       | $\bar{3}$     | 3       | $\bar{4}$     | 4       | $\bar{5}$     | 5       |
| $p = 1$         | 1             | 1/3     | 2/3           | 2/5     | 3/5           | 3/7     | 4/7           | 4/9     | 5/9           | 5/11    |
| $p = 2$         | 1/3           | 1/5     | 2/7           | 2/9     | 3/11          | 3/13    | 4/15          | 4/17    | 5/19          | 5/21    |
| $p = 3$         | 1/5           | 1/7     | 2/11          | 2/13    | 3/17          | 3/19    | 4/23          | 4/25    | 5/29          | 5/31    |
| $p = 4$         | 1/7           | 1/9     | 2/15          | 2/17    | 3/23          | 3/25    | 4/31          | 4/33    | 5/39          | 5/41    |

Table 3.1: Filling fractions  $\nu_{\pm r,p}$  for the first level of composite fermion states ( $\bar{r} \equiv -r$ ). Observed fractions are printed in red. Not shown are the corresponding particle-hole conjugate states, for which  $\nu_{\pm r,p}^C = 1 - \nu_{\pm r,p}$ .

One might worry about the effect of the projector, but Jain found that it has a rather weak

<sup>59</sup>J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989); J. K. Jain, *Phys. Rev. B* **41**, 7653 (1990). See also J. K. Jain, *Composite Fermions* (Cambridge, 2007).

effect, and *most* of the state  $\Psi_{\pm r,p}[N]$  is confined to the LLL, with

$$\begin{aligned} \nu = \frac{2}{5} \ (r = 2, p = 1) : \frac{\langle \Psi_{r,p} | \Pi_0 | \Psi_{r,p} \rangle}{\langle \Psi_{r,p} | \Psi_{r,p} \rangle} &\approx 0.05 \\ \nu = \frac{4}{9} \ (r = 4, p = 1) : \frac{\langle \Psi_{r,p} | \Pi_0 | \Psi_{r,p} \rangle}{\langle \Psi_{r,p} | \Psi_{r,p} \rangle} &\approx 0.01 \quad . \end{aligned} \quad (3.259)$$

With  $\tilde{\Psi}_{\pm r,p}[N]$  fully in the LLL, we can construct its particle-hole conjugate  $\Psi_{\pm r,p}^C[N]$ , with filling  $\nu_{\pm r,p}^C = 1 - \nu_{\pm r,p}$ . A table of values for small  $n$  and  $p$  is given in Tab. 3.1. States in this sequence may be thought of as describing an integer quantum Hall state of composite fermions. A touted success of the CF theory is that most of the observed states appear at early in the sequence.

Note that the  $r \rightarrow \infty$  limit of this sequence yields a filling

$$\lim_{r \rightarrow \infty} \nu_{\pm r,p} = \frac{1}{2p} \quad , \quad (3.260)$$

which is an *even denominator* fermionic state with  $r^{-1} = 0$  flux per composite fermion. If the wavefunction is  $\Psi[N] = V^{2p}[N] \Phi[N]$ , then the component  $\Phi[N]$  is a state corresponding to zero flux. One possibility is a Slater determinant of plane waves, *i.e.*

$$\Phi(\mathbf{r}_1, \dots, \mathbf{r}_N) = A^{-N/2} \det(e^{i\mathbf{k}_i \cdot \mathbf{r}_j}) \quad , \quad (3.261)$$

If the wavevectors  $\{\mathbf{k}_1, \dots, \mathbf{k}_N\}$  are arranged in a Fermi circle, then there must be *gapless particle-hole excitations* corresponding to removing a state just below  $k_F$  and replacing it with one just above  $k_F$ . We will discuss the theory of the half-filled Landau level in the next chapter.

The CF states described above represent only the first level of the CF hierarchy. At the second level, one can start, for instance, with the  $\nu = \frac{4}{3}$  state, which is  $\nu = \frac{1}{3}$  state in the  $n = 1$  LL sitting atop a filled  $n = 0$  LL<sup>60</sup>. Thus  $\nu = N/N_\phi = \frac{4}{3}$  and  $N_\phi = \frac{3}{4}N$ . The flux attachment operation results in  $\nu^{-1} \rightarrow \nu^{-2} + 2$  hence  $\nu' = \nu/(2\nu + 1)$  and with  $\nu = \frac{4}{3}$  we obtain  $\nu' = \frac{4}{11}$ , which is an observed fraction that is in fact not present among the states of the first level of the CF hierarchy. The states  $\frac{5}{13}$ ,  $\frac{7}{11}$ ,  $\frac{4}{13}$ ,  $\frac{6}{17}$ , and  $\frac{5}{17}$  are also observed (see Fig. 3.11 and are present only

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<sup>60</sup>We assume 100% spin polarization, so we work only with spinless fermion wavefunctions here.

at the second level of the hierarchy. Note that under flux addition we have<sup>61</sup>

$$\begin{aligned}
 \frac{4}{3} &\longrightarrow \frac{4/3}{(8p/3) \pm 1} \stackrel{p=1}{=} \begin{cases} 4/11 \\ 4/5 \end{cases} \\
 \frac{4}{3} &\longrightarrow \frac{4/3}{(8p/3) \pm 1} \stackrel{p=2}{=} \begin{cases} 4/19 \\ 4/13 \end{cases} \\
 \frac{6}{5} &\longrightarrow \frac{6/5}{(12p/5) \pm 1} \stackrel{p=1}{=} \begin{cases} 6/17 \\ 6/7 \end{cases} \\
 \frac{5}{3} &\longrightarrow \frac{5/3}{(10p/3) \pm 1} \stackrel{p=1}{=} \begin{cases} 5/13 \\ 5/7 \end{cases} \\
 \frac{7}{3} &\longrightarrow \frac{7/3}{(14p/3) \pm 1} \stackrel{p=1}{=} \begin{cases} 7/17 \\ 7/11 \end{cases}
 \end{aligned} \tag{3.262}$$

States in this sequence may be thought of as describing a fractional quantum Hall state of composite fermions. The complete set of operations generating the CF states is thus:

- (i) Landau level addition:  $\nu \rightarrow \nu + 1$
- (ii) flux attachment:  $\nu^{-1} \rightarrow \nu^{-1} + 2$
- (iii) LLL projection (preserves  $N$ ):  $\Psi \rightarrow \Pi_0 \Psi$
- (iv) particle-hole conjugation (LLL states only):  $\nu \rightarrow 1 - \nu$ .

### 3.5 Chern-Simons Ginzburg-Landau Theory

The understanding that the Laughlin states and their hierarchical descendants were especially stable gapped phases of matter with peculiar topological properties (*e.g.* FQHE, quasiparticle excitations obeying fractional exchange statistics) naturally led researchers to think of these states as some sort of novel condensates. If so, what is the corresponding order parameter and continuum field theory? In fact, today it is known that topological phases of matter such as the FQHE phases do not possess an order parameter of the usual sort and do not break any global symmetries. Nevertheless, a compelling field-theoretic description of the FQHE, due to Zhang, Hansson, and Kivelson<sup>62</sup>, based in part on earlier work by Girvin and MacDonald<sup>63</sup> has proven extremely useful and influential. I especially recommend the pellucid review by Zhang<sup>64</sup>.

<sup>61</sup>Red typeface indicates observed FQH fractions; blue typeface indicates fractions already present at the first level of the CF hierarchy.

<sup>62</sup>S.-C. Zhang, H. Hansson, and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989).

<sup>63</sup>S. M. Girvin and A. H. MacDonald, *Phys. Rev. Lett.* **58**, 1252 (1987).

<sup>64</sup>S.-C. Zhang, *Int. Jour. Mod. Phys. B* **6**, 25 (1992).

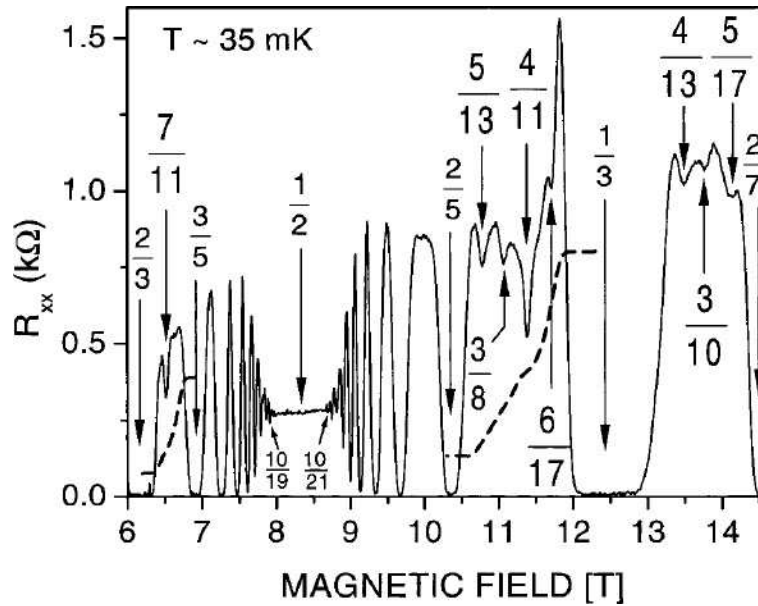


Figure 3.11: Experimental observation of second level composite fermion states:  $\nu = \frac{4}{11}, \frac{5}{13}, \frac{7}{11}, \frac{4}{13}, \frac{6}{17},$  and  $\frac{5}{17}$ . From W. Pan *et al.*, *Phys. Rev. Lett.* **90**, 016801 (2003).

### 3.5.1 Superfluids, vortices, and duality

As a warm-up, we will first consider the case of a  $2 + 1$ -dimensional superfluid. The Chern-Simons Ginzburg-Landau (CSGL) theory establishes a connection between the field theory of the superfluid and that of the FQHE. Of course, superfluidity is a phenomenon of bosonic systems, whereas the constituent particles of the FQHE are electrons, which are fermions. In superconductors, the electrons first pair before condensing, as do the helium atoms in liquid  $^3\text{He}$ <sup>65</sup>. But this is not what happens in the Laughlin states, for example. Rather, in order to describe the FQHE in terms of a bosonic field theory of an order parameter field, we will have to manufacture fermions from bosons – a trick known as *statistical transmutation*. As we shall see, at a mean field level, the implementation of the statistical transmutation, which is effected using a fictitious gauge field, can *cancel* with the physical magnetic field, leaving behind a purely bosonic theory in zero field, but with telltale fluctuation terms. The vortices of the bosonic superfluid then correspond to quasiparticles of the FQHE! But all good things to those who wait; first let's examine the case of superfluidity in  $(2 + 1)$ -dimensions<sup>66</sup>.

<sup>65</sup>The fact that electrons are charged, whereas  $^3\text{He}$  atoms are neutral, entails some essential corrections to the simplistic description of a superconductor as a Bose condensate.

<sup>66</sup>V. N. Popov, *Sov. Phys. JETP* **37**, 341 (1973); M. P. A. Fisher and D.-H. Lee, *Phys. Rev. B* **39**, 2756 (1989); G. E. Volovik, *JETP Lett.*, **62**, 65 (1995); E. Šimánek, *Inhomogeneous Superconductors* (Oxford, 1994); D. P. Arovas and J. A. Freire, *Phys. Rev. B* **55**, 1068 (1997).

Start with the Euclidean action

$$S_E = \int d\tau \int d^2x \left\{ \hbar \bar{\psi} \partial_\tau \psi + \frac{\hbar^2}{2m} |\nabla \psi|^2 \right\} + \frac{1}{2} \int d\tau \int d^2x \int d^2x' v(\mathbf{x} - \mathbf{x}') \delta n(\mathbf{x}) \delta n(\mathbf{x}') \quad , \quad (3.263)$$

where

$$\delta n(\mathbf{x}) \equiv n(\mathbf{x}) - n_0 \quad . \quad (3.264)$$

We write  $\psi = \sqrt{n} e^{i\phi}$ , in which case

$$\begin{aligned} \partial_\tau \psi &= \left( \frac{\partial_\tau n}{2\sqrt{n}} + i\sqrt{n} \partial_\tau \phi \right) e^{i\phi} . \\ |\nabla \psi|^2 &= \frac{(\nabla n)^2}{4n} + n (\nabla \phi)^2 \quad . \end{aligned} \quad (3.265)$$

Setting aside the interaction term for the moment, the Lagrangian density, other than the  $v$  term, is then given by

$$\begin{aligned} \mathcal{L}_0 &= i\hbar n \partial_\tau \phi + \frac{\hbar^2 n}{2m} (\nabla \phi)^2 + \frac{\hbar^2}{8mn} (\nabla n)^2 \\ &\xrightarrow{\text{HST}} i\hbar n \partial_\tau \phi + i\hbar \mathbf{Q} \cdot \nabla \phi + \frac{m \mathbf{Q}^2}{2n} + \frac{\hbar^2}{8mn} (\nabla n)^2 \quad , \end{aligned} \quad (3.266)$$

where  $\mathbf{Q}$  is a Hubbard-Stratonovich field. We now separate  $\phi = \phi_{\text{sw}} + \phi_v$  into the smooth spin-wave and singular vortex contributions. Integrating over  $\phi_{\text{sw}}$  yields the constraint

$$\partial_\tau n + \nabla \cdot \mathbf{Q} = 0 \quad , \quad (3.267)$$

which is solved by writing

$$\begin{aligned} n - n_0 &= n_0 \left( \frac{\partial \mathcal{W}_x}{\partial y} - \frac{\partial \mathcal{W}_y}{\partial x} \right) \equiv -n_0 \mathcal{B} \\ Q_x &= n_0 \left( \frac{\partial \mathcal{W}_y}{\partial \tau} - \frac{\partial \mathcal{W}_\tau}{\partial y} \right) \equiv -un_0 \mathcal{E}_y \\ Q_y &= n_0 \left( \frac{\partial \mathcal{W}_\tau}{\partial x} - \frac{\partial \mathcal{W}_x}{\partial \tau} \right) \equiv +un_0 \mathcal{E}_x \quad , \end{aligned} \quad (3.268)$$

where we have defined the dimensionless analog electromagnetic fields  $(\mathcal{E}, \mathcal{B})$  and where  $u$  is an as-yet undetermined constant with dimensions of speed. We also define the background gauge field  $w_\mu(\mathbf{x}, \tau) = -x \hat{y}$ . Regarding the vortex part, we have

$$i\hbar n \partial_\tau \phi_v + i\hbar \mathbf{Q} \cdot \nabla \phi_v = -2\pi i \hbar n_0 J^\mu (\mathcal{W}_\mu + w_\mu) + \partial(\cdot) \quad , \quad (3.269)$$

where

$$J^\mu(\mathbf{x}, \tau) = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda \phi_\nu = \sum_i q_i \left\{ \frac{1}{\dot{\mathbf{X}}_i} \right\} \delta(\mathbf{x} - \mathbf{X}_i(\tau)) \quad , \quad (3.270)$$

where  $q_i \in \mathbb{Z}$  is the integer vorticity of vortex  $i$ , and  $x^\mu = (\tau, x, y)$  for  $\mu = 0, 1, 2$ . There is no difference between raised and lowered indices here. Thus, dropping total derivatives, we have

$$\mathcal{L}_0 = \frac{\hbar^2}{8mn} (\nabla n)^2 + \frac{mQ^2}{2n} - 2\pi i \hbar n_0 J^\mu (\mathcal{W}_\mu + w_\mu) + \partial(\cdot) \quad (3.271)$$

We now expand to quadratic order in  $\delta n$  and  $Q$ . Including the  $v$  term in the potential, we have

$$\begin{aligned} S_E &= \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \left\{ \frac{m}{2n_0} |\hat{Q}(\mathbf{k}, \omega)|^2 + \left( \frac{1}{2} \hat{v}(\mathbf{k}) + \frac{\hbar^2 \mathbf{k}^2}{8mn_0} \right) |\delta \hat{n}(\mathbf{k}, \omega)|^2 \right\} - i \hbar n_0 \int d\tau \int d^2x J^\mu (\mathcal{W}_\mu + w_\mu) \\ &= \frac{1}{2} n_0 m u^2 \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \left( |\hat{\mathcal{E}}(\mathbf{k}, \omega)|^2 + \frac{\omega_p^2(\mathbf{k})}{u^2 \mathbf{k}^2} |\hat{\mathcal{B}}(\mathbf{k})|^2 \right) - i \hbar n_0 \int d\tau \int d^2x J^\mu (\mathcal{W}_\mu + w_\mu) \end{aligned} \quad (3.272)$$

where  $h = 2\pi \hbar$  and the phonon dispersion is

$$\omega_p^2(\mathbf{k}) = \frac{n_0}{m} \mathbf{k}^2 \hat{v}(\mathbf{k}) + \frac{\hbar^2 |\mathbf{k}|^4}{4m^2} \quad . \quad (3.273)$$

Thus if  $\hat{v}(\mathbf{0})$  is finite, we may define  $u = \sqrt{n_0 \hat{v}(\mathbf{0})/m}$ , which is the phonon velocity at long wavelengths. The Euclidean Lagrangian density in the long wavelength limit is then

$$\mathcal{L}_E = \frac{1}{2} n_0 m u^2 (\mathcal{E}^2 - \mathcal{B}^2) - 2\pi i \hbar n_0 J^\mu (\mathcal{W}_\mu + w_\mu) \quad , \quad (3.274)$$

which is to say  $(2+1)$ -dimensional electrodynamics in the presence of a background magnetic field  $b = \epsilon_{ij} \partial_i w_j = -1$ . Note that the Fourier transform of the vortex 3-current is

$$\hat{J}^\mu(\mathbf{k}, \omega) = \int d\tau \sum_i q_i \left\{ \frac{1}{\dot{\mathbf{X}}_i} \right\} e^{i\omega\tau} e^{-i\mathbf{k} \cdot \mathbf{X}_i(\tau)} \quad . \quad (3.275)$$

Note also that

$$-2\pi i \hbar n_0 \int_{-\infty}^{\infty} dt J^\mu w_\mu = 2\pi i \hbar n_0 \sum_i q_i \int d\tau X_i(\tau) \dot{Y}_i(\tau) \quad (3.276)$$

gives the geometric phase for vortices winding about the condensate. If  $\hat{v}(\mathbf{0})$  diverges, as is the case for the Coulomb potential  $v(r) = e^2/\epsilon r$ , for which  $\hat{v}(\mathbf{k}) = 2\pi e^2/\epsilon |\mathbf{k}|$ , then there is no long-wavelength low-frequency effective Lorentz invariance with speed  $u$ . For  $v(r) = e^2/\epsilon r$ , one has  $\hat{v}(\mathbf{k}) = 2\pi e^2/\epsilon |\mathbf{k}|$  and the phonon disperses as  $|\mathbf{k}|^{1/2}$  as  $\mathbf{k} \rightarrow 0$ .

### Charged bosons

If the bosons have a charge  $\tilde{e}$ , this may be accommodated by the substitutions

$$\partial_\tau \phi_\nu \rightarrow \partial_\tau \phi_\nu + \tilde{e}A^0 \quad , \quad \nabla \phi_\nu \rightarrow \nabla \phi_\nu + \frac{\tilde{e}}{c} \mathbf{A} \quad , \quad (3.277)$$

where  $A^\mu$  is the electromagnetic vector potential and  $c$  is the speed of light. Note that  $c \gg u$ . One may now define

$$K^\mu \equiv J^\mu + \frac{\tilde{e}}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = \begin{cases} J^0 - \tilde{e}B/2\pi c \\ J^x - \tilde{e}E_y/2\pi \\ J^y + \tilde{e}E_x/2\pi \end{cases} \quad , \quad (3.278)$$

where  $A^\mu \equiv (A^0, c^{-1} \mathbf{A})$ , and replace  $J^\mu$  by  $K^\mu$ , also including the Maxwell term  $(\mathbf{E}^2 - B^2) d/8\pi$  in the (2+1)-dimensional Euclidean Lagrangian density, where  $d$  is the thickness of the system.

### Integrating out the gauge field

Next we integrate out the gauge field  $\mathcal{W}_\mu$ . We define the fields  $\mathcal{W}_\parallel$  and  $\mathcal{W}_\perp$  by

$$\begin{aligned} \mathcal{W}_x(\mathbf{k}, \omega) &= i\hat{k}_x \mathcal{W}_\parallel(\mathbf{k}, \omega) - i\hat{k}_y \mathcal{W}_\perp(\mathbf{k}, \omega) \\ \mathcal{W}_y(\mathbf{k}, \omega) &= i\hat{k}_y \mathcal{W}_\parallel(\mathbf{k}, \omega) + i\hat{k}_x \mathcal{W}_\perp(\mathbf{k}, \omega) \quad , \end{aligned} \quad (3.279)$$

where  $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ . We then have

$$\delta\hat{n}(\mathbf{k}, \omega) = n_0(i\hat{k}_y \mathcal{W}_x - i\hat{k}_x \mathcal{W}_y) = n_0 |\mathbf{k}| \mathcal{W}_\perp(\mathbf{k}, \omega) \quad (3.280)$$

and

$$\begin{aligned} \hat{Q}_x(\mathbf{k}, \omega) &= n_0(-i\omega \mathcal{W}_y - ik_y \mathcal{W}_0) \\ &= n_0(-ik_y \mathcal{W}_0(\mathbf{k}, \omega) + \omega \hat{k}_y \mathcal{W}_\parallel(\mathbf{k}, \omega) + \omega \hat{k}_x \mathcal{W}_\perp(\mathbf{k}, \omega)) \end{aligned} \quad (3.281)$$

and

$$\begin{aligned} \hat{Q}_y(\mathbf{k}, \omega) &= n_0(i\omega \mathcal{W}_x + ik_x \mathcal{W}_0) \\ &= n_0(ik_x \mathcal{W}_0(\mathbf{k}, \omega) - \omega \hat{k}_x \mathcal{W}_\parallel(\mathbf{k}, \omega) + \omega \hat{k}_y \mathcal{W}_\perp(\mathbf{k}, \omega)) \quad . \end{aligned} \quad (3.282)$$

Thus,

$$\begin{aligned} |\delta\hat{n}(\mathbf{k}, \omega)|^2 &= n_0^2 \mathbf{k}^2 |\mathcal{W}_\perp(\mathbf{k}, \omega)|^2 \\ |\hat{Q}(\mathbf{k}, \omega)|^2 &= n_0^2 \omega^2 |\mathcal{W}_\perp(\mathbf{k}, \omega)|^2 + n_0^2 \left| \omega \mathcal{W}_\parallel(\mathbf{k}, \omega) - i|\mathbf{k}| \mathcal{W}_0(\mathbf{k}, \omega) \right|^2 \quad . \end{aligned} \quad (3.283)$$



Next, we write

$$\begin{aligned}
\int d\tau \int d^2x J^\mu \mathcal{W}_\mu &= \sum_i q_i \int d\tau \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} e^{-i\omega\tau} e^{i\mathbf{k}\cdot\mathbf{X}_i(\tau)} \left\{ i\hat{\mathbf{k}} \times \dot{\mathbf{X}}_i \cdot \hat{\mathbf{z}} \mathcal{W}_\perp(\mathbf{k}, \omega) \right. \\
&\quad \left. + \mathcal{W}_0(\mathbf{k}, \omega) + (i\hat{k}_x \dot{X}_i + i\hat{k}_y \dot{Y}_i) \mathcal{W}_\parallel(\mathbf{k}, \omega) \right\} \\
&= \sum_i q_i \int d\tau \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} e^{-i\omega\tau} e^{i\mathbf{k}\cdot\mathbf{X}_i(\tau)} \left\{ i\hat{\mathbf{k}} \times \dot{\mathbf{X}}_i \cdot \hat{\mathbf{z}} \mathcal{W}_\perp(\mathbf{k}, \omega) \right. \\
&\quad \left. + \frac{i}{|\mathbf{k}|} \left( \omega \mathcal{W}_\parallel(\mathbf{k}, \omega) - i|\mathbf{k}| \mathcal{W}_0(\mathbf{k}, \omega) \right) \right\} . \tag{3.284}
\end{aligned}$$

In obtaining the last line above we have used

$$(i\hat{k}_x \dot{X}_i + i\hat{k}_y \dot{Y}_i) e^{i\mathbf{k}\cdot\mathbf{X}_i(\tau)} = \frac{1}{|\mathbf{k}|} \frac{\partial}{\partial \tau} e^{i\mathbf{k}\cdot\mathbf{X}_i(\tau)} , \tag{3.285}$$

and then integrated by parts. We now integrate out the fields  $\mathcal{W}_\perp(\mathbf{k}, \omega)$  and the combination  $\omega \mathcal{W}_\parallel(\mathbf{k}, \omega) - i|\mathbf{k}| \mathcal{W}_0(\mathbf{k}, \omega)$ . Integrating out the latter yields a frequency-independent kernel  $|\mathbf{k}|^{-2}$  and an instantaneous logarithmic Coulomb interaction among the vortices. Thus,

$$\begin{aligned}
S_E &= 2\pi i \hbar n_0 \sum_i q_i \int d\tau X_i(\tau) \dot{Y}_i(\tau) - \frac{\pi \hbar^2 n_0}{m} \sum_{i,j} q_i q_j \int d\tau \ln |\mathbf{X}_i(\tau) - \mathbf{X}_j(\tau)| \\
&\quad + \frac{\pi \hbar^2 n_0}{m} \sum_{i,j} q_i q_j \int d\tau \int d\tau' \int \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \frac{\hat{\mathbf{k}} \wedge \dot{\mathbf{X}}_i(\tau) \hat{\mathbf{k}} \wedge \dot{\mathbf{X}}_j(\tau')}{\omega^2 + \omega_p^2(\mathbf{k})} e^{-i\omega(\tau-\tau')} e^{i\mathbf{k}\cdot(\mathbf{X}_i(\tau) - \mathbf{X}_j(\tau'))} , \tag{3.286}
\end{aligned}$$

where  $\mathbf{a} \wedge \mathbf{b} \equiv \mathbf{a} \times \mathbf{b} \cdot \hat{\mathbf{z}}$ .

Thus, we end up with a theory of logarithmically interacting vortices, whose dynamics are equivalent to those of electrons in the lowest Landau level due to the  $X_i \dot{Y}_i$  term in the Lagrangian, plus a retarded interaction described by the last term. This is  $(2+1)$ -dimensional electrodynamics, where vortices play the role of charges, and phonons play the role of photons.

### 3.5.2 Statistical transmutation

One usually does not think of quantum statistics as a continuous parameter, like a coupling constant. We are accustomed to the notion that many-particle wavefunctions are either symmetric or antisymmetric, *i.e.*  $\Psi(\dots j \dots i \dots) = e^{i\theta} \Psi(\dots i \dots j \dots)$  with  $\theta = 2n\pi$  for bosons and  $\theta = (2n+1)\pi$  for fermions. Other values of  $\theta$ , such as  $\theta = \frac{1}{2}\pi$ , seem to make no sense because iterating the relation twice gives  $\Psi(\dots i \dots j \dots) = e^{2i\theta} \Psi(\dots i \dots j \dots)$ , and thus  $e^{2i\theta} \neq 1$  contradicts single-valuedness of  $\Psi$ . One concludes that Bose and Fermi statistics exhaust all possible values of  $\theta$ .

What happens, though, if we relax the single-valuedness constraint and allow the wavefunction  $\Psi(\{\mathbf{r}_j\})$  to be a *multivalued* function of its arguments? One example of a multivalued function is the complex function  $f(z) = z^\alpha$ , which changes by a factor  $e^{2\pi i\alpha}$  when  $z$  moves counterclockwise around a circle enclosing the origin. Paths which wind around the origin  $n$  times accumulate a phase factor of  $e^{2\pi i n\alpha}$ . If  $\alpha$  is not an integer, then  $f(z)$  returns to its original value multiplied by a phase. Although it may seem strange to consider multivalued wavefunctions, there is nothing that prevents us from doing so. The Schrödinger equation is a differential equation and thus only requires that  $\Psi$  be *locally* well-defined. In addition, physical quantities, such as probability densities, always depend on  $|\Psi|^2$  and are appropriately single-valued, as the multivaluedness we consider will always be in the *phase* of the wavefunction.

In the example  $f(z) = z^\alpha$ ,  $z$  takes its values in the complex plane. In the case of many-particle quantum mechanics, the argument  $\mathbf{R} \equiv \{r_1, \dots, r_N\}$  of  $\Psi(\mathbf{R})$  lives in a more complicated space, called *configuration space*. It is the space of all  $N$ -tuples  $\mathbf{R}$  together with the equivalence relation  $\mathbf{R} \cong \sigma\mathbf{R}$ , where  $\sigma \in \mathcal{S}_N$  is any element of the permutation group, so that  $\sigma\mathbf{R} = \{r_{\sigma(1)}, \dots, r_{\sigma(N)}\}$ . The equivalence of  $\mathbf{R}$  and  $\sigma\mathbf{R}$  means that the particles are indistinguishable. For technical purposes, it is necessary to impose the restriction that no two particles ever lie at the same position – this is necessary for the multivaluedness to be meaningful. This is analogous to the situation in our simple example of  $f(z) = z^\alpha$  above, in which paths that intersect the origin cannot be assigned a definite winding number. Physically, the restriction that no two particles lie atop one another can be accomplished by imposing an infinitely repulsive hard core potential of vanishingly small range; this has no effect on any physical properties.

We now ask what sorts of multivalued functions can be defined on this configuration space. Recall that in the case of the simple example  $f(z) = z^\alpha$  that paths could be classified by an integer winding number  $n$ ; paths which have the same winding number are equivalent to one another in the sense that they can be smoothly deformed into each other without crossing the origin. Associated to each path of winding number  $n$  was a phase  $e^{2\pi i n\alpha}$ . If we append one path of winding number  $n'$  onto a path of winding number  $n$ , the resultant path has winding number  $n + n'$ . Thus, we can think of the space of paths as a mathematical group, and in this simple case, group addition of two paths of winding numbers  $n$  and  $n'$  produces a third path of winding number  $n + n'$ . Mathematically, this result is succinctly stated as

$$\pi_1(\mathbb{R}^2 \setminus \{\mathbf{0}\}) \cong \mathbb{Z} \quad , \quad (3.287)$$

which means that the group of paths (under the operation of path addition) on the punctured plane (the plane minus the origin) is isomorphic to the group of integers (under the operation of addition). Mathematicians refer to the group of paths  $\pi_1(\mathcal{M})$  as the *fundamental group*, or ‘first homotopy group’ *first homotopy group* of the manifold  $\mathcal{M}$ . The fundamental group of the punctured plane is isomorphic to the integers.

The configuration space for  $N$  identical particles living on a base manifold  $\mathcal{M}$  is

$$\mathcal{X}_N(\mathcal{M}) = (\mathcal{M}^N - \mathcal{D})/\mathcal{S}_N \quad , \quad (3.288)$$

where  $\mathcal{D} = \{(\mathbf{r}_1, \dots, \mathbf{r}_N) \mid \mathbf{r}_i = \mathbf{r}_j \text{ for some } i \neq j\}$ . Note as with the punctured plane, we exclude certain subspaces from our manifold, in this case corresponding to coincidences of the positions of at least two particles. This space is more complicated than the punctured plane. The difference is that rather than classifying paths by how they wind around the origin, we classify paths by how the particles wind around other particles. If the base space  $\mathcal{M}$  is  $d$ -dimensional, then  $\dim \mathcal{X}_N(\mathcal{M}) = dN$ . Consider a closed path in this configuration space from a point  $\mathbf{R}$  to an equivalent point  $\sigma\mathbf{R}$ . If  $d \geq 3$ , it is easy to see that any two paths from  $\mathbf{R}$  to  $\sigma\mathbf{R}$  are deformable into one another. Just as loops in  $\mathbb{R}^3$  cannot be classified by a winding number (they can be shrunk to a point without ever crossing the origin), any two configuration space paths  $\mathbf{R}$  to  $\sigma\mathbf{R}$  are *homotopically equivalent*, i.e. they can be deformed into one another. The paths are then classified by  $\sigma$  alone. The mathematicians would say that

$$\pi_1(\mathcal{X}_N(\mathcal{M})) \cong \mathcal{S}_N \quad (\dim(\mathcal{M}) > 2) \quad . \quad (3.289)$$

The phases associated with the paths form a unitary one-dimensional representation of the fundamental group  $\pi_1(\mathcal{X}_N(\mathcal{M}))$ , and so for  $d = \dim(\mathcal{M}) > 2$ , we are left with unitary one-dimensional representations of  $\mathcal{S}_N$ , of which there are only two: the symmetric (Bose) representation,  $e^{i\theta\sigma} = +1$ , and the antisymmetric (Fermi) representation,  $e^{i\theta\sigma} = \text{sgn } \sigma$ .

In two space dimensions, the notion of relative winding of particles becomes well-defined. As a consequence, the space of loops in configuration space becomes more complicated. Indeed, a path in which a particle winds completely around another particle can no longer be deformed to a point without crossing that particle. The fundamental group of configuration space is no longer  $\mathcal{S}_N$ , but rather is an infinite nonabelian group, known as the  $N$ -string *braid group*<sup>67</sup> on  $\mathcal{M}$ , i.e.  $\mathcal{B}_N(\mathcal{M})$ . As its name suggests, the algebra of this group is associated with the weaving of 'braids', which are world-lines for our particles. The phases associated with the paths in configuration space now form a unitary one-dimensional representation of the braid group: to each pairwise exchange of particles one associates a factor  $e^{i\theta}$ . If we let  $z_j = x_j + iy_j$  be the complex coordinate for particle  $j$ , the wavefunction takes the form

$$\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N) = \prod_{i < j} (z_i - z_j)^{\theta/\pi} \Phi(\mathbf{r}_1, \dots, \mathbf{r}_N), \quad (3.290)$$

where  $\Phi(\mathbf{R})$  is a totally symmetric function. Note that  $\theta = \pi$  leads to a function which satisfies Fermi statistics<sup>68</sup>. Intermediate values of  $\theta \in (0, \pi)$  correspond to fractional statistics, first clearly discussed by Leinaas and Myrheim in 1977<sup>69</sup>. The above configuration space analysis

<sup>67</sup>The braid groups were first introduced by E. Artin in 1928. For a review, see R. Fox and L. Neuwirth, *Math. Scand.* **10**, 119 (1962).

<sup>68</sup>It should be noted that conventional wavefunctions satisfying Fermi statistics *are* multivalued in configuration space, since their sign changes depending on the parity of the permutation associated with a given closed path connecting a point  $\mathbf{R}$  to  $\sigma\mathbf{R} \cong \mathbf{R}$ .

<sup>69</sup>J. M. Leinaas and J. Myrheim, *Il Nuovo Cimento B* **37**, 1 (1977). For a pedagogical review, see R. MacKenzie and F. Wilczek, *Int. Jour. Mod. Phys. A* **3**, 2827 (1988).

is due to Laidlaw and DeWitt<sup>70</sup>, who were mostly concerned with  $d = 3$ , and to Wu<sup>71</sup>, who considered the case  $d = 2$  in detail.

In general, if  $q$  is a coordinate on a multiply connected space  $\mathcal{M}$ , the propagator  $K(q_2, t_2 | q_1, t_1)$  may be written as

$$\begin{aligned} K(q_2, t_2 | q_1, t_1) &= \sum_{\mu \in \pi_1(\mathcal{M})} K_\mu(q_2, t_2 | q_1, t_1) \\ &= \sum_{\mu \in \pi_1(\mathcal{M})} \chi(\mu) \sum_{q(t) \in \mu} e^{iS[q(t)]/\hbar} \quad , \end{aligned} \quad (3.291)$$

where the sum over  $\mu$  is over all homotopy sectors in  $\pi_1(\mathcal{M})$ , and where  $\chi(\mu)$  is a unitary one-dimensional representation of  $\pi_1(\mathcal{M})$ <sup>72</sup>. Thus  $\chi(\mu' \circ \mu) = \chi(\mu') \chi(\mu)$ . In general,  $q_1 \neq q_2$  and  $q(t)$  is therefore not a closed loop. But by defining a standard set of paths from an arbitrary point  $q_0 \in \mathcal{M}$  (assuming  $\mathcal{M}$  is connected) to every other point<sup>73</sup>, one can append one of these paths or its inverse to the path  $q(t)$  to create a closed path. In this way, each path  $q(t)$  from  $q_1$  to  $q_2$  can be identified with a homotopy sector.

Paths in configuration space enter the Feynman path integral description of the many-particle propagator, *viz.*

$$K(\mathbf{R}', t_2 | \mathbf{R}, t_1) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \int_{\mathbf{R}}^{\sigma \mathbf{R}'} D\mathbf{R}(t) \exp \left\{ \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left( L(\mathbf{R}, \dot{\mathbf{R}}, t) + \hbar \frac{\theta}{\pi} \sum_{i < j} \dot{\varphi}_{ij} \right) \right\} \quad , \quad (3.292)$$

where the boundary conditions in the  $\sigma$  sector are given by  $\mathbf{R}(t_1) = \mathbf{R} = \{\mathbf{r}_1, \dots, \mathbf{r}_N\}$  and  $\mathbf{R}(t_2) = \sigma \mathbf{R}' = \{\mathbf{r}'_{\sigma(1)}, \dots, \mathbf{r}'_{\sigma(N)}\}$ . Here,  $\varphi_{ij} = \arg(z_i - z_j) = \tan^{-1}[(y_i - y_j)/(x_i - x_j)]$  is the relative angle between particles  $i$  and  $j$ . The  $\dot{\varphi}_{ij}$  term in the Lagrangian keeps track of the relative winding of particles, associating a phase factor  $e^{i\theta}$  to each interchange  $\Delta\varphi_{ij} = \pi$ . Thus, to shift the statistical angle by  $\theta$  one must alter the many-particle Lagrangian:

$$L(\theta) = L(0) + \hbar \frac{\theta}{\pi} \sum_{i < j} \dot{\varphi}_{ij} \quad . \quad (3.293)$$

Since the additional term is a total time derivative, the angle  $\theta$  does not appear in the equations of motion. However, the quantity  $\dot{\varphi}_{ij} dt$  cannot be regarded as an exact differential, because it is not the differential of a *single valued* function of the coordinates  $\mathbf{R}$ . Thus, the 'statistical' part of the action leads to additional phase interference between paths of differing winding number. This is the essence of statistical transmutation.

<sup>70</sup>M. G. G. Laidlaw and C. M. DeWitte, *Phys. Rev. D* **3**, 6 (1971)

<sup>71</sup>Y.-S. Wu, *Phys. Rev. Lett.* **52**, 2103 (1984); Y.-S. Wu, **53**, 111 (1984).

<sup>72</sup>Since all one-dimensional representations are abelian, it is really only the abelianized  $\pi_1(\mathcal{M})$  which matters here. This is called the *first homology group*,  $H_1(\mathcal{M})$ .

<sup>73</sup>Such a construction is known as a *standard path mesh*.

### Charge-flux composites

A compelling realization of fractional statistics was proposed by Wilczek<sup>74</sup> who noted that a composite object consisting of a particle of charge  $e$  and a flux tube of strength  $\phi = \theta\hbar c/e$  would possess fractional statistics. Recall that when a quantum-mechanical particle of charge  $q$  encircles a fixed solenoid of flux  $\phi$ , its wavefunction accrues a phase  $e^{iq\phi/\hbar c}$  – this is the celebrated Aharonov-Bohm effect. The same phase would result from a quantum-mechanical solenoid orbiting around a fixed charge. Now consider two of Wilczek's charge-flux composites and compute the phase they generate upon interchange, which is *half* a complete revolution. There are two contributions to the accumulated phase. A factor  $e^{ie\phi/2\hbar c} = e^{i\theta/2}$  is generated from the *charge* of particle 1 moving in the field of the *flux* of particle 2, and an identical factor arises from the *flux* of particle 1 moving in the field of the *charge* of particle 2. The net accrued phase is thus  $e^{i\theta}$ .

A generic Lagrangian  $L = \frac{1}{2}m\dot{\mathbf{R}}^2 - V(\mathbf{R})$ , altered to account for fractional statistics as in Eqn. 3.293, results in the many-body Hamiltonian

$$H = \sum_i \frac{1}{2m} \left( \mathbf{p}_i - \hbar \frac{\theta}{\pi} \sum_j' \frac{\hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \right)^2 + V(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad , \quad (3.294)$$

where the prime on the sums indicates that the  $j = i$  term is to be excluded. The  $\theta$ -dependent term resembles a 'statistical vector potential'

$$\mathcal{A}_i(\mathbf{R}) = \frac{\theta}{\pi} \cdot \frac{\phi_0}{2\pi} \sum_j' \frac{\hat{\mathbf{z}} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} = \frac{\theta}{\pi} \cdot \frac{\hbar c}{e} \sum_j' \nabla_i \varphi_{ij} \quad , \quad (3.295)$$

where  $\phi_0 = \hbar c/e$  is the Dirac flux quantum. The form of the statistical vector potential is the same as the vector potential of a flux tube of strength  $\phi = 2\theta\hbar c/e$ , which is *twice* the flux of Wilczek's composite. The reason for this is that the statistical vector potential accounts for both the charge-flux and the flux-charge interactions. Note that

$$\mathbf{p}_i - \frac{e}{c} \mathcal{A}_i(\mathbf{R}) = \exp \left( +i \frac{\theta}{\pi} \sum_j' \varphi_{ij} \right) \mathbf{p}_i \exp \left( -i \frac{\theta}{\pi} \sum_j' \varphi_{ij} \right) \quad , \quad (3.296)$$

indicating that the statistical vector potential is a *pure gauge*, although a *topologically nontrivial* one, because the gauge factor is not single-valued as a function of the coordinates  $\mathbf{R}$ . Application of this singular gauge transformation to a symmetric wavefunction yields a multivalued wavefunction of the kind in Eqn. 3.290.

There are thus two equivalent ways to formulate the implementation of fractional statistics in  $d = 2$  space dimensions. We can work with single-valued wavefunctions and include a statistical vector potential in our many-body Hamiltonian. This leads to long-ranged two- and (from

<sup>74</sup>F. Wilczek, *Phys. Rev. Lett.* **48**, 1144 (1982); F. Wilczek, *Phys. Rev. Lett.* **49**, 957 (1982).

the  $\mathcal{A}^2$  term) three-body interactions. Equivalently, we can employ a singular gauge transformation to ‘gauge away’ the statistical vector potential at the cost of requiring multivalued wavefunctions, as in Eqn. 3.290. Wilczek named particles obeying fractional statistics *anyons*, presumably because they can have *any* statistics.

### Statistical transmutation in field theory

Suppose we have a theory with a conserved current  $j^\mu$ , which means  $\partial_\mu j^\mu = 0$ . Here we use the Minkowski metric  $\eta^{\mu\nu} = (+, -, -)$  to raise and lower indices, with  $x^\mu = (t, x, y)$  and  $d^3x = dt dx dy$ . Given a field theory with a conserved matter current  $j^\mu$ , one can transmute statistics to the matter field by coupling this current to a U(1) gauge field  $a^\mu$  and adding a Chern-Simons term<sup>75</sup> to the action, *viz.*

$$S_{\text{mat}}(\theta) = S_{\text{mat}}(0) + \frac{e}{c} \int d^3x j^\mu a_\mu + \frac{e^2}{4\theta\hbar c^2} \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \quad . \quad (3.297)$$

Although the bare  $a_\mu$  field is present in the Chern-Simons term, and not only its field strength  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ , the action remains gauge-invariant because  $\mathcal{J}^\mu = \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$  is a conserved current. Thus, if we make the gauge transformation  $a_\mu \rightarrow a_\mu + \partial_\mu f$ , the change in the Chern-Simons term is

$$S_{\text{CS}} \rightarrow S_{\text{CS}} + \frac{e^2}{4\theta\hbar c^2} \int d^3x \epsilon^{\mu\nu\lambda} \partial_\mu (f \partial_\nu a_\lambda) \quad , \quad (3.298)$$

which vanishes if taken over a closed surface. When taken over a manifold with boundary, an extra contribution must be included at the edge in order to render the action gauge-invariant. We shall discuss this feature further on below.

For example, with nonrelativistic particles we have

$$S_{\text{mat}} \int d^3x \left\{ \sum_{i=1}^N \frac{1}{2} m \dot{\mathbf{x}}_i^2 - V(\mathbf{x}_1, \dots, \mathbf{x}_N) \right\} \quad (3.299)$$

$$j^\mu(x) = \int d\tau \sum_{i=1}^N \delta^{(3)}(x - x_i(\tau)) \frac{dx_i^\mu}{d\tau} \quad .$$

Since the action in Eqn. 3.297 is quadratic in the  $a_\mu$  fields, they can be integrated out simply by solving the equations of motion,

$$\frac{c}{e} \frac{\delta S}{\delta a_\mu} = j^\mu + \frac{e}{4\theta\hbar c} \epsilon^{\mu\nu\lambda} f_{\nu\lambda} = 0 \quad . \quad (3.300)$$

<sup>75</sup>S. Deser, R. Jackiw, and S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982); J. Schonfeld, *Nucl. Phys.* **B185**, 157 (1981).

Thus,

$$j^0 = -\frac{e}{2\theta\hbar c} f_{12} = \frac{eb}{2\theta\hbar c} = \frac{\pi}{\theta} \frac{b}{\phi_0} , \quad (3.301)$$

where  $b = \partial_2 a_1 - \partial_1 a_2 = \partial_x a^y - \partial_y a^x$ . One can now integrate out the  $a_\mu$  fields by manipulating Eqn. 3.300 to obtain

$$\epsilon_{\mu\nu\lambda} j^\lambda = \frac{e}{2\theta\hbar c} (\partial_\nu a_\mu - \partial_\mu a_\nu) . \quad (3.302)$$

Working in the Lorentz gauge  $\partial_\mu a^\mu = 0$ , we can invert the above relation to yield

$$\square a_\mu = \frac{2\theta\hbar c}{e} \eta_{\mu\nu\lambda} \partial^\nu j^\lambda , \quad (3.303)$$

where  $\square = c^{-2} \partial_t^2 - \nabla^2$  is the wave operator<sup>76</sup>. When substituted into the action, this yields

$$\begin{aligned} S_{\text{eff}}(\theta) &= S_{\text{mat}}(0) + \hbar\theta \int d^3x \eta_{\mu\nu\lambda} j^\mu(x) \frac{\partial^\nu}{\square} j^\lambda(x) \\ &= S_{\text{mat}}(0) + 2\hbar\theta N_{\text{link}} , \end{aligned} \quad (3.304)$$

where  $N_{\text{link}}$  is the *linking number* of the particle trajectories. For a complete revolution of one particle around another,  $N_{\text{link}} = 1$ , and thus we associate  $\theta$  with the statistical angle for particle interchange (*i.e.* half a complete revolution).

An explicit calculation is instructive. Define the formally nonlocal operator

$$K^\mu(x - x') = \frac{\partial^\mu}{\square} , \quad (3.305)$$

which satisfies  $\tilde{\partial}_\nu K^\nu(x - x') = \delta^{(3)}(x - x')$ , where  $\tilde{\partial}_\mu = (c^{-2} \partial_t, \nabla)$ . Gauge freedom allows us to take  $K^0 = 0$  and  $K^i(x - x') = k^i(\mathbf{x} - \mathbf{x}') \delta(x^0 - x'^0)$ , with

$$k^i(\mathbf{x} - \mathbf{x}') = \frac{1}{2\pi} \frac{x^i - x'^i}{|\mathbf{x} - \mathbf{x}'|} . \quad (3.306)$$

The function  $k^i(\mathbf{x} - \mathbf{x}')$  is recognized as the vector potential of a flux tube of unit strength. Now let's wind one particle ( $X$ ) around another which stays fixed at the origin. The particle currents are then

$$\begin{aligned} j^0(x) &= \delta(\mathbf{x}) + \delta(\mathbf{x} - \mathbf{X}) \\ \mathbf{j}(x) &= \delta(\mathbf{x} - \mathbf{X}) \dot{\mathbf{X}} , \end{aligned} \quad (3.307)$$

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<sup>76</sup>We have to be a bit careful here since  $\partial_0 = \partial_t$  and  $\partial_{1,2} = \partial_{x,y}$  don't have the same units. The wave operator should be written as  $\square = c^{-2} \partial_t^2 - \nabla^2$ . The symbol  $\eta_{\mu\nu\lambda}$  in Eqn. 3.303 is given by  $\eta_{012} = -\eta_{021} = \eta_{120} = -\eta_{210} = 1$  and  $\eta_{201} = -\eta_{102} = c^{-2}$ , with all other elements vanishing. The simplest way to make everything work out is to pour yourself a nice glass of bourbon and measure space and time in the same units, *i.e.* take  $c = 1$ .

where  $\mathbf{X} = \mathbf{X}(\tau)$ . The linking number term in Eqn. 3.304 then gives

$$- \int d^3x \int d^3x' j^\mu(x) \epsilon_{\mu\nu\lambda} K^\nu(x-x') j^\lambda(x') = -2 \int d\tau \eta_{ij} k^i(-\mathbf{X}) \dot{X}^j = 2\epsilon_{ij} \oint dX^j \frac{X^i}{X^2} \quad (3.308)$$

which is indeed  $2N_{\text{link}}$ . Another way to see it: the geometric flux enclosed by  $\mathbf{X}(\tau)$  as it winds around the origin is

$$\phi = \oint dt \cdot \mathbf{a} = \int dS b = \frac{2\theta\hbar c}{e} \int dS j^0 = \frac{\nu\theta}{\pi} \phi_0 \quad , \quad (3.309)$$

where the charge is taken to be  $\nu e$ . The Aharonov-Bohm phase is then  $e^{2\pi i\nu\phi/\phi_0} = e^{2i\nu^2\theta}$ , which is just what we expect.

Another example comes from the  $(2+1)$ -dimensional  $O(3)$  nonlinear sigma model, with

$$S_{\text{mat}} = \frac{1}{2g} \int d^3x (\partial_\mu n^a)(\partial^\mu n^a) \quad (3.310)$$

$$j^\mu(x) = \frac{1}{8\pi} \epsilon_{abc} \epsilon^{\mu\nu\lambda} n^a \partial_\nu n^b \partial_\lambda n^c \quad ,$$

where  $\hat{n}(x)$  is a unit vector lying along the surface of a two-dimensional sphere. For this model, which possesses a Lorentz invariance, we define  $x^\mu = (ct, \mathbf{x})$ ,  $\partial_\mu = \partial/\partial x^\mu$ , and  $d^3x = c dt dx dy$ . The conservation of  $j^\mu$  licenses us to write

$$j^\mu = \frac{1}{8\pi} \epsilon_{abc} \epsilon^{\mu\nu\lambda} n^a \partial_\nu n^b \partial_\lambda n^c \equiv \epsilon^{\mu\nu\lambda} \partial_\nu \mathcal{A}_\lambda \quad , \quad (3.311)$$

where  $\mathcal{A}_\mu[j]$  is a gauge field. Integrating out the CS gauge field by its equations of motion, we obtain  $a_\mu = -(e/2\theta\hbar c) \mathcal{A}_\mu$ , and hence  $S_{\text{eff}}(\theta) = S_{\text{mat}}(0) + \theta S_{\text{Hopf}}$ , where

$$S_{\text{Hopf}}/\hbar = - \int d^3x j^\mu(x) \mathcal{A}_\mu[j(x)] = - \int d^3x \epsilon^{\mu\nu\lambda} \mathcal{A}_\mu \partial_\nu \mathcal{A}_\lambda \quad , \quad (3.312)$$

which is the so-called *Hopf term*. Note that it is nonlocal in the  $n^a$  fields because  $\mathcal{A}_\mu[j]$  is a *functional* of its argument. At any fixed time  $t$ , we may demand that the field  $\hat{n}(x, t)$  approaches the same value  $\hat{n}(\infty)$  as  $|x| \rightarrow \infty$ . This compactifies  $\mathbb{R}^2 \rightarrow S^2$ . Now consider the function  $\hat{n}(x, t)$  as a function of  $t$  for  $t \in [0, T]$ . Everywhere along this time interval,  $\hat{n}(x, t)$  takes its values on the unit sphere  $S^2$  (we assume  $\hat{n}(\infty, t) = \hat{n}(\infty)$  is fixed as a function of  $t$ ). The function  $\hat{n}(x, t)$  with  $t$  fixed may be regarded as a map taking the compactified real space  $S^2$  to the internal  $\hat{n}$  space  $S^2$ . The space of such maps is called  $\mathcal{Q} = \text{Map}_0(S^2, S^2)$ . Now there is a general result which says that<sup>77</sup>

$$\pi_k(\mathcal{Q}) \cong \pi_{k+2}(S^2) \quad (3.313)$$

<sup>77</sup>T. R. Govindarajan, R. Shankar, N. Shaji, and M. Sivajumar, *Int. Jour. Mod. Phys. A* 8, 3965 (1993).



for all nonnegative integers  $k$ . Recall that  $\pi_k(\mathcal{M})$  is the group of equivalence classes of maps from  $\mathbb{S}^k$  to  $\mathcal{M}$ . In particular,  $\pi_0(\mathcal{Q}) \cong \pi_2(S^2) \cong \mathbb{Z}$ , which says that the configuration space of the  $(2+1)$ -dimensional nonlinear sigma model is disconnected, and separates into individual soliton sectors  $\mathcal{Q}_n$  where  $n \in \mathbb{Z}$ . We also have that  $\pi_1(\mathcal{Q}) \cong \pi_3(S^2) \cong \mathbb{Z}$ , which says that we can associate a phase  $e^{in\theta}$  in the configuration space path integral of Eqn. 3.291 to paths of winding number  $n$ . This is precisely what the Hopf term accomplishes.

### Many-body theory of the anyon gas

The field theory of the many anyon problem is then given by the Lagrangian density

$$\mathcal{L} = \bar{\psi} (i\hbar D_0 - \tilde{\mu}) \psi - \frac{\hbar^2}{2m} \mathbf{D}^* \bar{\psi} \cdot \mathbf{D} \psi - v(\bar{\psi} \psi) + \frac{e^2}{4\theta \hbar c^2} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \quad , \quad (3.314)$$

where  $D_\mu = \partial_\mu - i(e/\hbar c)(A_\mu + a_\mu)$  is the covariant derivative,  $A_\mu$  is the physical electromagnetic vector potential,  $v(\bar{\psi} \psi)$  is the potential energy, and  $\tilde{\mu}$  is the chemical potential for the  $\psi$  field, which may be either fermionic or bosonic. The statistics of the bosons are transmuted to anyons of statistical angle  $\theta$  (or  $\pi + \theta$ ) by the CS term. With fermionic  $\psi$ , the above action serves as a point of departure for the study of the anyon gas. If the physical electromagnetic fields are weak, one may separate the statistical vector potential  $a_\mu = a_\mu^{\text{MF}} + \delta a_\mu$  into a mean field contribution satisfying  $\epsilon^{ji} \partial_i a_j^{\text{MF}} = (\theta/\pi)n\phi_0$ , where  $n$  is the bulk density, and a fluctuating part  $\delta a_\mu$ . One then integrates out the fermion fields  $\psi$  and  $\bar{\psi}$ , generating an effective action in terms of  $A_\mu$  and  $\delta a_\mu$ . Finally, one may attempt to integrate out the  $\delta a_\mu$  fields, generating an effective action in terms of the physical  $A_\mu$  fields alone, from which one can directly obtain the electromagnetic response functions of anyon gases. These developments are clearly discussed in the article by Fradkin<sup>78</sup>. A particularly interesting conclusion is that the anyon gas for statistical angle  $\theta \neq 0, \pi$  should be a superconductor<sup>79</sup>!

To peek just a little bit into how the sausage is made, consider the case of statistical angle  $\theta = \pi + \pi/q$  where  $q \in \mathbb{Z}$ . We now consider Eqn. 3.314 with *fermionic* fields  $\psi$  and  $\bar{\psi}$ , and with  $\theta = \pi/q$ . At the mean field level we have  $b = n\phi_0/q$  where  $n$  is the number density of anyons. In the absence of any external field  $B$ , we have a gas of fermions ( $\psi$  particles) in a uniform magnetic field  $b$ , hence the cyclotron frequency and magnetic length are given by

$$\hbar\omega_c = \frac{\hbar eb}{mc} = \frac{2\pi\hbar^2}{m} \frac{n}{q} \quad , \quad \ell = \sqrt{\frac{\hbar c}{eb}} = \sqrt{\frac{q}{2\pi n}} \quad . \quad (3.315)$$

The LL filling fraction is then  $\nu = 2\pi\ell^2 n = q$ , which means we have an integer number  $q$  of filled LLs. So far, so good.

<sup>78</sup>E. Fradkin, *Phys. Rev. B* **42**, 570 (1990).

<sup>79</sup>A. Fetter, C. Hanna, and R. B. Laughlin, *Phys. Rev. B* **39**, 9679 (1989); Y. Chen, F. Wilczek, E. Witten, and B. I. Halperin, *Int. Jour. Mod. Phys. B* **3**, 1001 (1989).

The ground state energy is then<sup>80</sup>

$$E_0 = N_\phi \sum_{k=0}^{N_\phi-1} (k + \frac{1}{2}) \hbar\omega_c = A \frac{\pi\hbar^2}{m} n^2, \quad (3.316)$$

where  $A$  is the area and  $N_\phi = bA/\phi_0$ . Note that the ground state energy per particle is  $\varepsilon_0 = E_0/N = \pi\hbar^2/mv$  where  $v = 1/n$  is the specific volume (*i.e.* the area per particle). This result is identical to that of a free Fermi gas of the same density. We thus obtain a finite bulk modulus  $\mathcal{B}$  and velocity  $c$  of first (thermodynamic) sound:

$$\mathcal{B} = v \frac{\partial^2 \varepsilon_0}{\partial v^2} = \frac{2\pi\hbar^2}{m} n^2, \quad c = \sqrt{\frac{\mathcal{B}}{mn}} = \frac{\hbar}{m} \sqrt{2\pi n}. \quad (3.317)$$

Note that these expressions are independent of  $q$  and are identical to the corresponding free Fermi gas values. What is missing here is a description of the compressional sound wave; our single particle energy spectrum has a gap of  $\hbar\omega_c$ . The sound wave appears in a more sophisticated random phase approximation (RPA) treatment, as first shown by Fetter, Hanna, and Laughlin (1989).

A simple calculation, due to Chen, Wilczek, Witten, and Halperin (1989), shows that despite the breaking of time-reversal symmetry for  $\theta = \pi + \pi/q$  (and for all  $\theta \neq 0, \pi$ ), the anyon gas wants to expel magnetic flux. To this end, consider our anyon gas in the presence of an external applied magnetic field  $B$  (now parallel to the statistical field  $b$ ). At the mean field level, the effective field strength is  $b + B$ , and as  $B$  is increased, the LL degeneracy  $N_\phi = (b + B)A/\phi_0$  increases. Since the particle number  $N$  remains constant, a fraction  $x$  of the states in the  $q^{\text{th}}$  LL, *i.e.* with LL index  $q - 1$ , will be empty. Number conservation then gives  $(q - x)(b + B) = qb$ , which determines  $x$ . Summing the single particle energies, we obtain

$$E_0(B > 0) = A \frac{\pi\hbar^2 n}{m} \left[ 1 + \frac{B}{qb} - \left( 1 - \frac{1}{q} \right) \frac{B^2}{b^2} \right] \quad (3.318)$$

When  $B < 0$  and the external field is anti-aligned with  $b$ , the effective field strength is  $b + B < b$ . Now the LL degeneracy  $N_\phi$  is smaller, and number conservation requires that a fraction  $y$  of states in the  $(q + 1)^{\text{th}}$  LL (*i.e.* with LL index  $q$ ) are occupied, with  $(n + y)(b + B) = nb$ . Summing once again the single-particle energies, one finds

$$E_0(B < 0) = A \frac{\pi\hbar^2 n}{m} \left[ 1 - \frac{B}{qb} - \left( 1 + \frac{1}{q} \right) \frac{B^2}{b^2} \right]. \quad (3.319)$$

Thus for general  $B$  we have

$$E_0(B) = A \frac{\pi\hbar^2 n}{m} \left[ 1 + \frac{|B|}{qb} - \left( 1 - \frac{1}{q} \text{sgn} B \right) \frac{B^2}{b^2} \right] \quad (3.320)$$

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<sup>80</sup>We assume our anyons are spinless.

and any finite  $B$  initially increases the total energy. Thus the system always wants to expel a weak external field, despite the fact that time-reversal symmetry is explicitly broken!

To demonstrate the Meissner effect, one must perform substantially more refined calculations. RPA calculations yield a London penetration depth of  $\lambda_L = \sqrt{mc^2 d / 4\pi n e^2}$ , where  $d$  is the distance between two-dimensional planes, and a sound wave velocity of  $c = \frac{\hbar}{m} \sqrt{2\pi n}$ , exactly as found above. Thus, the excitation spectrum of the  $\theta = \pi + \pi/q$  anyon gas for  $q \in \mathbb{Z}_+$  is qualitatively different from the spectrum of a Fermi liquid. While the latter exhibits a gapless sound mode, it also exhibits a continuum of particle-hole excitations extending down to zero energy, and which are responsible for various dissipative processes. In the anyon gas, the particle-hole continuum begins at a finite energy  $\hbar\omega_c = 2\pi\hbar^2 n / qm$ , which properly tends to zero in the fermion limit  $q \rightarrow \infty$ , and the *only* gapless excitation is the density wave. This density wave is a Goldstone mode which in an *uncharged* superconductor (*i.e.* a superfluid) would correspond to phase fluctuations of the order parameter in the phase where  $U(1)$  is spontaneously broken. When minimally coupled to electromagnetism, this mode is "eaten"<sup>81</sup> via the Anderson-Higgs mechanism, and the photon becomes massive (*i.e.* the Meissner effect).

### 3.5.3 Cultural interlude

Essentially we have done all the work to derive the CSGL action, which is given in Eqn. 3.314 for a particular choice  $\theta = q\pi$  where  $q$  is an odd integer and where  $\psi$  is a bosonic field. The CS term then transmutes the bosons into fermions, and different choices of odd  $q$ , while representing the same theory at the level of  $\mathcal{L}$ , yield different theories at the mean field level, as we shall soon see. Before engaging with the CSGL theory of the FQHE, though, we take a stroll down memory lane to recall two highlights of the Heroic Era of the FQHE.

#### Girvin-MacDonald order

It was first suggested by Girvin and MacDonald<sup>82</sup> that Laughlin's wavefunction could be understood as a condensate of composite objects consisting of both charge and flux. Specifically, Girvin and MacDonald showed that if one were to adiabatically pierce each electron in the  $\nu = 1/q$  state with a flux tube of strength  $q\phi_0$ , the resulting off-diagonal density matrix,

$$\begin{aligned} \tilde{n}_1(\mathbf{r}, \mathbf{r}') &= \int d^2r_2 \cdots \int d^2r_N \Psi_q^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \exp\left(-\frac{ie}{\hbar c} \int_r^{\mathbf{r}'} d\mathbf{s} \cdot \mathcal{A}(\mathbf{s})\right) \Psi_q(\mathbf{r}', \mathbf{r}_2, \dots, \mathbf{r}_N) \\ &= \int d^2r_2 \cdots \int d^2r_N \tilde{\Psi}_q^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \tilde{\Psi}_q(\mathbf{r}', \mathbf{r}_2, \dots, \mathbf{r}_N) \end{aligned} \quad (3.321)$$

<sup>81</sup>Om nom nom. See E. S. Abers and B. W. Lee, *Phys. Rep.* **9**, 1 (1973).

<sup>82</sup>S. M. Girvin in *The Quantum Hall Effect*, R. Prange and S. M. Girvin, eds. (Springer, 1986); S. M. Girvin and A. H. MacDonald, *Phys. Rev. Lett.* **58**, 1252 (1987).

where  $\mathcal{A}(s) = (q\hbar c/e) \sum_{j=2}^N \nabla \varphi(s - \mathbf{r}_j)$  and

$$\tilde{\Psi}_q(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) = \exp\left(-iq \sum_{j=2}^N \varphi(\mathbf{r} - \mathbf{r}_j)\right) \Psi_q(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \quad , \quad (3.322)$$

decays only *algebraically* at long distances, as  $|\mathbf{r} - \mathbf{r}'|^{-q/2}$ . This follows from the plasma analogy applied to the product

$$\tilde{\Psi}_q^*(\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_N) \tilde{\Psi}_q(\mathbf{r}', \mathbf{r}_2, \dots, \mathbf{r}_N) \equiv \exp\left[-\beta \tilde{H}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_2, \dots, \mathbf{r}_N)\right] \quad , \quad (3.323)$$

again with  $\beta = 1/m$ . One has

$$\begin{aligned} \tilde{H}(\mathbf{r}, \mathbf{r}'; \mathbf{r}_2, \dots, \mathbf{r}_N) = & -2q^2 \sum_{2 \leq i < j}^N \ln |\mathbf{r}_i - \mathbf{r}_j| + \frac{q}{2\ell^2} \sum_{i=2}^N r_i^2 + \frac{q}{4\ell^2} (r^2 + r'^2) \\ & - q^2 \sum_{i=2}^N \left( \ln |\mathbf{r} - \mathbf{r}_i| + \ln |\mathbf{r}' - \mathbf{r}_i| \right) \quad . \end{aligned} \quad (3.324)$$

This corresponds to a system of  $(N + 1)$  logarithmically interacting charges, and a uniform background with the  $(N - 1)$  charge  $\sqrt{2}q$  particles at positions  $\{\mathbf{r}_2, \dots, \mathbf{r}_N\}$ , and two charge  $\frac{1}{\sqrt{2}}q$  particles at positions  $\mathbf{r}$  and  $\mathbf{r}'$  which do not interact with each other. Adding back this interaction gives the desired result,

$$\tilde{n}_1(\mathbf{r}, \mathbf{r}') = C |\mathbf{r} - \mathbf{r}'|^{-q/2} \quad , \quad (3.325)$$

where  $C$  is a dimensionful constant proportional to  $\exp(-\beta F)$ , where  $F$  is the classical free energy of the *fully interacting*  $(N + 1)$ -particle system, *i.e.* where we include the  $-2(\frac{1}{2}q)^2 \ln |\mathbf{r} - \mathbf{r}'|$  interaction between charge  $\frac{1}{\sqrt{2}}q$  test particles at  $\mathbf{r}$  and  $\mathbf{r}'$ . Note that the Girvin-MacDonald result establishes power-law, or *quasi-long-ranged* order. By contrast, the off-diagonal one-body density matrix in the Laughlin state  $\Psi_q$  is given by

$$n_1(\mathbf{r}, \mathbf{r}') = \langle \Psi_q | \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}') | \Psi_q \rangle = \frac{\nu}{2\pi\ell^2} G(\mathbf{r}, \mathbf{r}') \quad , \quad (3.326)$$

where  $G(\mathbf{r}, \mathbf{r}')$  is given in Eqn. 3.242. This follows from the relation  $\langle c_m^\dagger c_n \rangle = \nu \delta_{m,n}$  in any homogeneous state, where  $m$  and  $n$  are angular momentum indices. Thus,  $n_1(\mathbf{r}, \mathbf{r}')$  falls off as a Gaussian in the Laughlin state  $\Psi_q$ , which is very different from the quasi-ODLRO exhibited by  $\tilde{n}(\mathbf{r}, \mathbf{r}')$  in the state  $\tilde{\Psi}_q$ .

### Read's order parameter

Nicholas Read<sup>83</sup> proposed the order parameter operator

$$\phi^\dagger(\mathbf{r}) = \psi^\dagger(\mathbf{r}) U^q(z) \quad (3.327)$$

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<sup>83</sup>N. Read, *Phys. Rev. Lett.* **62**, 86 (1989).

for the Laughlin state  $\Psi_q$ , where  $\psi^\dagger(\mathbf{r})$  is an electron creation operator in second quantized form and  $U(z) = \prod_{i=1}^N (z - z_i)$  is the quasihole creation operator in first quantized form. Defining the  $(N + 1)$ -particle state,

$$|\Phi_q[N + 1]\rangle = \int d^2r e^{-r^2/4\ell^2} \psi^\dagger(\mathbf{r}) U^q(z) |\Psi_q[N]\rangle \quad , \quad (3.328)$$

Now let's calculate the overlap  $\Phi_q(\mathbf{r}_1, \dots, \mathbf{r}_{N+1}) = \langle \mathbf{r}_1, \dots, \mathbf{r}_{N+1} | \Phi_q[N + 1] \rangle$ . We note that

$$\begin{aligned} \psi(\mathbf{r}) |\mathbf{r}_1, \dots, \mathbf{r}_{N+1}\rangle &= \frac{1}{\sqrt{(N+1)!}} \psi(\mathbf{r}) \psi^\dagger(\mathbf{r}_{N+1}) \cdots \psi^\dagger(\mathbf{r}_1) |0\rangle \\ &= \frac{1}{2\pi\ell^2 \sqrt{(N+1)}} \left\{ G(\mathbf{r}, \mathbf{r}_{N+1}) |\mathbf{r}_1, \dots, \mathbf{r}_N\rangle \right. \\ &\quad \left. - G(\mathbf{r}, \mathbf{r}_N) |\mathbf{r}_1, \dots, \mathbf{r}_{N-1}, \mathbf{r}_N\rangle + \dots + (-1)^N G(\mathbf{r}, \mathbf{r}_1) |\mathbf{r}_2, \dots, \mathbf{r}_{N+1}\rangle \right\} \quad , \end{aligned} \quad (3.329)$$

where  $G(\mathbf{r}, \mathbf{r}') = \{\psi(\mathbf{r}), \psi^\dagger(\mathbf{r}')\}$  is given in Eqn. 3.242. Recall that  $G(\mathbf{r}, \mathbf{r}')$  is the Girvin-Jach reproducing kernel for analytic functions, *viz.*

$$\int d^2r' G(\mathbf{r}, \mathbf{r}') f(z') e^{-|z'|^2/4\ell^2} = f(z) e^{-|z|^2/4\ell^2} \quad . \quad (3.330)$$

Thus,

$$\begin{aligned} \Phi_q(\mathbf{r}_1, \dots, \mathbf{r}_{N+1}) &= \frac{1}{\sqrt{(N+1)}} \int d^2r e^{-r^2/4\ell^2} \left\{ G(\mathbf{r}, \mathbf{r}_{N+1}) \Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_N) \prod_{\substack{i=1 \\ (i \neq N+1)}}^{N+1} (z - z_i)^q \right. \\ &\quad \left. - G(\mathbf{r}, \mathbf{r}_N) \Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}, \mathbf{r}_{N+1}) \prod_{\substack{i=1 \\ (i \neq N)}}^{N+1} (z - z_i)^q + \dots \right\} \\ &= (N+1)^{-1/2} \left\{ \prod_{\substack{i=1 \\ (i \neq N+1)}}^{N+1} (z_{N+1} - z_i)^q \Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_N) e^{-r_{N+1}^2/4\ell^2} \right. \\ &\quad \left. - \prod_{\substack{i=1 \\ (i \neq N)}}^{N+1} (z_N - z_i)^q \Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_{N-1}, \mathbf{r}_{N+1}) e^{-r_N^2/4\ell^2} + \dots \right\} \\ &= (N+1)^{1/2} \Psi_q(\mathbf{r}_1, \dots, \mathbf{r}_{N+1}) \quad . \end{aligned} \quad (3.331)$$

This says that the Laughlin state may be written as

$$|\Psi_q[N]\rangle = \frac{1}{\sqrt{N!}} \left( \int d^2r e^{-r^2/4\ell^2} \psi^\dagger(\mathbf{r}) U^q(z) \right)^N |0\rangle \quad , \quad (3.332)$$

which is a condensate of the composite  $\phi^\dagger(\mathbf{r})$ , which creates  $q$  quasiholes and fills them with one electron, thus increasing the electron number by 1. What makes Read's operator so natural is that it is truly a boson and thus can condense. Let's compute the statistical angle  $\Theta$  resulting from the interchange of two Read composites. There are four contributions:

$$\Theta = \pi + q\pi + q\pi + q^2\theta = (q+1)\pi \bmod 2\pi \quad . \quad (3.333)$$

The first of these contributions arises from the exchange of the fermions created by the electron creation operators  $\psi^\dagger$ . Next, there is a phase accrued by the fermions moving in the field of the flux tubes effectively added by the  $U^q$  operators. A single electron encircling  $q$  Dirac flux tubes accrues a phase angle of  $2\pi q$ , and so an exchange gives us half this value, or  $q\pi$ . However, there is an equal phase arising from the flux of one composite orbiting the charge of the other. Finally, there is the statistical angle due to the exchange of the quasiparticles themselves. Since each composite consists of an electron plus  $q$  quasiholes, this last contribution to the statistical angle is  $q^2(\pi/q) = q\pi$  since  $\theta = \pi/q$ . The net statistical angle is thus  $\Theta = (q+1)\pi$ , which is bosonic when  $q$  is odd<sup>84</sup>.

### 3.5.4 The CSGL action

Read's order parameter  $\phi^\dagger(\mathbf{r}) = \psi^\dagger(\mathbf{r}) U^q(z)$  describes an electron bound to  $q$  flux quanta. We may now define a fictitious gauge field  $\mathbf{a}(\mathbf{r})$  whose curl,  $b = \hat{z} \cdot \nabla \times \mathbf{a}$ , satisfies  $\nabla \times \mathbf{a} = q\phi_0 n(\mathbf{r}) \hat{z}$ , hence

$$b(\mathbf{r}) = q\phi_0 \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i) \hat{z} \quad . \quad (3.334)$$

We thus arrive at the cartoon sketched in Fig. 3.12. Suppose  $\dot{N}$  is the number current of charge-flux composites moving across the surface  $\Sigma$ . From Faraday's law  $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ , we have

$$\int_{\Sigma} d\mathbf{l} \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial \Phi}{\partial t} = -\frac{\phi}{c} \dot{N} = V_H \quad (3.335)$$

with  $\phi = q\phi_0$ . We conclude that the Hall conductance is quantized:

$$G_H = \frac{I_c}{V_H} = \frac{-e\dot{N}}{-q(h/e)\dot{N}} = \frac{e^2}{qh} \quad . \quad (3.336)$$

The binding of flux to charge also qualitatively explains why the FQH system is *incompressible*: if the wavefunction  $\Psi$  condenses into a superconducting state, *excess* 'magnetic' flux is expelled. But the magnetic flux is proportional to the particle density, hence there must be a fixed uniform density of particles.

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<sup>84</sup>Note that the generalization of Read's order parameter to the even denominator bosonic FQHE gives  $\Theta = q\pi$  with  $q$  even, because the field operator  $\psi^\dagger$  then creates a boson. So again the order parameter is bosonic.

The CSGL action functional is given by

$$S_{\text{CSGL}}[\Psi, \Psi^*, A^\mu, a^\mu] = S_{\text{mat}}[\Psi, \Psi^*, A^\mu, a^\mu] + S_{\text{CS}}[a^\mu] \quad , \quad (3.337)$$

where

$$S_{\text{mat}}[\Psi, \Psi^*, A^\mu, a^\mu] = \int dt \int d^2x \left\{ \Psi^* \left( i\hbar \partial_t + \frac{e}{c} A^0 + \frac{e}{c} a^0 \right) \Psi - \frac{1}{2m^*} \left| \left( \frac{\hbar}{i} \nabla + \frac{e}{c} \mathbf{A} + \frac{e}{c} \mathbf{a} \right) \Psi \right|^2 \right\} \\ - \frac{1}{2} \int dt \int d^2x \int d^2x' (|\Psi(\mathbf{x}, t)|^2 - n_0) v(\mathbf{x} - \mathbf{x}') (|\Psi(\mathbf{x}', t)|^2 - n_0) \quad (3.338)$$

and

$$S_{\text{CS}}[a^\mu] = \frac{\pi e}{2\theta c \phi_0} \int dt \int d^2x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \quad . \quad (3.339)$$

The CS term transmutes the statistical angle of the  $\Psi$  field from bosonic to the value  $\theta$ , which we may choose as suits our nefarious purposes. For fermionic statistics, we require  $\theta = q\pi$  with  $q$  an odd integer. The background number density is  $n_0$ , and the terms linearly proportional to  $n_0$  determine the chemical potential  $\mu = n_0 \int d^2x v(\mathbf{x}) = \hat{v}(\mathbf{0}) n_0$ ; in Coulomb systems where  $\hat{v}(\mathbf{0})$  diverges, the average number density  $\langle |\Psi|^2 \rangle$  must be  $n_0$  in order to enforce global charge neutrality. Note that the variation of the CS term is given by

$$\delta S_{\text{CS}}[a^\mu] = \frac{\pi e}{\theta c \phi_0} \int dt \int d^2x \left( -b \delta a^0 + c e^y \delta a^x - c e^x \delta a^y \right) \quad , \quad (3.340)$$

where

$$\mathbf{e} = -\frac{1}{c} \left( \nabla a^0 + \frac{\partial \mathbf{a}}{\partial t} \right) \quad , \quad b = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{a} = \frac{\partial a^y}{\partial x} - \frac{\partial a^x}{\partial y} \quad (3.341)$$

are the fictitious electric and magnetic field strengths associated with the CS gauge field. Note that we have chosen our factors such that  $e$  and  $b$  have the same dimensions as the physical electromagnetic field strengths  $\mathbf{E}$  and  $B$  (in cgs units<sup>85</sup>).

The bosonic number density and number current are given by

$$n(\mathbf{x}, t) = +\frac{c}{e} \frac{\delta S}{\delta A^0(\mathbf{x}, t)} = |\Psi(\mathbf{x}, t)|^2 \quad (3.342) \\ \mathbf{j}(\mathbf{x}, t) = -\frac{c}{e} \frac{\delta S}{\delta \mathbf{A}(\mathbf{x}, t)} = \frac{\hbar}{m^*} \text{Im} [\Psi^*(\mathbf{x}, t) \nabla \Psi(\mathbf{x}, t)] + \frac{e}{m^* c} |\Psi(\mathbf{x}, t)|^2 (\mathbf{A}(\mathbf{x}, t) + \mathbf{a}(\mathbf{x}, t))$$

Thus the functional variation of the complete CSGL action with respect to the components of the gauge field  $a^\mu$  is given by

$$\frac{c}{e} \frac{\delta S}{\delta a^0(\mathbf{x}, t)} = n(\mathbf{x}, t) - \frac{\pi}{\theta} \frac{b(\mathbf{x}, t)}{\phi_0} \quad (3.343) \\ -\frac{c}{e} \frac{\delta S}{\delta \mathbf{a}(\mathbf{x}, t)} = \mathbf{j}(\mathbf{x}, t) - \frac{\pi}{\theta} \frac{c \mathbf{e}(\mathbf{x}, t) \times \hat{\mathbf{z}}}{\phi_0} \quad ,$$

<sup>85</sup>Which are God's units.

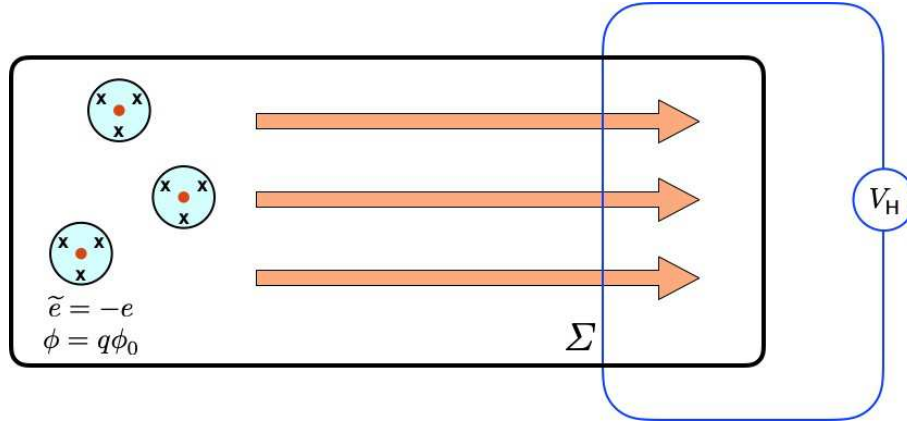


Figure 3.12: Schematic picture of transport for charge-flux composites. The charge current across the surface  $\Sigma$  is  $I_c = -e \partial_t N$  while the vortex current is  $I_v = \phi \partial_t N$ .

We may also vary the action with respect to the field  $\Psi^*(\mathbf{x})$ :

$$\begin{aligned} \frac{\delta S}{\delta \Psi^*(\mathbf{x}, t)} = & \left[ i\hbar \frac{\partial}{\partial t} + \frac{e}{c} (A^0 + a^0) - \frac{1}{2m^*} \left( \frac{\hbar}{i} \nabla + \frac{e}{c} (\mathbf{A} + \mathbf{a}) \right)^2 \right] \Psi(\mathbf{x}, t) \\ & - \left[ \int d^2 x' v(\mathbf{x} - \mathbf{x}') (|\Psi(\mathbf{x}', t)|^2 - n_0) \right] \Psi(\mathbf{x}, t) \quad . \end{aligned} \quad (3.344)$$

### 3.5.5 Mean field solution

We now write  $a_\mu = a_\mu^{\text{MF}} + \delta a_\mu$  and  $\Psi = \Psi^{\text{MF}} + \delta \Psi$ . The mean field level solution is obtained by setting  $\Psi^{\text{MF}} = \sqrt{n_0} e^{i\phi}$  where  $\phi$  is an arbitrary phase, and setting  $A_\mu + a_\mu^{\text{MF}} = 0$ . In the Ginzburg-Landau theory of superconductivity,  $\phi$  would be the phase of the superconducting condensate. We are free to choose a gauge in which  $A_0 = a_0^{\text{MF}} = 0$ , but for the spatial components the condition  $\mathbf{A} + \mathbf{a}^{\text{MF}} = 0$  requires that  $B + b^{\text{MF}} = 0$ , *i.e.* the Chern-Simons magnetic field  $b$  cancels the applied magnetic field  $-B$ . Note that we are again taking the applied field as  $\mathbf{B} = -B\hat{z}$ , whereas  $\mathbf{b} = +b\hat{z}$ . So  $b^{\text{MF}} = B$ . But we also have the condition from the first of Eqns. 3.343, which demands that

$$b = B = \frac{\theta}{\pi} n_0 \phi_0 = q n_0 \phi_0 \quad , \quad (3.345)$$

where  $\theta = \pi q$  for Fermi statistics, with  $q$  odd. Note that in the original field theory, the CS term attaches an infinitely narrow flux tube to each point particle. Therefore the Aharonov-Bohm flux is only sensitive to  $\theta \bmod 2\pi$ , and all odd integer  $q$  are equivalent. Not so at the mean field level! Each odd  $q$  results in a *different* mean field solution, with  $N_\phi = BA/\phi_0 = qN$ , *i.e.*  $\nu = N/N_\phi = q^{-1}$ .

Suppose the condensate field  $\Psi(\mathbf{x})$  contains a vortex centered at the origin. Asymptotically,



as  $r = |\mathbf{x}| \rightarrow \infty$ , we must then have  $\Psi(r, \varphi) = \sqrt{n_0} \exp(\pm i\varphi)$  and  $\delta \mathbf{a}(r, \varphi) = \pm(\hbar c/e) \hat{\varphi}/r$ , where  $\varphi$  is the azimuthal angle. But this corresponds to a variation in the total CS flux  $\Phi_{\text{CS}}$ , with

$$\delta \Phi_{\text{CS}} = \delta \int d^2x b(\mathbf{x}) = \oint_{r=\infty} d\mathbf{l} \cdot \delta \mathbf{a} = \phi_0 \quad . \quad (3.346)$$

But then the first of Eqns. 3.343 says that there is a concomitant change in particle number, with

$$\delta N = \frac{\pi}{\theta} \frac{\delta \Phi_{\text{CS}}}{\phi_0} = \frac{1}{q} \quad . \quad (3.347)$$

This is the fractionally charged quasiparticle! The antivortex corresponds to the quasihole.

### 3.5.6 Fluctuations about the mean field

Let us write  $A^\mu = \bar{A}^\mu + \delta A^\mu$  where  $\bar{A}^\mu$  is the electromagnetic 3-vector potential corresponding to uniform magnetic field  $\mathbf{B} = -B\hat{z}$  and electric field  $\mathbf{E} = 0$ , with  $B$  fixed by the condition  $\nu = n\phi_0/B = 1/q$  with  $q$  an odd integer. Similarly we write  $a^\mu = \bar{a}^\mu + \delta a^\mu$  with  $\bar{a}_\mu = -\bar{A}_\mu = a_\mu^{\text{MF}}$ . The action is then

$$\begin{aligned} S = \int d^3x \left\{ \hbar \Psi^* (i\partial_t + \frac{e}{\hbar c} \delta \mathbf{a}_0) \Psi - \frac{\hbar^2}{2m} \left| (i\nabla - \frac{e}{\hbar c} \delta \vec{\mathbf{a}}) \Psi \right|^2 \right\} \\ - \frac{1}{2} \int dt \int d^2x \int d^2x' (|\Psi(\mathbf{x}, t)|^2 - n_0) v(\mathbf{x} - \mathbf{x}') (|\Psi(\mathbf{x}', t)|^2 - n_0) \\ + \frac{\pi e}{2\theta c \phi_0} \int d^3x \epsilon^{\mu\nu\lambda} \left[ (\delta \mathbf{a}_\mu - \delta A_\mu) \partial_\nu (\delta \mathbf{a}_\lambda - \delta A_\lambda) - 2 \delta a_\mu \partial_\nu \bar{A}_\lambda \right] \quad , \end{aligned} \quad (3.348)$$

where  $\delta a_\mu \equiv \delta a_\mu + \delta A_\mu$ . We now follow the method outlined in §3.5.1, now in real rather than Euclidean time, writing  $\Psi = \sqrt{n} e^{i\phi}$ , which results in the matter component of the action

$$\begin{aligned} S_{\text{mat}} = \int d^3x \left\{ -\hbar n (\partial_t \phi - \frac{e}{\hbar c} \delta \mathbf{a}_0) - \frac{\hbar^2 n}{2m^*} (\nabla \phi + \frac{e}{\hbar c} \delta \vec{\mathbf{a}})^2 - \frac{\hbar^2}{8m^* n} (\nabla n)^2 \right\} \\ - \frac{1}{2} \int dt \int d^2x \int d^2x' (n(\mathbf{x}, t) - n_0) v(\mathbf{x} - \mathbf{x}') (n(\mathbf{x}', t) - n_0) \quad . \end{aligned} \quad (3.349)$$

We again use a Hubbard-Stratonovich transformation to replace

$$-\frac{\hbar^2 n}{2m^*} (\nabla \phi + \frac{e}{\hbar c} \delta \vec{\mathbf{a}})^2 \quad \longrightarrow \quad \frac{m^* \mathbf{Q}^2}{2n} - \hbar \mathbf{Q} \cdot (\nabla \phi + \frac{e}{\hbar c} \delta \vec{\mathbf{a}}) \quad (3.350)$$

Integrating by parts, the contribution from  $\phi_{\text{sw}}$  in the Lagrangian density is  $\hbar(\partial_t n + \nabla \cdot \mathbf{Q}) \phi_v$ , hence integrating out  $\phi_v$  results in the constraint  $\partial_t n + \nabla \cdot \mathbf{Q} = 0$ , which we solve by writing

$$K^\mu \equiv (n, \mathbf{Q}) \equiv n_0 \epsilon^{\mu\nu\lambda} \partial_\nu (\mathcal{W}_\lambda + w_\lambda) \quad . \quad (3.351)$$

Here  $n$ ,  $Q_x$ , and  $Q_y$  are the three components of the 3-vector  $K^\mu$ .<sup>86</sup> Here  $w^\mu = (0, 0, -x)$  is a background vector potential corresponding to electric field  $e = 0$  and magnetic field  $b = -1$  which will help us keep track of the Berry phase as a vortex in the  $\Psi$  field moves in the presence of the background condensate, as in §3.5.1. We may write

$$\begin{aligned} n - n_0 &= n_0 (\partial_x \mathcal{W}_y - \partial_y \mathcal{W}_x) \equiv -n_0 \mathcal{B} \\ Q_x &= n_0 (\partial_y \mathcal{W}_t - \partial_t \mathcal{W}_y) \equiv -n_0 u \mathcal{E}_y \\ Q_y &= n_0 (\partial_t \mathcal{W}_x - \partial_x \mathcal{W}_t) \equiv +n_0 u \mathcal{E}_x \quad , \end{aligned} \quad (3.352)$$

where  $u$ , to be determined, has dimensions of speed. The dimensionless quantities  $\mathcal{E}$  and  $\mathcal{B}$  are the "electric" and "magnetic" fields derived from the vector potential  $\mathcal{W}^\mu$ .

Again we separate  $\phi = \phi_{\text{sw}} + \phi_v$  into a smooth 'spin-wave' part and a singular vortex part, where

$$J_v^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda \phi_v = \sum_i q_i \left\{ \frac{1}{\dot{\mathbf{X}}_i} \right\} \delta(\mathbf{x} - \mathbf{X}_i(\tau)) \quad . \quad (3.353)$$

The matter component of the action may now be written

$$\begin{aligned} S_{\text{mat}} &= \int d^3x \left\{ \frac{en_0}{c} \epsilon^{\mu\nu\lambda} \delta a_\mu \partial_\nu (\mathcal{W}_\lambda + w_\lambda) + \frac{1}{2} n_0 m^* u^2 \frac{\mathcal{E}^2}{1 - \mathcal{B}} - \frac{\hbar^2 n_0}{8m^*} \frac{(\nabla \mathcal{B})^2}{1 - \mathcal{B}} \right. \\ &\quad \left. - 2\pi \hbar n_0 J_v^\mu (\mathcal{W}_\mu + w_\mu) \right\} - \frac{1}{2} n_0^2 \int dt \int d^2x \int d^2x' \mathcal{B}(\mathbf{x}, t) v(\mathbf{x} - \mathbf{x}') \mathcal{B}(\mathbf{x}', t) \quad . \end{aligned} \quad (3.354)$$

If the Fourier transform of the potential  $\hat{v}(\mathbf{k})$  is finite at  $\mathbf{k} = 0$ , the constant  $u$  may be defined by  $u \equiv \sqrt{n_0 \hat{v}(\mathbf{0})/m^*}$ , as in §3.7. The term proportional to  $(\nabla \mathcal{B})^2$  is fourth order in derivatives of  $\mathcal{W}^\mu$ ; this term of course yields the crossover to a ballistic dispersion in the phonon spectrum of the superfluid in the ultraviolet regime. In the infrared, we may neglect it, and we have

$$\begin{aligned} S_{\text{mat}} &= \int d^3x \left\{ \frac{1}{2} n_0 m^* u^2 (\mathcal{E}^2 - \mathcal{B}^2) + \frac{en_0}{c} \epsilon^{\mu\nu\lambda} \delta a_\mu \partial_\nu (\mathcal{W}_\lambda + w_\lambda) - 2\pi \hbar n_0 J_v^\mu (\mathcal{W}_\mu + w_\mu) \right\} \\ S_{\text{CS}} &= \frac{\pi e}{2\theta c \phi_0} \int d^3x \epsilon^{\mu\nu\lambda} \left\{ (\delta a_\mu - \delta A_\mu) \partial_\nu (\delta a_\lambda - \delta A_\lambda) - 2 \delta a_\mu \partial_\nu \bar{A}_\lambda \right\} \end{aligned} \quad (3.355)$$

We now integrate out the gauge field  $\delta a_\mu$  using the equations of motion, which yield

$$\begin{aligned} \delta a_\mu &= \delta A_\mu - \frac{\theta}{\pi} n_0 \phi_0 (\mathcal{W}_\mu + w_\mu) + \bar{A}_\mu \\ &= \delta A_\mu - \frac{\theta}{\pi} n_0 \phi_0 \mathcal{W}_\mu \quad , \end{aligned} \quad (3.356)$$

<sup>86</sup>On spatial 2-vectors like  $\mathbf{Q}$  and the electric field  $\mathbf{E}$ , we do not distinguish between raised and lowered indices, hence  $Q^x = Q_x$  etc. We shall endeavor to always use lowered indices for such 2-vectors.

since  $B = \frac{\theta}{\pi} n_0 \phi_0$  yields  $\bar{A}_\mu = \frac{\theta}{\pi} n_0 \phi_0 w_\mu$ . Substituting this back into the action, we obtain the Lagrangian density

$$\mathcal{L}_{\text{eff}} = \frac{1}{2} n_0 m^* u^2 (\mathcal{E}^2 - \mathcal{B}^2) - \hbar \theta n_0^2 \epsilon^{\mu\nu\lambda} \mathcal{W}_\mu \partial_\nu \mathcal{W}_\lambda + \frac{en_0}{c} \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu \mathcal{W}_\lambda - 2\pi \hbar n_0 J_\nu^\mu (\mathcal{W}_\mu + w_\mu) \quad . \quad (3.357)$$

Note that there are two quantities here with dimensions of speed:  $c$  and  $u$ , with  $u \ll c$ .

If  $\hat{v}(\mathbf{0})$  diverges, which is indeed the case when  $v(\mathbf{r}) = e^2/\epsilon r$  and  $\hat{v}(\mathbf{k}) = 2\pi e^2/\epsilon k$ , then there is no effective Lorentz symmetry in the superfluid component of the matter Lagrangian. In this case we have

$$S_{\text{eff}} = \int d^3x \left\{ \frac{1}{2} n_0 m^* (\nabla \mathcal{W}^0 + \partial_t \mathcal{W})^2 - \hbar \theta n_0^2 \epsilon^{\mu\nu\lambda} \mathcal{W}_\mu \partial_\nu \mathcal{W}_\lambda + \frac{en_0}{c} \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu \mathcal{W}_\lambda - 2\pi \hbar n_0 J_\nu^\mu (\mathcal{W}_\mu + w_\mu) \right\} - \frac{1}{2} n_0^2 \int dt \int \frac{d^2q}{(2\pi)^2} \hat{v}(\mathbf{q}) |\mathbf{q} \times \hat{\mathcal{W}}(\mathbf{q})|^2 \quad . \quad (3.358)$$

At this point, we may integrate out the gauge field  $\mathcal{W}_\mu$ . Since the Maxwell term in Eqn. 3.357 or its corrected version in Eqn. 3.358 both involve one higher derivative than the induced Chern-Simons term  $\mathcal{W}d\mathcal{W}$ , we will ignore the former. Varying with respect to  $\mathcal{W}_\mu$ , we obtain

$$\epsilon^{\mu\nu\lambda} \partial_\nu \mathcal{W}_\lambda = -\frac{\pi}{\theta n_0} \left( J_\nu^\mu - \frac{1}{2\phi_0} \epsilon^{\mu\nu\lambda} \delta F_{\nu\lambda} \right) \quad , \quad (3.359)$$

where  $\delta F_{\nu\lambda} = \partial_\nu \delta A_\lambda - \partial_\lambda \delta A_\nu$  is the field strength tensor corresponding to  $\delta A_\mu$ . Thus

$$\mathcal{W}_\mu = \frac{\pi}{\theta n_0} \left( \frac{1}{\phi_0} \delta A_\mu - \eta_{\mu\nu\lambda} \frac{\partial^\nu}{\square} J_\nu^\lambda \right) \quad . \quad (3.360)$$

Inserting this into  $\mathcal{L}_{\text{eff}}$  yields our final result<sup>87</sup>,

$$\mathcal{L}(J_\nu, \delta A) = -2\pi \hbar n_0 J_\nu^\mu w_\mu - \frac{\pi}{\theta} \frac{e}{c} J_\nu^\mu \delta A_\mu + \frac{e^2}{4\theta \hbar c^2} \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu \delta A_\lambda + \hbar \frac{\pi^2}{\theta} \eta_{\mu\nu\lambda} J_\nu^\mu \frac{\partial^\nu}{\square} J_\nu^\lambda \quad . \quad (3.361)$$

Thus, we find:

- The vortices of the CSGL condensate accrue a Berry phase in traversing a loop  $\mathcal{C}$  of  $\gamma_{\mathcal{C}} = -2\pi N_0(\mathcal{C})$ , where  $N_0(\mathcal{C}) = n_0 A(\mathcal{C})$  is the condensate number density times the area enclosed by the loop, which is to say the average number of condensate particles encircled.
- The vortex current  $J_\nu^\mu$  is minimally coupled to fluctuations  $\delta A_\mu$  in the physical electromagnetic field, with an effective charge  $e^* = \pi e/\theta = e/q$  in the Laughlin state at  $\nu = 1/q$ .

<sup>87</sup>Recall that  $\square = c^{-2} \partial_t^2 - \nabla^2$  and  $\eta_{012} = -\eta_{021} = \eta_{120} = -\eta_{210} = 1$  with  $\eta_{201} = -\eta_{102} = c^{-2}$ .

- There is an induced Chern-Simons term in the physical electromagnetic vector potential. Varying with respect to the physical electromagnetic field, we have<sup>88</sup>

$$J^\mu = -c \frac{\delta S}{\delta A_\mu} = -\frac{\pi}{\theta} e J_V^\mu + \frac{e^2}{2\hbar c} \epsilon^{\mu\nu\lambda} \partial_\mu \delta A_\lambda \quad , \quad (3.362)$$

and with the physical electric field given by  $\mathbf{E} = -c^{-1}(\nabla\delta A^0 + \partial_0 \delta \mathbf{A})$ , we obtain

$$\mathbf{J} = \frac{e^2}{qh} \hat{\mathbf{z}} \times \mathbf{E} - \frac{e}{q} \mathbf{J}_V \quad . \quad (3.363)$$

This tells us that in the absence of vortices (*i.e.* quasiparticles) there is a quantized Hall effect  $\sigma_{yx} = e^2/qh$ . Since vortices are charged, when present they carry an electrical current. However, the random potential from the displaced dopant ions produces many *pinning sites* for vortices, and a pinned vortex, which remains spatially localized, carries zero current.

- Comparing with Eqn. 3.304, we see that the vortices of our theory are anyons with a statistical angle  $\vartheta = \pi^2/\theta$ . Thus for  $\theta = q\pi$  we have  $\vartheta = \pi/q$ , exactly as the adiabatic calculation of §3.3.6 concluded. Had we included the Maxwell term, the combination of CS and Maxwell terms would have attached a smeared flux tube to each of the vortices, where the length scale of the smearing is  $d = m^*u^2/\theta\hbar n_0c$ .

What we *don't* have here is the  $1/r$  Coulomb interaction between the vortices, which as we have seen possess finite charge  $\pm e/q$ . In fact, this is indeed included in the last term of the action of Eqn. 3.355, but we have neglected the long range part of  $v(\mathbf{r} - \mathbf{r}')$  in deriving Eqn. 3.361. More on this below in §3.5.8.

### Remarks on units

In our units,  $x^\mu = (t, x, y)$  and  $d^3x = dt dx dy$ . The units for the components of particle 3-current and vector potentials are

$$[n] = [j^0] = L^{-2} \quad , \quad [j] = L^{-1}T^{-1} \quad , \quad [eA^0] = ELT^{-1} \quad , \quad [e\mathbf{A}] = E \quad , \quad (3.364)$$

where  $L$  stands for length,  $T$  for time, and  $E$  for energy. Thus  $[\frac{e}{c} j_\mu A^\mu] = EL^{-2}$ , *i.e.* energy density. Since  $\alpha^{-1} = \hbar c/e^2 \approx 137.036$  is the inverse fine structure constant, we have

$$[e^2] = [\hbar c] = EL \quad \Rightarrow \quad [e] = E^{1/2}L^{1/2} \quad . \quad (3.365)$$

Note that the physical electric and magnetic fields have dimensions  $[\mathbf{E}] = [B] = EL^{-3}$ , which agrees with

$$EL^{-3} = [B^2] = [n]^2 [\phi_0]^2 = L^{-4} \cdot \frac{[\hbar c]^2}{[e^2]} = \frac{E^2L^{-2}}{EL} = EL^{-3} \quad . \quad (3.366)$$

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<sup>88</sup>Note that  $J^\mu = -ej^\mu$  is the *electrical current* of the Chern-Simons bosons, while  $J_V^\mu$  is the *number current* of the vortices.

### 3.5.7 Superfluid response and CSGL theory

The action  $S_{\text{mat}}$  in Eqn. 3.338 corresponds to the (first quantized) Hamiltonian

$$H(\delta\mathbf{a}) = \frac{1}{2m^*} \sum_{i=1}^N \left( \mathbf{p}_i + \frac{e}{c} \delta\vec{\mathbf{a}}_i \right)^2 - \frac{e}{c} \sum_{i=1}^N \delta a_i^0 + \sum_{i<j} v(\mathbf{r}_i - \mathbf{r}_j) \quad . \quad (3.367)$$

Here we are invoking the shifted CS gauge potential,  $\delta\mathbf{a}_\mu = \delta a_\mu + \delta A_\mu$  and  $A_\mu = \bar{A}_\mu + \delta A_\mu$ , as in §?. The CS action in terms of  $\delta\mathbf{a}$  and  $\delta A$  is given by

$$S_{\text{CS}} = \frac{\pi e}{2\theta c \phi_0} \int d^3x \epsilon^{\mu\nu\lambda} \left\{ (\delta\mathbf{a}_\mu - \delta A_\mu) \partial_\nu (\delta\mathbf{a}_\lambda - \delta A_\lambda) - 2 \delta\mathbf{a}_\mu \partial_\nu \bar{A}_\lambda \right\} \quad . \quad (3.368)$$

Thus, we have

$$H(\delta\mathbf{a}) = H(0) - \frac{e}{c} \int d^2x j_\mu^{\text{p}}(\mathbf{x}) \delta a^\mu(\mathbf{x}) + \frac{e^2}{2m^*c^2} \int d^2x n(\mathbf{x}) [\delta\vec{\mathbf{a}}(\mathbf{x})]^2 \quad , \quad (3.369)$$

where  $j_\mu^{\text{p}}$  is the paramagnetic current. A review of the linear response formalism is given in §3.8 below.

The effective action, once matter fields are integrated away, is given by<sup>89</sup>

$$\begin{aligned} S_{\text{eff}}[\delta a, \delta A] &= \frac{1}{8\pi} \int d^3x \int d^3x' \delta a^\mu(\mathbf{x}, t) K_{\mu\nu}(\mathbf{x} - \mathbf{x}', t - t') \delta a^\nu(\mathbf{x}', t') + S_{\text{CS}}[\delta a, \delta A] \\ &= \frac{1}{8\pi} \int \frac{d^3q}{(2\pi)^3} \begin{pmatrix} \delta a^\mu(-q) & \delta A^\mu(-q) \end{pmatrix} \begin{pmatrix} \hat{K}_{\mu\nu}(q) + \hat{L}_{\mu\nu}(q) & -\hat{L}_{\mu\nu}(q) \\ -\hat{L}_{\mu\nu}(q) & \hat{L}_{\mu\nu}(q) \end{pmatrix} \begin{pmatrix} \delta a^\nu(q) \\ \delta A^\nu(q) \end{pmatrix} \end{aligned} \quad (3.370)$$

where  $K_{\mu\nu}(\mathbf{x} - \mathbf{x}', t - t')$  is the electromagnetic response tensor derived in §3.8, and where we retain only finite wavevector components of the gauge fields. Here it proves useful to redefine  $\delta a^0 \rightarrow \delta a^0/c$  and  $\delta A^0 \rightarrow \delta A^0/c$  so that all the components of  $\delta a^\mu$  and  $\delta A^\mu$  have the same dimensions. Similarly, we take  $x^\mu = (ct, \mathbf{x})$  and  $q^\mu = (c^{-1}\omega, \mathbf{q})$ . Then

$$\hat{K}_{\mu\nu}(q) = \begin{pmatrix} (c^2\mathbf{q}^2/\omega^2) \hat{K}_{\parallel}(q) & (cq_j/\omega) \hat{K}_{\parallel}(q) \\ (cq_i/\omega) \hat{K}_{\parallel}(q) & \hat{q}_i \hat{q}_j \hat{K}_{\parallel}(q) + (\delta_{ij} - \hat{q}_i \hat{q}_j) \hat{K}_{\perp}(q) \end{pmatrix} \quad (3.371)$$

and

$$\hat{L}_{\mu\nu}(q) = \frac{2\pi i \alpha}{\theta c} \begin{pmatrix} 0 & -cq_2 & +cq_1 \\ +cq_2 & 0 & +\omega \\ -cq_1 & -\omega & 0 \end{pmatrix} \quad , \quad (3.372)$$

where  $\alpha = e^2/\hbar c$  is the fine structure constant. Note that both matrices are Hermitian.

<sup>89</sup>We only include fields at nonzero wavelength and/or frequency.

Now we integrate out  $\delta a^\mu$ , resulting in the new reduced effective action

$$S_{\text{red}}[\delta A^\mu] = \frac{1}{8\pi} \int \frac{d^3q}{(2\pi)^3} \delta A^\mu(-q) \left( \hat{L}(q) - \hat{L}(q) [\hat{K}(q) + \hat{L}(q)]^{-1} \hat{L}(q) \right)_{\mu\nu} \delta A^\nu(q) \quad . \quad (3.373)$$

To inver the matrices  $\hat{K}_{\mu\nu}$  and  $\hat{L}_{\mu\nu}$ , it is convenient to work solely with components of  $q$  with lowered indices, since these matrices are expressed above in those variables. We define the orthonormal triad,

$$\psi_{0,\mu} = \frac{1}{\sqrt{\omega^2 + c^2\mathbf{q}^2}} \begin{pmatrix} \omega \\ -cq_1 \\ -cq_2 \end{pmatrix} \quad , \quad \psi_{1,\mu} = \frac{1}{|\mathbf{q}|} \begin{pmatrix} 0 \\ -q_2 \\ +q_1 \end{pmatrix} \quad , \quad \psi_{2,\mu} = \frac{1}{|\mathbf{q}|\sqrt{\omega^2 + c^2\mathbf{q}^2}} \begin{pmatrix} c\mathbf{q}^2 \\ \omega q_1 \\ \omega q_2 \end{pmatrix} \quad , \quad (3.374)$$

which satisfy the orthogonality relations  $\sum_\mu \psi_{a,\mu}(q) \psi_{b,\mu}(q) = \delta_{ab}$  and the completeness relations  $\sum_a \psi_{a,\mu}(q) \psi_{a,\nu}(q) = \delta_{\mu\nu}$  for all  $q$ . Suppressing  $q$ , one readily obtains

$$\hat{K} |0\rangle = 0 \quad , \quad \hat{K} |1\rangle = \hat{K}_\perp |1\rangle \quad , \quad \hat{K} |2\rangle = \tilde{K}_\parallel |2\rangle \quad , \quad (3.375)$$

where

$$\tilde{K}_\parallel(\mathbf{q}, \omega) \equiv \left( \frac{c^2\mathbf{q}^2}{\omega^2} + 1 \right) \hat{K}_\parallel(\mathbf{q}, \omega) = - \left( 1 + \frac{\omega^2}{c^2\mathbf{q}^2} \right) 4\pi e^2 \hat{\chi}(\mathbf{q}, \omega) \quad , \quad (3.376)$$

where  $\hat{\chi}(\mathbf{q}, \omega)$  is the scalar susceptibility of the corresponding neutral superfluid (see §3.8.3); note that  $\hat{\chi}(\mathbf{q} \rightarrow 0, 0) = n^2 \kappa_T$  is finite. We also have

$$\hat{L} |0\rangle = 0 \quad , \quad \hat{L} |1\rangle = i\beta |2\rangle \quad , \quad \hat{L} |2\rangle = -i\beta |1\rangle \quad , \quad (3.377)$$

where

$$\beta(\mathbf{q}, \omega) = \frac{2\pi\alpha}{\theta} \sqrt{\frac{\omega^2}{c^2} + \mathbf{q}^2} \quad . \quad (3.378)$$

Note that  $|0\rangle$  is annihilated by both  $\hat{K}$  and  $\hat{L}$  – this is a consequence of gauge invariance. We may now write

$$\begin{aligned} \hat{K} &= \hat{K}_\perp |1\rangle\langle 1| + \tilde{K}_\parallel |2\rangle\langle 2| \\ \hat{L} &= i\beta \left( |2\rangle\langle 1| - |1\rangle\langle 2| \right) \quad . \end{aligned} \quad (3.379)$$

Thus, in the truncated  $|a\rangle$  basis ( $a = 1, 2$ ), we have

$$\hat{K} + \hat{L} = \begin{pmatrix} \hat{K}_\perp & -i\beta \\ +i\beta & \tilde{K}_\parallel \end{pmatrix} \quad . \quad (3.380)$$

We may now construct the pseudo-inverse

$$(\hat{K} + \hat{L})^{-1} = \frac{1}{\hat{K}_\perp \tilde{K}_\parallel - \beta^2} \begin{pmatrix} \tilde{K}_\parallel & i\beta \\ -i\beta & \hat{K}_\perp \end{pmatrix} \quad , \quad (3.381)$$

and we find

$$\hat{Q} \equiv \hat{L} - \hat{L}(\hat{K} + \hat{L})^{-1}\hat{L} = -\frac{\beta^2}{\hat{K}_\perp \tilde{K}_\parallel - \beta^2} \hat{K} + \frac{\hat{K}_\perp \tilde{K}_\parallel}{\hat{K}_\perp \tilde{K}_\parallel - \beta^2} \hat{L} \quad . \quad (3.382)$$

In the low frequency, long wavelength limit,  $\hat{K}_\perp(\mathbf{q} \rightarrow 0, 0)$  and  $\tilde{K}_\parallel(\mathbf{q} \rightarrow 0, 0)$  are both constant and dominate over  $\beta^2 \propto c^{-2}\omega^2 + \mathbf{q}^2$ . Thus  $\hat{Q} \rightarrow \hat{L}$  and we obtain the long wavelength action

$$S_{\text{red}}[\delta A^\mu] = \frac{e^2}{4\theta\hbar c} \int d^3x \epsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu \delta A_\lambda \quad , \quad (3.383)$$

exactly as in Eqn. 3.361.

Now you may ask: where are the vortices? Our description of superfluid response doesn't include them! To account for vortices, consider the vortex 3-current,

$$J_\nu^\mu = \sum_{i=1}^{N_\nu} q_i \left\{ \begin{array}{c} 1 \\ \dot{\mathbf{x}}_i(t) \end{array} \right\} \delta(\mathbf{x} - \mathbf{x}_i(t)) \quad . \quad (3.384)$$

Conservation of vorticity means  $\partial_\mu J_\nu^\mu = 0$ , which licenses us to define a gauge field  $\mathcal{V}^\mu$  whose curl is the vortex current, *viz.*

$$J_\nu^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu \mathcal{V}_\lambda \quad . \quad (3.385)$$

We now add a term to the Lagrangian for the superfluid particles,

$$\Delta L = 2\pi\hbar \int d^2x j_\mu^p \mathcal{V}^\mu \quad , \quad (3.386)$$

which in the action provides a Berry phase of  $2\pi q_i$  for each time a bosonic particle of the superfluid executes a closed path encircling the  $i^{\text{th}}$  vortex<sup>90</sup>. In the Hamiltonian description of the superfluid, this amounts to the replacement

$$\delta \mathbf{a}^\mu \longrightarrow \delta \mathbf{a}^\mu - \phi_0 \mathcal{V}^\mu \quad . \quad (3.387)$$

With this refinement, *all* the terms in Eqn. 3.361 are recovered<sup>91</sup>. Ta da!

Why do we need to mess with this tedious response function formalism? Because the interaction potential  $v(\mathbf{r} - \mathbf{r}')$  may be very strong at short distances. The correlations of the underlying superfluid may not be adequately described by a simple Gross-Pitaevskii  $|\Psi|^4$  interaction.

<sup>90</sup>See D. P. Arovas and J. A. Freire, *Phys. Rev. B* **55**, 1068 (1997).

<sup>91</sup>See if you can trace the appearance of the first term on the RHS of Eqn. 3.361, which accounts for the Berry phase of each vortex as it winds around the background condensate.

### 3.5.8 Kohn mode and collective excitations

If we hold the field fixed at  $B = qn\phi_0$  and set  $\delta A^\mu = 0$ , then the action of Eqn. 3.370 is given by

$$S_{\text{eff}}[\delta a] = \frac{1}{8\pi} \int \frac{d^3q}{(2\pi)^3} \delta a^\mu(-q) [\hat{K}_{\mu\nu}(q) + \hat{L}_{\mu\nu}(q)] \delta a^\nu(q) \quad . \quad (3.388)$$

The frequencies of the elementary excitations are given by solving the equation

$$\begin{aligned} 0 &= \det [\hat{K} + \hat{L}] = \hat{K}_\perp \tilde{K}_\parallel - \beta^2 \\ &= -\hat{K}_\perp(\mathbf{q}, \omega) \left(1 + \frac{\omega^2}{c^2 \mathbf{q}^2}\right) 4\pi e^2 \hat{\chi}(\mathbf{q}, \omega) - \left(\frac{2\pi\alpha}{\theta}\right)^2 \left(\mathbf{q}^2 + \frac{\omega^2}{c^2}\right) \quad , \end{aligned} \quad (3.389)$$

which says

$$4\pi e^2 \hat{K}_\perp(\mathbf{q}, \omega) \hat{\chi}(\mathbf{q}, \omega) + \left(\frac{2\pi\alpha}{\theta}\right)^2 \mathbf{q}^2 = 0 \quad . \quad (3.390)$$

At  $T = 0$ , there is no normal component to the superfluid to produce a transverse response, and we have

$$\lim_{\mathbf{q} \rightarrow 0} \hat{K}_\perp(\mathbf{q}, 0) = \frac{4\pi e^2 n}{m^* c^2} \quad . \quad (3.391)$$

For the density response function, we will use the SMA formula,

$$\hat{\chi}_{\text{SMA}}(\mathbf{q}, \omega) = \frac{n\mathbf{q}^2/m}{\omega^2(\mathbf{q}) - (\omega + i\epsilon)^2} \quad . \quad (3.392)$$

If  $\omega(\mathbf{q}) = c|\mathbf{q}|$ , we obtain the dispersion relation

$$\omega^2 = c^2 \mathbf{q}^2 + \left(\frac{4\pi n e^2}{m^*}\right)^2 \left(\frac{\theta}{2\pi\alpha}\right)^2 = \omega_c^2 + c^2 \mathbf{q}^2 \quad , \quad (3.393)$$

where  $\omega_c = eB/m^*c$  is the cyclotron frequency. We have found the Kohn mode.

However, the presumption of a long-wavelength dispersion  $\omega(\mathbf{q}) = c|\mathbf{q}|$  for the phonons of the superfluid is not correct here, due to the long range of the interaction potential  $v(r)$ . As we derive in §3.7 below, the long-wavelength phonon dispersion is rather given by

$$\omega(\mathbf{q}) = \sqrt{\frac{n_0 \hat{v}(\mathbf{q})}{m^*}} |\mathbf{q}| \quad , \quad (3.394)$$

which for  $\hat{v}(\mathbf{q}) = 2\pi e^2/\epsilon|\mathbf{q}|$  behaves as  $|\mathbf{q}|^{1/2}$ . This, you may recall, is the form for the L-phonon in the two-dimensional Wigner crystal with  $1/r$  Coulomb interactions. Thus, we should expect

$$\omega^2 = \omega_c^2 + \frac{2\pi n_0 e^2}{\epsilon m^*} |\mathbf{q}| \quad (3.395)$$



for the Kohn mode at long wavelengths.

In the original Zhang-Hansson-Kivelson paper on the CSL theory, this inter-LL Kohn mode was misidentified as an intra-LL collective mode. This interpretation was subsequently revisited by Lee and Zhang<sup>92</sup>, who provided a revised understanding of the intra-LL collective mode in the CSL theory in terms of vortex-antivortex pairs and quadrupoles. If we add the long-ranged instantaneous Coulomb interaction between vortices to the theory of Eqn. 3.361, *i.e.* a term

$$\Delta S_{CS} = -\frac{\pi^2}{2\theta^2} \int dt \int d^2x \int d^2x' J_v^0(\mathbf{x}, t) v(\mathbf{x} - \mathbf{x}') J_v^0(\mathbf{x}', t) \quad , \quad (3.396)$$

where  $v(r) = e^2/\epsilon r$ , then with  $\delta A^\mu = 0$  we obtain a theory of vortices and antivortices *confined to the lowest Landau level*, since there is no vortex mass term to set a scale for a vortex cyclotron energy. Interpreting the first term as measuring the enclosed area swept out by each vortex in units of  $l^2$ , where  $l$  is a vortex "magnetic length", we have  $l = (2\pi n_0)^{-1/2} = \nu^{-1/2} \ell$ , since the number density is given by  $n_0 = \nu/2\pi\ell^2$ . The remaining terms are the Coulomb interaction from  $\Delta S_{CS}$ , and the topological term proportional to the vortex linking numbers which endows the vortices with fractional exchange statistics. As the linking number is always an integer, this can only change discontinuously due to the crossing of vortex world lines, and cannot affect the vortex equations of motion. Thus, the Lagrangian is<sup>93</sup>

$$S_{\text{vor}} = \frac{\nu}{2\ell^2} \sum_{i=1}^{N_V} q_i \epsilon_{ab} X_i^a \dot{X}_i^b - \frac{\nu^2 e^2}{\epsilon} \sum_{i<j}^{N_V} \frac{q_i q_j}{|\mathbf{X}_i - \mathbf{X}_j|} - \sum_{i=1}^{N_V} q_i U(\mathbf{X}_i) - \tilde{\epsilon}^{\text{QE}} N_{\text{QE}} - \tilde{\epsilon}^{\text{QH}} N_{\text{QH}} \quad , \quad (3.397)$$

where we have also included a one-body potential  $U(\mathbf{r})$  for the vortices which reflects the random potential coupling to the density  $n = n_0 + \delta n$ , recognizing that each vortex produces a local surplus or deficit of physical electrons. Recall from §3.3.7 that a quasielectron-quasihole pair is an exciton whose wavevector  $\mathbf{k}$  is related to the qe-qh separation  $\mathbf{r}$  according to  $\mathbf{k} = \nu \hat{z} \times \mathbf{r} / \ell^2$ . The energy of a single exciton is  $\Delta_{\text{EX}} = \tilde{\epsilon}^{\text{QE}} + \tilde{\epsilon}^{\text{QH}} + v(\mathbf{r})$  where for Coulomb interactions  $v(\mathbf{r}) = (\nu e)^2 / \epsilon r$ . To create an excitation at zero wavevector, one can make a quadrupole with zero net dipole moment. This suggests that the  $\mathbf{k} = 0$  magnetophonon is a quadrupole with energy  $\Delta(\mathbf{0})$  should be on the order of  $2\tilde{\epsilon}^{\text{QE}} + 2\tilde{\epsilon}^{\text{QH}}$ , whereas the magnetoroton, which is a finite  $\mathbf{k}$  excitation, is a dipole with energy on the order of  $\tilde{\epsilon}^{\text{QE}} + \tilde{\epsilon}^{\text{QH}}$ . Indeed, whereas the dipoles have a definite energy-momentum relationship,  $\mathbf{k} = 0$  quadrupoles are available in a continuum of states. Consider a configuration with two quasielectrons at positions  $\pm(x, y)$  and two quasiholes at positions  $\pm(x, -y)$ . For every choice of  $(x, y)$  the net dipole moment is zero, and the energy is

$$\Delta_{\text{QUAD}} = 2\tilde{\epsilon}^{\text{QE}} + 2\tilde{\epsilon}^{\text{QH}} + 2v\left(2\sqrt{x^2 + y^2}\right) - 2v(2x) - 2v(2y) \quad . \quad (3.398)$$

From numerical calculations, the  $q = 0$  portion of the collective excitation spectrum is indeed a continuum, the bottom edge of which lies above the magnetoroton minimum (see Fig. 3.10).

<sup>92</sup>D.-H. Lee and S.-C. Zhang, *Phys. Rev. Lett.* **66**, 1220 (1991).

<sup>93</sup>For a more general interaction potential, replace  $e^2/\epsilon r$  with  $v(r)$ .

Read's version of the CSLG theory has the virtue of describing only LLL physics. His coefficient of the covariant derivative squared term is proportional to the Laplacian of a Hartree-type potential whose energy scale is set by  $e^2/\epsilon\ell$ . In the CSLG theory of ZHK, the coefficient is  $\hbar^2/2m^*$ . The electron mass  $m^*$  enters nowhere within Read's theory, which is apposite since in any LLL-projected theory we should be able to set  $m^* \rightarrow 0$ . However, the theory is unwieldy for other reasons and in fact does not yield a magnetoroton minimum in its collective excitation branch<sup>94</sup>.

### 3.5.9 Quasi-LRO and CSLG theory

Recall that by varying the action in Eqn. 3.348 with respect to the gauge field  $\delta A^0$  we obtain the condition  $\delta n = \pi b/\theta\phi_0$ , where  $b$  is the magnetic field strength corresponding to the shifted gauge field  $\delta\vec{a}$ . Since the action is linear in  $\delta A^0$ , this result is exact, and we may substitute it back into the remaining terms of the action with no approximations. In Fourier space, we have, taking  $\theta = \pi q = \pi/\nu$ ,

$$\delta\hat{a}^i(k) = \frac{\hbar c}{e} \frac{2\pi}{\nu k^2} i\epsilon_{ij} k^j \delta\hat{n}(k) \quad , \quad (3.399)$$

where, following Zhang (1992), we work in the transverse gauge  $\vec{\nabla} \cdot \delta\vec{a} = 0$ . We substitute the above result into the rest of the Lagrangian density,

$$\begin{aligned} \mathcal{L} = & -\hbar \delta n \partial_t \phi - \frac{\hbar^2}{2m} (n_0 + \delta n) \left( (\nabla\phi)^2 + \frac{2e}{\hbar c} \nabla\phi \cdot \delta\vec{a} + \frac{e^2}{\hbar^2 c^2} (\delta\vec{a})^2 \right) \\ & - \frac{\hbar^2}{8m(n_0 + \delta n)} (\nabla\delta n)^2 - \frac{\nu e^2}{4\pi\hbar c^2} \epsilon_{ij} \delta a^i \partial_t \delta a^j - \frac{1}{2} \delta n v \delta n \quad , \end{aligned} \quad (3.400)$$

where  $\Psi = \sqrt{n} e^{i\phi}$  and where the last term is shorthand for what is in the action written as a double integral over  $x$  and  $x'$ , with potential  $v(x - x')$ , as previously. We assume that no vortices are present and thus that  $\phi = \phi_{sw}$ . In the transverse gauge, this means that  $\nabla\phi \cdot \delta\vec{a} = 0$ . We also neglect terms cubic in the density. The result of the substitution is then

$$\hat{\mathcal{L}} = i\hbar\omega \delta\hat{n}(-k) \hat{\phi}(k) - \frac{\nu}{4\pi} \hbar\omega_c k^2 |\hat{\phi}(k)|^2 - \frac{1}{2} \left( \hat{v}(\mathbf{k}) + \frac{\hbar^2 k^2}{4mn_0} + \frac{2\pi}{\nu} \frac{\hbar\omega_c}{k^2} \right) |\delta\hat{n}(k)|^2 \quad . \quad (3.401)$$

We now integrate out the density fluctuations, using their equation of motion. Varying with respect to  $\delta\hat{n}(-k)$  yields

$$\delta\hat{n}(k) = \frac{i\hbar\omega \hat{\phi}(k)}{\hat{v}(\mathbf{k}) + \frac{\hbar^2 k^2}{4mn_0} + \frac{2\pi}{\nu} \frac{\hbar\omega_c}{k^2}} \quad , \quad (3.402)$$

which results in the effective Fourier space Lagrangian density for the  $\phi$  field,

$$\hat{\mathcal{L}}(\phi) = \frac{\nu}{4\pi} \hbar\omega_c k^2 \left( \frac{\omega^2 - \Omega_K^2(\mathbf{k})}{\Omega_K^2(\mathbf{k})} \right) |\hat{\phi}(k)|^2 \quad , \quad (3.403)$$

<sup>94</sup>Read obtains, correctly, a magnetophonon gap on the order of  $e^2/\epsilon\ell$ , but upwardly dispersing as a function of wavevector.

where

$$\Omega_{\mathbf{k}}(\mathbf{k}) = \left[ \omega_c^2 + \frac{n_0}{m} \hat{v}(\mathbf{k}) \mathbf{k}^2 + \left( \frac{\hbar \mathbf{k}^2}{2m} \right)^2 \right]^{1/2} \quad (3.404)$$

is the frequency of the Kohn mode derived in §3.5.8 above<sup>95</sup>.

We may now compute the expectation of the phase fluctuations at  $T = 0$ :

$$\langle |\hat{\phi}(\mathbf{k})|^2 \rangle = \hbar \int_0^\infty \frac{d\omega}{2\pi i} \frac{2\pi}{\nu} \frac{1}{\hbar \omega_c \mathbf{k}^2} \frac{\Omega_{\mathbf{k}}^2(\mathbf{k})}{(\omega + i\epsilon)^2 - \Omega_{\mathbf{k}}^2(\mathbf{k})} = \frac{\pi}{\nu} \frac{\Omega_{\mathbf{k}}(\mathbf{k})}{\omega_c \mathbf{k}^2} \approx \frac{\pi}{\nu \mathbf{k}^2} . \quad (3.405)$$

Thus,

$$\begin{aligned} \langle \Psi^*(\mathbf{r}) \Psi(\mathbf{r}') \rangle &\simeq n_0 \langle e^{i\phi(\mathbf{r}')} e^{-i\phi(\mathbf{r})} \rangle \\ &= n_0 \exp \left[ -\frac{1}{2} \langle [\phi(\mathbf{r}) - \phi(\mathbf{r}')]^2 \rangle \right] , \end{aligned} \quad (3.406)$$

and with

$$\langle [\phi(\mathbf{r}) - \phi(\mathbf{0})]^2 \rangle = \frac{2\pi}{\nu} \int \frac{d^2k}{(2\pi)^2} \frac{1 - e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = \nu^{-1} \ln r , \quad (3.407)$$

we recover the algebraic Girvin-MacDonald order<sup>96</sup>,

$$\langle \Psi^*(\mathbf{r}) \Psi(\mathbf{r}') \rangle \propto n_0 |\mathbf{r} - \mathbf{r}'|^{-1/2\nu} . \quad (3.408)$$

(Compare with Eqn. 3.325.)

## 3.6 Global Phase Diagram of the Quantum Hall Effect

Finally, we discuss the issue of phase transitions between different quantum Hall phases, following the "global phase diagram" picture of Kivelson, Lee, and Zhang (KLZ)<sup>97</sup>. As we saw in chapter 2 (§2.1.7), for weak disorder, the extended single-particle states at the center of each disorder-broadened Landau level are separated in energy by a mobility gap in which all states are localized. This provided a quantum percolation picture of the IQH transition, which could be investigated via the Chalker-Coddington network or disordered Hofstadter models, and when disorder is increased, the extended states "float up" in energy. Still, within this picture all direct IQH transitions involve  $\Delta\sigma_{xy} = \pm e^2/h$ . Despite some problems with noninteracting

<sup>95</sup>We have included the term proportional to  $|\mathbf{k}|^4$  in  $\Omega_{\mathbf{k}}^2(\mathbf{k})$  here for completeness. This arises from the  $(\nabla\delta n)^2$  term in the original Lagrangian.

<sup>96</sup>The integral in Eqn. 3.407 diverges in the ultraviolet and a cutoff at  $k \approx \ell^{-1}$  must be imposed.

<sup>97</sup>S. Kivelson, D.-H. Lee, and S.-C. Zhang, *Phys. Rev. B* **46**, 2223 (1992).

models of the IQHE<sup>98</sup>, if we ignore LL mixing then the transition between  $\nu = n$  and  $\nu = n + 1$  plateaus occurs for  $\nu = n + \frac{1}{2}$ , *i.e.* in the center of each disorder-broadened LL. The transition is marked by a crossing of  $\sigma_{xy}(B, T, L)$  curves at the value  $\sigma_{xy} = (n + \frac{1}{2}) e^2/\hbar$ . Recall the relations, valid in isotropic systems,

$$\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} = \frac{1}{\sigma_{xx}^2 + \sigma_{xy}^2} \begin{pmatrix} \sigma_{xx} & -\sigma_{xy} \\ \sigma_{xy} & \sigma_{xx} \end{pmatrix} = \sigma^{-1} \quad (3.409)$$

and

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} = \frac{1}{\rho_{xx}^2 + \rho_{xy}^2} \begin{pmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{xy} & \rho_{xx} \end{pmatrix} = \rho^{-1} \quad (3.410)$$

Thus the  $n \rightarrow n + 1$  IQH transition lying at  $\sigma_{xy} = (n + \frac{1}{2}) e^2/\hbar$  entails a relation between the longitudinal and transverse components of the resistivity,  $\rho_{xx} \equiv r h/e^2$  and  $\rho_{yx} \equiv s h/e^2$ :

$$n \text{ to } n + 1 \quad : \quad r^2 + \left( s - \frac{1}{2n + 1} \right)^2 = \frac{1}{(2n + 1)^2} \quad (3.411)$$

In the quadrant ( $s > 0, r > 0$ ) of the  $(s, r)$  plane, for each  $n$  the above equation describes a half-circle, centered at  $(s, r) = (\frac{1}{2n+1}, 0)$ , of radius  $\frac{1}{2n+1}$ . The maximum value for each  $n$  occurs at the center, and is given by  $r^* = \frac{1}{2n+1}$ .

The KLZ picture is based on a “law of corresponding states” which posits that the physics of a QH state at filling fraction  $\nu$  is related to that at other fillings related by the operations of LL addition ( $\nu \rightarrow \nu + 1$ ), particle-hole transformation within the LLL ( $\nu \rightarrow 1 - \nu$ ), and flux addition ( $\nu^{-1} \rightarrow \nu^{-1} + 2$ ). We discussed these operations toward the end of §3.4.4, and one can define explicit mappings at the level of wavefunctions for each of them. KLZ provide a nonrigorous but well-motivated argument for this based on the CSLG theory. It is important to note that their procedure accommodates disorder as well.

It is convenient to define the dimensionless components of the conductivity tensor  $u$  and  $v$  according to  $\sigma_{xx} \equiv u e^2/h$  and  $\sigma_{xy} \equiv v e^2/h$ . Thus,

$$r = \frac{u}{u^2 + v^2} \quad , \quad s = \frac{v}{u^2 + v^2} \quad , \quad u = \frac{r}{r^2 + s^2} \quad , \quad v = \frac{s}{r^2 + s^2} \quad (3.412)$$

Suppose a phase boundary between QH states at fillings  $\nu$  and  $\nu'$  is expressed as a relation between  $r$  and  $s$  as  $F(\nu, \nu' | r, s) = 0$ , with  $\lambda F(\nu, \nu' | r, s) \cong F(\nu, \nu' | r, s)$ , *i.e.* multiplication by a constant does not change the condition  $F(\nu, \nu' | r, s) = 0$ . Eqn. 3.411 may be written as

$$F(n, n + 1 | r, s) = r^2 + s^2 - \frac{2s}{2n + 1} \quad , \quad (3.413)$$

From this expression we may derive the phase boundaries for all other QH transitions within the KLZ scheme via a combination of the following operations:

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<sup>98</sup>Recall that such noninteracting descriptions apparently don't properly recover the experimentally observed value for the correlation length exponent  $\nu \approx 2.35$  and instead give  $\nu \approx 2.58$ ; the difference is now large enough to rule out the noninteracting theory, which also fails to give  $z = 1$  for the dynamic critical exponent, which is what experimental scaling analysis supports.

- (i) *Landau level addition* : Under LL addition, one has  $\nu = \nu_0 + 1$ ,  $u = u_0$ , and  $v = v_0 + 1$ . Suppose we do this  $n$  times, so  $\nu = \nu_0 + n$ ,  $u = u_0$  and  $v = v_0 + n$ . Then with

$$u_0 = u = \frac{r}{r^2 + s^2} \quad , \quad v_0 = v - n = \frac{s - n(r^2 + s^2)}{r^2 + s^2} \quad , \quad (3.414)$$

we have

$$\begin{aligned} r_0^{\text{LLA}}(r, s) &= \frac{r(r^2 + s^2)}{r^2 + [s - n(r^2 + s^2)]^2} \\ s_0^{\text{LLA}}(r, s) &= \frac{[s - n(r^2 + s^2)](r^2 + s^2)}{r^2 + [s - n(r^2 + s^2)]^2} \end{aligned} \quad (3.415)$$

and we may write

$$F(\nu + n, \nu' + n | r, s) = F(\nu, \nu' | r_0^{\text{LLA}}(r, s), s_0^{\text{LLA}}(r, s)) \quad . \quad (3.416)$$

If we start with the  $0 \rightarrow 1$  transition, where we may take  $F_0(r, s) = r^2 + s^2 - 2s$ , then we obtain

$$F(n, n + 1 | r, s) = \frac{r^2 + s^2}{r^2 + [s - n(r^2 + s^2)]^2} [(2n + 1)(r^2 + s^2) - 2s] \quad , \quad (3.417)$$

which is congruent to the form in Eqn. 3.413.

- (ii) *Particle-hole conjugation* : Under the PHC operation, assuming  $\nu < 1$ , one has  $\nu = 1 - \nu_0$ ,  $u = u_0$ , and  $v = 1 - v_0$ . Suppose we do this  $n$  times, so  $\nu = \nu_0 + n$ ,  $u = u_0$ , and  $v = v_0 + n$ . Thus

$$\begin{aligned} r_0^{\text{PHC}}(r, s) &= \frac{r(r^2 + s^2)}{r^2 + [s - r^2 - s^2]^2} \\ s_0^{\text{PHC}}(r, s) &= \frac{[r^2 + s^2 - s](r^2 + s^2)}{r^2 + [s - r^2 - s^2]^2} \end{aligned} \quad (3.418)$$

and we may write

$$F(1 - \nu', 1 - \nu | r, s) = F(1 - \nu', 1 - \nu | r_0^{\text{PHC}}(r, s), s_0^{\text{PHC}}(r, s)) \quad (3.419)$$

- (iii) *Flux attachment* : Under the flux attachment operation, one has  $\nu^{-1} = \nu_0^{-1} + 2$ ,  $r = r_0$ , and  $s = s_0 + 2$ . Suppose we do this  $p$  times, so  $\nu = \nu_0 / (2p\nu_0 + 1)$ ,  $r = r_0$ , and  $s = s_0 + 2p$ . Then

$$F\left(\frac{\nu}{2p\nu + 1}, \frac{\nu'}{2p\nu' + 1} \middle| r, s\right) = F(\nu, \nu' | r, s - 2p) \quad . \quad (3.420)$$

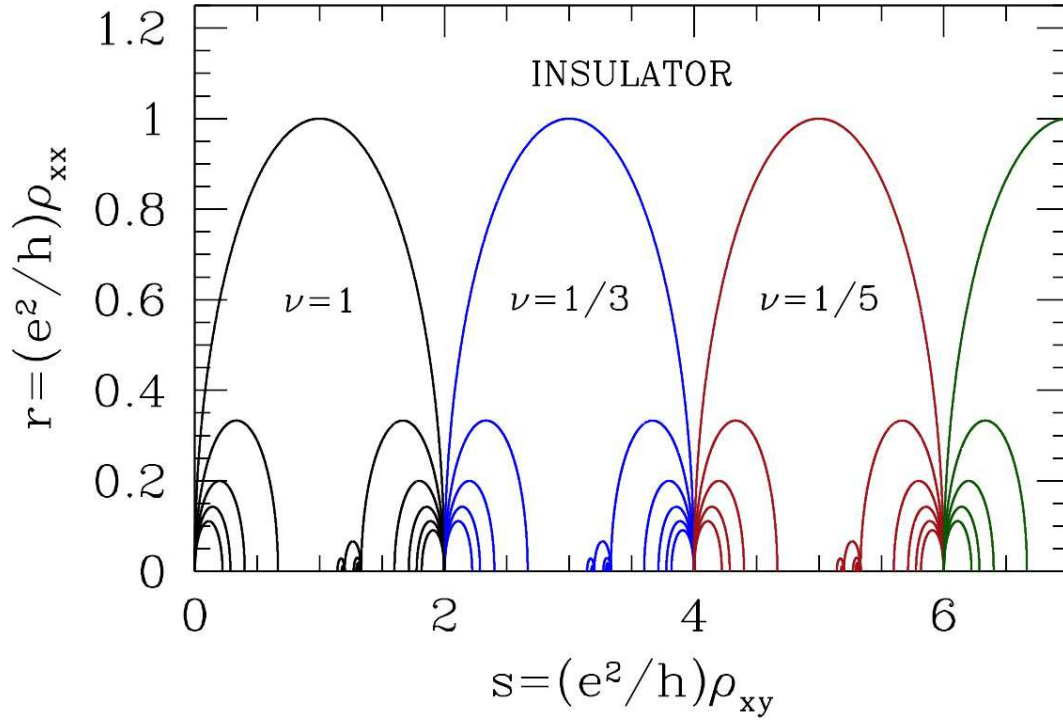


Figure 3.13: Conjectured global phase diagram of the quantum Hall effect.

Thus, we have

$$F\left(\frac{n}{2pn+1}, \frac{n+1}{2p(n+1)+1} \middle| r, s\right) = r^2 + (s - s_{n,p})^2 - a_{n,p}^2 \quad (3.421)$$

with

$$s_{n,p} = 2p + \frac{1}{2n+1}, \quad a_{n,p} = \frac{1}{2n+1}. \quad (3.422)$$

It is a useful exercise to compute the effect of the PHC operation on the  $\nu = n/(2np+1)$ . After a straightforward but slightly tedious calculation, one obtains

$$F\left(\frac{(2p-1)(n+1)+1}{2p(n+1)+1}, \frac{(2p-1)n+1}{2pn+1} \middle| r, s\right) = r^2 + (s - \tilde{s}_{n,p})^2 - \tilde{a}_{n,p}^2, \quad (3.423)$$

where

$$\tilde{s}_{n,p} = \frac{2p(2p-1)(2n+1) + 4p - 1}{(2p-1)[(2p-1)(2n+1) + 2]} \quad (3.424)$$

and

$$\tilde{a}_{n,p}^2 = s_{n,p}^2 - \frac{4p[p(2n+1)+1]}{(2p-1)[(2p-1)(2n+1)+2]}. \quad (3.425)$$

For the case  $n = 1$  and  $p = 1$ , for example, we find

$$F\left(\frac{3}{5}, \frac{2}{3} \mid r, s\right) = r^2 + \left(s - \frac{9}{5}\right)^2 - \frac{1}{25} . \quad (3.426)$$

A few iterations of the various operations yields the phase diagram shown in Fig. 3.13.

As KLZ stress, one should not take the details of Fig. 3.13 too seriously. Each of the three transformations results in a different effective disorder potential. Rather, it is the topology of the global phase diagram which is alleged to be robust. This tells us that there are direct IQH transitions only from  $n \rightarrow n \pm 1$  and never with  $\Delta n > 1$ . Similarly, one may observe the transitions  $\frac{1}{3} \leftrightarrow \frac{2}{5} \leftrightarrow \frac{3}{7}$ , but not  $\frac{1}{3} \leftrightarrow \frac{3}{7}$ . Another feature of the global phase diagram is the singularities at even denominators, where there is a confluence of an infinite number of phases. In general, disorder will kill off all but a relatively small number of these phases, but rather than the insulating state extending down to the  $r = 0$  axis in Fig. 3.13 at even values of  $s$ , at some point the Fermi liquid like physics of the  $\nu = \frac{1}{2}$  etc. states sets in. We shall discuss the half-filled Landau level in the next chapter.

The insulating phase in Fig. 3.13 is identified as a *Hall insulator*, in which  $\sigma_{xx} \rightarrow 0$ ,  $\sigma_{xy} \rightarrow 0$ ,  $\rho_{xx} \rightarrow \infty$ , but  $\rho_{yz} < \infty$  is a constant value roughly given by  $B/nec$ . It differs from the band insulator and Mott insulator phases, where  $\rho_{yx} \rightarrow \infty$ . The disordered Wigner crystal phase with finite size Imry-Ma domains could be a Hall insulator.

### 3.7 Appendix I: Density Correlations in a Superfluid

As a model of a vanilla superfluid, consider the Gross-Pitaevskii field theory, with Euclidean Lagrangian density

$$\mathcal{L}_E = \hbar \bar{\psi} \partial_\tau \psi + \frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{1}{2} g (|\psi|^2 - n_0) \quad (3.427)$$

in  $d = 2$  space dimensions. Write  $\psi = \sqrt{n} \exp(i\phi)$ , so that

$$\mathcal{L}_E = i\hbar n \partial_\tau \phi + \frac{\hbar^2 n}{2m} (\nabla \phi)^2 + \frac{\hbar^2}{8mn} (\nabla n)^2 + \frac{1}{2} g (n - n_0)^2 . \quad (3.428)$$

We write  $n = n_0 + \delta n$  and expand in the small quantities  $\delta n$ ,  $\nabla \delta n$ , and  $\nabla \phi$ , and adding a source term  $j$  with respect to which we may differentiate. Thus,

$$\mathcal{L}_E = i\hbar n_0 \partial_\tau \phi + i\hbar \delta n \partial_\tau \phi + \frac{\hbar^2 n_0}{2m} (\nabla \phi)^2 + \frac{\hbar^2}{8mn_0} (\nabla \delta n)^2 + \frac{1}{2} g (\delta n)^2 + j \delta n , \quad (3.429)$$

which is valid to quadratic order in small quantities. The first term on the RHS is important when vortices are present. Else, since it is a total derivative, in the action it integrates to zero.



We shall be interested in the case when there are no vortices, so we will drop this term. Going now to Fourier space, we have

$$\tilde{\mathcal{L}}_E = \left( \hbar\omega \hat{\phi}(\mathbf{k}, \omega) + \hat{j}(\mathbf{k}, \omega) \right) \delta\hat{n}(-\mathbf{k}, -\omega) + n_0 \varepsilon_{\mathbf{k}} |\hat{\phi}(\mathbf{k}, \omega)|^2 + \left( \frac{\varepsilon_{\mathbf{k}}}{4n_0} + \frac{1}{2}g \right) |\delta\hat{n}(\mathbf{k}, \omega)|^2 \quad (3.430)$$

where  $\varepsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m$ . Now vary the action with respect to  $\delta\hat{n}^*(\mathbf{k}, \omega) = \delta\hat{n}(-\mathbf{k}, -\omega)$  to obtain

$$\left( \frac{\varepsilon_{\mathbf{k}}}{2n_0} + g \right) \delta\hat{n}(\mathbf{k}, \omega) + \left( \hbar\omega \hat{\phi}(\mathbf{k}, \omega) + \hat{j}(\mathbf{k}, \omega) \right) = 0 \quad . \quad (3.431)$$

Integrating out the density fluctuations using the above equation of motion, we obtain the Lagrangian density

$$\begin{aligned} \tilde{\mathcal{L}}_E = n_0 \left( \frac{(\hbar\omega)^2}{\varepsilon_{\mathbf{k}} + 2gn_0} + \varepsilon_{\mathbf{k}} \right) |\hat{\phi}(\mathbf{k}, \omega)|^2 - \frac{n_0 |\hat{j}(\mathbf{k}, \omega)|^2}{\varepsilon_{\mathbf{k}} + 2gn_0} \\ + \frac{\hbar\omega n_0}{\varepsilon_{\mathbf{k}} + 2gn_0} \left( \hat{\phi}^*(\mathbf{k}, \omega) \hat{j}(\mathbf{k}, \omega) - \hat{\phi}(\mathbf{k}, \omega) \hat{j}^*(\mathbf{k}, \omega) \right) \quad . \end{aligned} \quad (3.432)$$

From the coefficient of the  $|\hat{\phi}|^2$  term we can read off the phonon dispersion,

$$\omega(\mathbf{k}) = \frac{1}{\hbar} \sqrt{\varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2gn_0)} \quad , \quad (3.433)$$

which shows up as a pole in the  $\hat{\phi}$  propagator at  $\omega = i\omega(\mathbf{k})$ <sup>99</sup>. As  $k \rightarrow 0$  we obtain  $\omega(\mathbf{k}) = u|\mathbf{k}|$  with a phonon velocity  $u = \sqrt{gn_0/m}$ . As expected, in the ultraviolet limit  $k \rightarrow \infty$  we recover the ballistic dispersion  $\omega(\mathbf{k}) = \varepsilon_{\mathbf{k}}/\hbar$ .

We can now integrate out the phase fluctuations  $\hat{\phi}(\mathbf{k}, \omega)$  using the same method to obtain the Euclidean action as a function of the source  $\hat{j}(\mathbf{k}, \omega)$ :

$$S_E[j] = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^2k}{(2\pi)^2} \frac{n_0 \varepsilon_{\mathbf{k}}}{(\hbar\omega)^2 + \varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2gn_0)} |\hat{j}(\mathbf{k}, \omega)|^2 \quad . \quad (3.434)$$

In essence we have just done two Gaussian functional integrals. As a sanity check, note that setting  $m \rightarrow \infty$  and then  $\hbar \rightarrow 0$  kills off all but the last two terms on the RHS of Eqn. 3.429, and accordingly in this limit the integrand becomes  $|\hat{j}|^2/2g$ .

Differentiating now with respect to the source, we obtain the equal-time correlator for density fluctuations

$$\begin{aligned} s(\mathbf{k}) &= \frac{1}{n_0} \int d^2r e^{i\mathbf{k}\cdot\mathbf{r}} \langle \delta n(\mathbf{r}, 0) \delta n(\mathbf{0}, 0) \rangle \\ &= - \frac{\hbar}{n_0} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\delta^2 S[j]}{\delta \hat{j}^*(\mathbf{k}, \omega) \delta \hat{j}(\mathbf{k}, \omega)} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{2\hbar \varepsilon_{\mathbf{k}}}{(\hbar\omega)^2 + \varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2gn_0)} = \sqrt{\frac{\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}} + 2gn_0}} \quad . \end{aligned} \quad (3.435)$$

<sup>99</sup>Had we been working in real time, rather than Euclidean time, the pole would have been at  $\omega = \omega(\mathbf{k})$ .



At long wavelengths we have  $s(\mathbf{k}) = \frac{1}{\sqrt{2}} k\lambda$  where  $\lambda = \hbar/mu$  is the Compton wavelength (with  $u$  the speed of sound in the superfluid). In the ultraviolet limit,  $s(\mathbf{k}) \rightarrow 1$  as is always the case<sup>100</sup>. Note that there is no superfluidity at any finite temperature  $T > 0$  in  $d = 2$ , as a consequence of the Hohenberg-Mermin-Wagner theorem. Rather, for a model with  $O(2)$  symmetry, there is a Kosterlitz-Thouless phase transition at a critical temperature  $T_{\text{KT}}$ , below which thermally excited vortices and antivortices are bound. For  $T < T_{\text{KT}}$ , there is a finite superfluid stiffness  $\rho_s$ , but the condensate fraction  $n_0$  is rigorously zero, in accordance with the HMW theorem<sup>101</sup>. For  $T > T_{\text{KT}}$ , there is a vortex-antivortex plasma.

As we've seen, the dispersion  $\omega(\mathbf{k})$  in the Gross-Pitaevskii model crosses over from the acoustic phonon behavior  $u|\mathbf{k}|$  in the infrared to the ballistic  $\hbar^2\mathbf{k}^2/2m$  in the ultraviolet. It is thus a convex function of  $k$  and shows no hint of a roton dip. The reason is that the contact interaction  $v(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$  is purely repulsive. If we replace it with a more general interaction potential  $g(\mathbf{r} - \mathbf{r}')$ , then we can accommodate a roton in our model through the behavior of its Fourier transform  $\hat{g}(\mathbf{k})$ . The GP dispersion and structure factor are then given by

$$\omega(\mathbf{k}) = \frac{1}{\hbar} \sqrt{\varepsilon_{\mathbf{k}} (\varepsilon_{\mathbf{k}} + 2\hat{g}(\mathbf{k}) n_0)} \quad , \quad s(\mathbf{k}) = \sqrt{\frac{\varepsilon_{\mathbf{k}}}{\varepsilon_{\mathbf{k}} + 2\hat{g}(\mathbf{k}) n_0}} \quad . \quad (3.436)$$

Thus, a phonon-roton dispersion curve is modeled if  $\hat{g}(\mathbf{k})$  has a pronounced dip in the vicinity of  $k \approx k_{\text{R}}$ .

### 3.8 Appendix II: Linear Response and Correlation Functions

We now present a litany of useful definitions and results which are applied to CSGL theory of the FQHE in §3.5.7. The Hamiltonian for a system of particles of charge ( $-e$ ) is given by

$$H(\mathcal{A}^\mu) = \frac{1}{2m^*} \sum_{i=1}^N \left( \mathbf{p}_i + \frac{e}{c} \mathcal{A}_i \right)^2 - \frac{e}{c} \sum_{i=1}^N \mathcal{A}_i^0 + \sum_{i<j} v(\mathbf{r}_i - \mathbf{r}_j) \quad . \quad (3.437)$$

We begin with the definitions

$$\begin{aligned} j_0^{\text{p}}(\mathbf{x}) &= n(\mathbf{x}) = \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{x}_i) \\ j^{\text{p}}(\mathbf{x}) &= \frac{1}{2m^*} \sum_{i=1}^N \left[ \mathbf{p}_i \delta(\mathbf{x} - \mathbf{x}_i) + \delta(\mathbf{x} - \mathbf{x}_i) \mathbf{p}_i \right] \quad , \end{aligned} \quad (3.438)$$

<sup>100</sup>Recall  $s(\mathbf{k}) = N^{-1} \sum_{i,j} \langle e^{i\mathbf{k}\cdot\mathbf{r}_i} e^{-i\mathbf{k}\cdot\mathbf{r}_j} \rangle$ . For  $k \rightarrow \infty$ , only terms with  $i = j$  contribute and  $s(\mathbf{k}) \rightarrow 1$ .

<sup>101</sup>The superfluid stiffness  $\rho_s(T)$  vanishes for  $T > T_{\text{KT}}$ , but rather than vanishing as a power law, as in conventional second order phase transitions, it exhibits a *universal jump* such that  $\rho_s(T_{\text{KT}}^-) = 2k_{\text{B}}T_{\text{KT}}/\pi$ .

where  $j_\mu^p$  is called the *paramagnetic number current*<sup>102</sup>. For charged systems in the presence of an electromagnetic field with vector potential  $\mathcal{A}^\mu(\mathbf{x}, t)$ , there is also a *diamagnetic number current*,

$$j_0^d = 0 \quad , \quad \mathbf{j}^d(\mathbf{x}) = \frac{e}{m^*c} n(\mathbf{x}) \mathcal{A}(\mathbf{x}) \quad , \quad (3.439)$$

where the particle charge is  $(-e)$ , and the *gauge-invariant current operator* is given by

$$j_\mu = -\frac{c}{e} \frac{\delta H}{\delta \mathcal{A}^\mu} = j_\mu^p + j_\mu^d \quad . \quad (3.440)$$

### 3.8.1 Linear response theory

Consider a quantum Hamiltonian  $H(t) = H_0 - \sum_i Q_i \phi_i(t)$  where  $\{Q_i\}$  are operators and the  $\{\phi_i(t)\}$  are fields or potentials<sup>103</sup>. From first order perturbation theory, one derives the linear response relation

$$\langle Q(t) \rangle = \int_{-\infty}^{\infty} dt' \chi_{ij}(t-t') \phi_j(t') + \mathcal{O}(\phi^2) \quad , \quad (3.441)$$

where the *response functions*  $\chi_{ij}(t-t')$  are given by

$$\chi_{ij}(t-t') = \frac{i}{\hbar} \langle [Q_i(t), Q_j(t')] \rangle \Theta(t-t') \quad , \quad (3.442)$$

where  $\langle \dots \rangle$  denotes a thermal average. Thus, the Fourier transform  $\hat{\chi}_{ij}(\omega)$  is given by

$$\begin{aligned} \hat{\chi}_{ij}(\omega) &= \frac{i}{\hbar} \int_0^{\infty} dt \langle [Q_i(t), Q_j(0)] \rangle e^{i\omega t} \\ &= \frac{1}{\hbar Z} \sum_{m,n} e^{-\beta E_m} \left\{ \frac{\langle m | Q_j | n \rangle \langle n | Q_i | m \rangle}{\omega - E_m + E_n + i\epsilon} - \frac{\langle m | Q_i | n \rangle \langle n | Q_j | m \rangle}{\omega + E_m - E_n + i\epsilon} \right\} . \end{aligned} \quad (3.443)$$

At  $T = 0$ , with  $\omega_n \equiv E_n - E_0$ , we have

$$\hat{\chi}_{ij}(\omega) = \frac{1}{\hbar} \sum_n \left\{ \frac{\langle 0 | Q_j | n \rangle \langle n | Q_i | 0 \rangle}{\omega + \omega_n + i\epsilon} - \frac{\langle 0 | Q_i | n \rangle \langle n | Q_j | 0 \rangle}{\omega - \omega_n + i\epsilon} \right\} . \quad (3.444)$$

<sup>102</sup>Note that with our  $(+ - -)$  metric that  $V^0 = V_0$  for any 3-vector  $V^\mu$ , and hence  $(j^p)^0 = j_0^p$ .

<sup>103</sup>We also assume that  $\langle Q_i \rangle = 0$  when all the  $\phi_i$  are set to zero.

### 3.8.2 Electromagnetic response

In the case of electromagnetic response for charge ( $-e$ ) objects, we define  $J_\mu^p \equiv -e j_\mu^p$ . In the presence of an electromagnetic vector potential  $\mathcal{A}^\mu$  we have

$$H(\mathcal{A}) = H(0) + \frac{1}{c} \int d^2x J_\mu^p \mathcal{A}^\mu - \frac{e}{2m^*c^2} \int d^2x J_0^p \mathcal{A}^2 \quad . \quad (3.445)$$

Linear response theory then says

$$\langle J_\mu(\mathbf{x}, t) \rangle = \frac{c}{4\pi} \int d^2x' K_{\mu\nu}(\mathbf{x} - \mathbf{x}', t - t') \mathcal{A}^\nu(\mathbf{x}', t') \quad , \quad (3.446)$$

where the electromagnetic response tensor is

$$K_{\mu\nu}(\mathbf{x} - \mathbf{x}', t - t') = \frac{4\pi}{i\hbar c^2} \langle [J_\mu^p(\mathbf{x}, t), J_\nu^p(\mathbf{x}', t')] \rangle \Theta(t - t') + \frac{4\pi e}{m^*c^2} \langle J_0^p(\mathbf{x}) \rangle g_{\mu\nu} (1 - \delta_{\mu 0}) \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \quad , \quad (3.447)$$

where, recall, the metric is  $g = \text{diag}(+, -, -)$ . Taking the Fourier transform,

$$\hat{K}_{\mu\nu}(\mathbf{q}, \omega) = \frac{4\pi e^2}{i\hbar c^2} \int_0^\infty dt \langle [\hat{j}_\mu^p(\mathbf{q}, t), \hat{j}_\nu^p(-\mathbf{q}, 0)] \rangle e^{i\omega t} + \frac{4\pi e^2}{c^2} \frac{n}{m^*} \delta_{\mu\nu} (1 - \delta_{\mu 0}) \quad . \quad (3.448)$$

Linear response says

$$\langle J_\mu(\mathbf{q}, \omega) \rangle = \frac{c}{4\pi} \hat{K}_{\mu\nu}(\mathbf{q}, \omega) \mathcal{A}^\nu(\mathbf{q}, \omega) \quad . \quad (3.449)$$

Gauge invariance requires that the physical current  $J_\mu$  is unchanged if  $\mathcal{A}^\mu \rightarrow \mathcal{A}^\mu + \partial^\mu f$ . Charge conservation requires  $\partial^\mu J_\mu = 0$ . These two conditions therefore guarantee

$$q^\mu \hat{K}_{\mu\nu}(\mathbf{q}, \omega) = \hat{K}_{\mu\nu}(\mathbf{q}, \omega) q^\nu = 0 \quad , \quad (3.450)$$

with  $q^\mu = (\omega, \mathbf{q})$ . In fact, these two conditions are equivalent, as a consequence of Onsager reciprocity, which guarantees

$$\begin{aligned} \text{Re } \hat{K}_{\mu\nu}(\mathbf{q}, \omega) &= +\text{Re } \hat{K}_{\nu\mu}(-\mathbf{q}, -\omega) \\ \text{Im } \hat{K}_{\mu\nu}(\mathbf{q}, \omega) &= -\text{Im } \hat{K}_{\nu\mu}(-\mathbf{q}, -\omega) \quad . \end{aligned} \quad (3.451)$$

Furthermore, spatial isotropy requires that

$$\hat{K}_{ij}(\mathbf{q}, \omega) = \hat{K}_{\parallel}(\mathbf{q}, \omega) \hat{q}_i \hat{q}_j + \hat{K}_{\perp}(\mathbf{q}, \omega) (\delta_{ij} - \hat{q}_i \hat{q}_j) \quad , \quad (3.452)$$

where  $\hat{q} = \mathbf{q}/|\mathbf{q}|$ . We may now invoke gauge invariance and charge conservation to establish

$$\begin{aligned} \hat{K}_{0j}(\mathbf{q}, \omega) &= \hat{K}_{j0}(\mathbf{q}, \omega) = \frac{q_j}{\omega} \hat{K}_{\parallel}(\mathbf{q}, \omega) \\ \hat{K}_{00}(\mathbf{q}, \omega) &= \frac{q^2}{\omega^2} \hat{K}_{\parallel}(\mathbf{q}, \omega) \quad . \end{aligned} \quad (3.453)$$

If we choose the gauge  $\mathcal{A}^0 = 0$ , then  $\mathcal{E} = -c^{-1}\dot{\mathcal{A}} = i\omega c^{-1}\mathcal{A}$  and the conductivity tensor is given by

$$\hat{\sigma}_{ij}(\mathbf{q}, \omega) = \frac{c^2}{4\pi i\omega} \hat{K}_{ij}(\mathbf{q}, \omega) \quad . \quad (3.454)$$

If  $\hat{\sigma}(\mathbf{q}, 0)$  is not to diverge, when we must have  $\hat{K}_{ij}(\mathbf{q}, 0) = 0$ , which implies the following sum rule:

$$\frac{i}{\hbar} \int_0^\infty dt \langle [J_i^p(\mathbf{x}, t), J_j^p(0, 0)] \rangle = -\frac{ne^2}{m^*} \delta(\mathbf{x}) \delta_{ij} \quad , \quad (3.455)$$

where  $n = \langle n(\mathbf{x}) \rangle$  is presumed constant. This sum rule is violated in superconductors.

### 3.8.3 Neutral systems

In neutral systems, linear response theory may be applied to the particle 3-current,  $j^\mu = (n, \mathbf{j})$ , and we define the susceptibility matrix  $\chi_{\mu\nu}(\mathbf{x}, t)$  as

$$\chi_{\mu\nu}(\mathbf{x}, t) = \frac{i}{\hbar} \langle [j_\mu(\mathbf{x}, t), j_\nu(0, 0)] \rangle \Theta(t) = \begin{pmatrix} \chi_{00} & \chi_{0j} \\ \chi_{i0} & \chi_{ij} \end{pmatrix} \quad . \quad (3.456)$$

The component  $\chi_{00}(\mathbf{x}, t) \equiv \chi(\mathbf{x}, t)$  describes the density response to a potential  $U(\mathbf{x}, t)$ , with perturbing Hamiltonian  $H_1 = -\int d^2x n(\mathbf{x}) U(\mathbf{x}, t)$ . The Fourier transform is written as

$$\hat{\chi}_{\mu\nu}(\mathbf{q}, \omega) = \int_{-\infty}^\infty dt \int d^2x \chi_{\mu\nu}(\mathbf{x}, t) e^{-i\mathbf{q}\cdot\mathbf{x}} e^{i\omega t} \quad . \quad (3.457)$$

Note that

$$\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = \frac{1}{n^2} \left( \frac{\partial n}{\partial \mu} \right)_T = n^{-2} \hat{\chi}(\mathbf{q} \rightarrow 0, 0) \quad . \quad (3.458)$$

Spatial isotropy says that we may write the spatial tensor

$$\hat{\chi}_{ij}(\mathbf{q}, \omega) \equiv \hat{\chi}_{\parallel}(\mathbf{q}, \omega) \hat{q}_i \hat{q}_j + \hat{\chi}_{\perp}(\mathbf{q}, \omega) (\delta_{ij} - \hat{q}_i \hat{q}_j) \quad . \quad (3.459)$$

Continuity  $\partial_t n + \nabla \cdot \mathbf{j} = 0$  then guarantees

$$\begin{aligned} \hat{\chi}_{00}(\mathbf{q}, \omega) &\equiv \hat{\chi}(\mathbf{q}, \omega) = \frac{\mathbf{q}^2}{\omega^2} \left( \hat{\chi}_{\parallel}(\mathbf{q}, \omega) - \frac{n}{m^*} \right) \\ \hat{\chi}_{0j}(\mathbf{q}, \omega) &= \hat{\chi}_{j0}(\mathbf{q}, \omega) = \frac{q_j}{\omega} \hat{\chi}_{\parallel}(\mathbf{q}, \omega) \quad . \end{aligned} \quad (3.460)$$

Note that at  $T = 0$  we have

$$\hat{\chi}(\mathbf{q}, \omega) = \frac{1}{\hbar A} \sum_j \left[ \frac{\langle 0 | \rho_{-\mathbf{q}} | j \rangle \langle j | \rho_{\mathbf{q}} | 0 \rangle}{\omega + \omega_j + i\epsilon} - \frac{\langle 0 | \rho_{\mathbf{q}} | j \rangle \langle j | \rho_{-\mathbf{q}} | 0 \rangle}{\omega - \omega_j + i\epsilon} \right] \quad (3.461)$$

where  $\rho_{\mathbf{q}} = \sum_{i=1}^N e^{-i\mathbf{q}\cdot\mathbf{x}_i}$  is the Fourier component of the density. If the (normalized) SMA state,

$$|\mathbf{q}\rangle \equiv [Ns(\mathbf{q})]^{-1/2} \rho_{\mathbf{q}} |0\rangle \quad (3.462)$$

is an eigenstate, this means that  $|\mathbf{q}\rangle$  saturates all the oscillator strength at this wavevector, in which case  $\hat{\chi}(\mathbf{q}, \omega) = \hat{\chi}_{\text{SMA}}(\mathbf{q}, \omega)$ , with

$$\hat{\chi}_{\text{SMA}}(\mathbf{q}, \omega) = \frac{n}{\hbar} s(\mathbf{q}) \left\{ \frac{1}{\omega + \omega(\mathbf{q}) + i\epsilon} - \frac{1}{\omega - \omega(\mathbf{q}) + i\epsilon} \right\} = \frac{n\mathbf{q}^2/m}{\omega^2(\mathbf{q}) - (\omega + i\epsilon)^2}, \quad (3.463)$$

where  $\omega(\mathbf{q}) = \langle \mathbf{q} | H | \mathbf{q} \rangle - E_0$  is the SMA energy.

### Relation between charged and neutral system response

If we endow each of our neutral particles with a charge ( $-e$ ), then

$$\hat{K}_{\mu\nu}(\mathbf{q}, \omega) = \frac{4\pi e^2}{c^2} \left( \frac{n}{m^*} \delta_{\mu\nu} (1 - \delta_{\mu 0}) - \hat{\chi}_{\mu\nu}(\mathbf{q}, \omega) \right). \quad (3.464)$$

Thus,

$$\begin{aligned} \hat{K}_{00}(\mathbf{q}, \omega) &= \frac{4\pi e^2}{c^2} \frac{\mathbf{q}^2}{\omega^2} \left( \frac{n}{m^*} - \hat{\chi}_{\parallel}(\mathbf{q}, \omega) \right) = -\frac{4\pi e^2}{c^2} \hat{\chi}(\mathbf{q}, \omega) \\ \hat{K}_{0i}(\mathbf{q}, \omega) &= \hat{K}_{i0}(\mathbf{q}, \omega) = \frac{4\pi e^2}{c^2} \frac{q_i}{\omega} \left( \frac{n}{m^*} - \hat{\chi}_{\parallel}(\mathbf{q}, \omega) \right) \\ \hat{K}_{\parallel}(\mathbf{q}, \omega) &= \frac{4\pi e^2}{c^2} \left( \frac{n}{m^*} - \hat{\chi}_{\parallel}(\mathbf{q}, \omega) \right) = \frac{\omega^2}{\mathbf{q}^2} \hat{K}_{00}(\mathbf{q}, \omega) \\ \hat{K}_{\perp}(\mathbf{q}, \omega) &= \frac{4\pi e^2}{c^2} \left( \frac{n}{m^*} - \hat{\chi}_{\perp}(\mathbf{q}, \omega) \right). \end{aligned} \quad (3.465)$$

### 3.8.4 Meissner effect and superfluid density

Suppose the electric field is  $\mathcal{E}$  and the magnetic field is  $\mathcal{B} = \nabla \times \mathcal{A}$  in three space dimensions. In  $d = 2$  space dimensions, the magnetic field is a scalar  $\mathcal{B} = \hat{z} \cdot \nabla \times \mathcal{A}$ , but we can also for convenience define  $\mathcal{B} = \mathcal{B}\hat{z}$ . We may work in the  $\mathcal{A}^0 = 0$  gauge, in which case  $\mathcal{E} = -c^{-1}\partial_t \mathcal{A}$ , which entails  $\nabla \cdot \mathcal{A} = 0$  at any finite frequency. We now have

$$\begin{aligned} \nabla \times \mathcal{A} &= \frac{4\pi}{c} \mathbf{J} + \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} = \nabla(\nabla \cdot \mathcal{A}) - \nabla^2 \mathcal{A} \\ &= -\hat{K}_{\perp}(-i\nabla, i\partial_t) \mathcal{A} - \frac{1}{c^2} \frac{\partial^2 \mathcal{A}}{\partial t^2}, \end{aligned} \quad (3.466)$$

which is valid for  $d = 3$ , but for  $d = 2$  we must replace  $\hat{K}_\perp \rightarrow \hat{K}_\perp/d_z$  where  $d_z$  is the thickness of the sample in the  $\hat{z}$  direction. One has that the units of  $\hat{K}_{ij}(\mathbf{q}, \omega)$  are  $L^{1-d}$ . Thus

$$\nabla^2 \mathcal{A} - \frac{1}{c^2} \frac{\partial^2 \mathcal{A}}{\partial t^2} = \hat{K}_\perp (-i\nabla, i\partial_t) \mathcal{A} \quad , \quad (3.467)$$

again with  $\hat{K}_\perp$  replaced by  $\hat{K}_\perp/d_z$  in two space dimensions. Setting  $\omega = 0$  and taking the limit  $q \rightarrow 0$ , we obtain the formula for the London penetration depth,

$$\lambda_L^{-2} = \left\{ \begin{array}{c} d_z^{-1} \\ 1 \end{array} \right\} \times \lim_{q \rightarrow 0} \hat{K}_\perp(\mathbf{q}, 0) \quad . \quad (3.468)$$

For a purely two-dimensional system, the distance between consecutive layers is  $d_z = \infty$ , and we have  $\lambda_L = \infty$ , which says that a purely two-dimensional system cannot screen a three-dimensional electromagnetic field. In three-dimensions, the *superfluid density* is defined by

$$n_s \equiv \frac{m^* c^2}{4\pi e^2 \lambda_L^2} = n - m^* \lim_{q \rightarrow 0} \hat{\chi}_\perp(\mathbf{q}, 0) \quad . \quad (3.469)$$

In the three-dimensional ideal Bose gas, for example, one finds

$$\hat{\chi}_{ij}(\mathbf{q} \rightarrow 0, 0) = \frac{n_0}{m^*} \hat{q}_i \hat{q}_j + \frac{n - n_0}{m^*} \delta_{ij} \quad , \quad (3.470)$$

where  $n_0(T)$  is the number density of condensed bosons and  $n' \equiv n - n_0$  is that of uncondensed bosons. Thus  $\hat{\chi}_{\parallel}(\mathbf{q} \rightarrow 0, 0) = n/m^*$  and  $\hat{\chi}_\perp(\mathbf{q} \rightarrow 0, 0) = n'/m^*$ . The superfluid number density is  $n_s(T) = n_0(T)$ . In fact, as Landau first showed, an ideal Bose gas is in fact *not* a superfluid because its excitation spectrum, which follows the ballistic dispersion  $\omega(\mathbf{q}) = \hbar q^2/2m$  is too 'soft', and any nonzero superflow is unstable to decay into single particle excitations.