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**THE INTEGER QUANTUM HALL EFFECT AND
ANDERSON LOCALISATION**

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Contents

1	Introduction	881
2	Scaling theory and localisation transitions	882
3	The plateau transitions as quantum critical points	885
4	Single particle models	887
5	Numerical studies	890
6	Discussion and outlook	892

THE INTEGER QUANTUM HALL EFFECT AND ANDERSON LOCALISATION

J.T. Chalker

1 Introduction

The existence of the Integer Quantum Hall Effect (IQHE) depends crucially on Anderson localisation, and, conversely, many aspects of the delocalisation transition have been studied in most detail in quantum Hall systems. The following article is intended to provide a introduction to the IQHE from this viewpoint, as a supplement to the broader accounts of quantum Hall experiment and theory by Shayegan [1] and by Girvin [2], which appear elsewhere in this volume.

To give an overview of what is to come, we start by outlining some of the experimental facts and some of the theoretical ideas used in their interpretation. Consider a two-dimensional electron gas (2DEG) in a magnetic field, with some (but not too much) scattering from disorder. Suppose the filling factor ν (the ratio of electron density to flux density) is increased from one integer value to the next, at low temperature. The Hall conductivity, σ_{xy} , as a function of ν has the form of a staircase, consisting of broad plateaus at integer multiples of e^2/h (with e the electron charge and h Planck's constant), separated by narrow risers. The dissipative conductivity, σ_{xx} , is very small at electron densities for which the Hall conductivity is quantised, but has Shubnikov – de Haas peaks which coincide with the risers in σ_{xy} . Both the transitions between Hall plateaus and the Shubnikov – de Haas peaks become sharper at lower temperatures, as indicated schematically in Figure 1.

It was appreciated [3] rather quickly after von Klitzing's [4] discovery of the IQHE that some understanding of this observed dependence of the conductivity tensor on ν can be got by starting from a single-particle description of the electron states which involves an Anderson delocalisation transition. More specifically, the picture is as follows. In a clean system, the spectrum for a charged particle moving in two dimensions in a perpendicular magnetic field consists of a series of macroscopically degenerate Landau levels. A realistic model of the experimental sample should include scattering from impurities and inhomogeneities, which lifts the degeneracy and

broadens the Landau levels. Provided the magnetic field is strong enough, however, the disorder-broadening is less than the cyclotron energy, and the levels retain their identity. It turns out that the states within each disorder-broadened Landau level fall into two categories, for reasons that we shall outline heuristically in Section 4. States in the tails of Landau levels are Anderson localised, meaning that each is trapped within a particular, microscopic region of the system. By contrast, states near the centre of each Landau level have wavefunctions that extend throughout that sample. The distinction is crucial for the conductivity, since, within linear response, a constant external electric field cannot induce transitions between different localised states. Thus, only those electrons occupying extended states participate in current flow.

Given these features of the eigenstates, one can deduce the essentials in the dependence of conductivity on electron density. To do so, imagine adding electrons at zero temperature to a system with the spectrum sketched in Figure 1b. Initially, all occupied states are localised and transport is impossible. Addition of more electrons brings the Fermi energy into a region of extended states, so that σ_{xy} increases with ν . In this situation, σ_{xx} is non-zero, because scattering of electrons close to the Fermi energy gives rise to dissipation. A further increase in ν brings the Fermi energy into the localised states in the upper tail of the Landau level. Now, a variation in electron density does not alter the number of extended, current-carrying states that are occupied. Correspondingly, the Hall conductivity exhibits a plateau. Also, since the occupied, current-carrying states are buried at some depth beneath the Fermi energy, dissipation is necessarily activated, and vanishes in the low temperature limit. In turn, this implies that the Hall angle is 90° , and thus that $\sigma_{xx} = 0$. The sequence is repeated on moving the Fermi energy through higher Landau levels, and generates the behaviour represented in Figure 1a.

Several questions arise from this account of the IQHE. In particular, one can ask what evidence there is for the suggested delocalisation transition, how the transition fits into the general theory of critical phenomena, and how interactions might change the single-particle picture. We summarise some of the current answers in the remaining sections of this article. More detail can be found in the following sources: for the quantum Hall effect in general, the book edited by Prange and Girvin [5]; for the IQHE and localisation, the article by Huckestein [6]; for the IQHE as a quantum phase transition, the article by Sondhi *et al.* [7].

2 Scaling theory and localisation transitions

Scaling theory provides a framework for discussing delocalisation transitions, both in systems of real, interacting electrons, and in single-particle

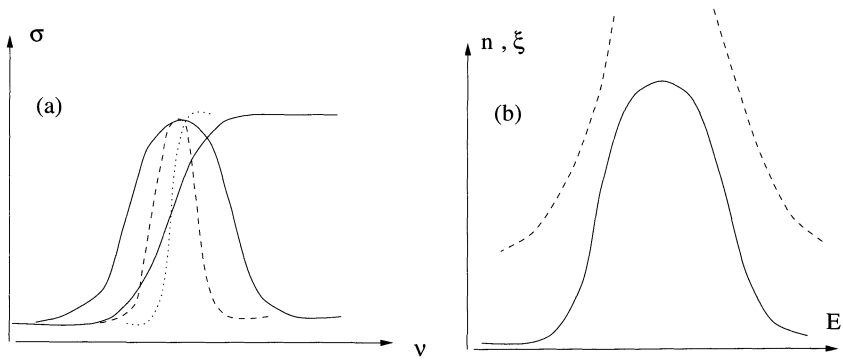


Fig. 1. (a) Schematic behaviour σ_{xy} and σ_{xx} as a function of filling factor (full lines); the plateau transition becomes sharper at lower temperature (dashed lines). (b) Density of states, n , (full line) and localisation length, ξ , (dashed line) as a function of energy, E , in a disorder-broadened Landau level.

models. It was first developed for transitions in zero or weak magnetic field, in a celebrated paper by Abrahams *et al.* [8] shortly before the discovery of the IQHE. Although a substantial extension [9] of these original ideas is necessary in order to encompass the IQHE, we discuss first the zero field problem in order to provide some perspective. A general review of disordered electronic systems is given in the article by Lee and Ramakrishnan [10]; a more recent account is by Kramer and MacKinnon [11].

The basis of scaling theory is to identify a quantity that plays the role of a coupling constant for the problem under consideration, and to discuss the change in this coupling constant with a change in the length scale at which it is measured. Both of these steps are most transparently described for a non-interacting system, though most of the ideas developed in that context should also apply to interacting systems. To be concrete, suppose that a change in length scale is accomplished in d -dimensions by combining 2^d systems, individually of size L^d , to form a single system of size $(2L)^d$. Each single-particle state of the large system can be considered as a superposition of states from the smaller systems. The states entering one such superposition will be drawn mainly from a window in energy around the level in question. Let the width of this window be ϵ , and let the mean level spacing (in, say, the smaller systems) be Δ . From a microscopic point of view, the dimensionless ratio ϵ/Δ is a good candidate for a coupling constant: if $\epsilon/\Delta \gg 1$, each state of the large system should be spread over all 2^d subsystems, while if each state of the large system is localised in a particular subsystem, one expects $\epsilon/\Delta \ll 1$. Crucially, this ratio also has a macroscopic significance. As Thouless argued [12], ϵ should be related

by the uncertainty principle to the time required for a particle to travel a distance L : with diffusion constant D , $\epsilon \sim \hbar D/L^2$. Expressing Δ in terms of the density of states, n , and recalling the Einstein expression for the conductivity, $\sigma = e^2 n D$, one has

$$\frac{\epsilon}{\Delta} \sim \frac{(\hbar D/L^2)}{(L^d n)^{-1}} \sim \frac{\hbar}{e^2} \sigma L^{d-2} \equiv g(L) \quad (1)$$

where $g(L)$ is the conductance of a hypercube of size L^2 , measured in units of the conductance quantum, e^2/h .

Given $g(L)$ as a coupling constant, the next step is to consider its scale-dependence, which (treating L now as a continuous variable) is characterised by the function

$$\beta(g) \equiv \frac{d \ln(g)}{d \ln(L)}. \quad (2)$$

The approach rests on the assumption that $\beta(g)$ is a function *only* of g , and independent of L or the microscopic details of the system. A formal justification for this assumption comes from the derivation of a field-theoretic description for the localisation transition from microscopic models of disordered conductors (see [10]); this route also provides a way to calculate $\beta(g)$, at least close to two dimensions. The essential features of the function $\beta(g)$ can be determined [8], however, from its asymptotic behaviour at large and small g , and the requirement that, since it is defined in a system of finite size, it should interpolate smoothly between these limits. For $g \gg 1$, one expects the size-dependence of the conductance to be given correctly by Ohm's law, so that $g \propto L^{d-2}$ and $\beta(g) = d - 2$, while for strong disorder, one expects localisation on a length scale ξ , $g \sim \exp(-L/\xi) \ll 1$ for $L \gg \xi$, and $\beta(g) \sim \ln(g) + \text{const}$. These ideas (together with perturbation theory in g^{-1} at large g) lead (in the absence of spin-orbit scattering) to the behaviour sketched in Figure 2 [8]. In this way, the two dimensional problem is identified as marginal: in more than two dimensions, $\beta(g)$ has a zero at $g = g_c$. Flow is towards metallic behaviour for $g > g_c$, and to insulating behaviour for $g < g_c$. In or below two dimensions, flow as illustrated is always towards the insulator.

These ideas have been confirmed by extensive experiments in zero or weak magnetic field (see [10]). The existence of the IQHE, however, shows that a strong magnetic field must be capable of changing the behaviour. The mechanism was identified, at the level of a field-theoretic description, by Pruisken *et al.* [13], who showed that σ_{xy} appears as a second coupling constant. A scaling flow diagram in the $\sigma_{xx} - \sigma_{xy}$ plane, incorporating the IQHE, was suggested by Khmel'nitskii [9], and is illustrated in Figure 3. In this diagram each flow line indicates how the two components of the conductivity tensor change as a quantum Hall system is probed at increasing length scales, for fixed ν , and different flow lines correspond to different values of

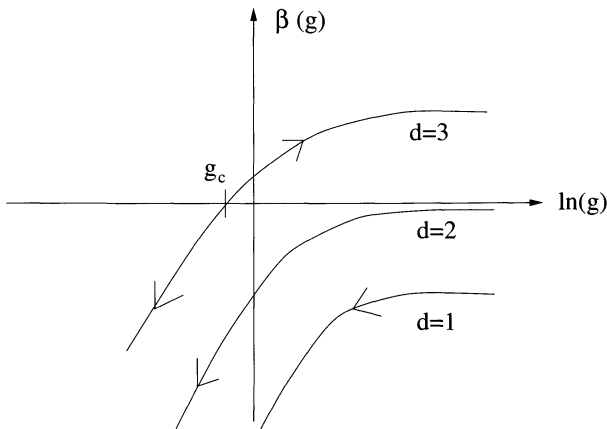


Fig. 2. The scaling function $\beta(g)$ as a function of $\ln(g)$ for dimensions $d = 1, 2,$ and 3 , in the absence of a magnetic field.

ν . Experimentally, the length scale of measurements can be increased, and scaling flow lines traced out, by reducing the temperature [14], while by varying ν at fixed high temperature, the system is swept along a trajectory in the conductance plane that intersects a range of flow lines. The flow as a whole is periodic in σ_{xy} , and its asymptotic behaviour is controlled by fixed points, which are of two kinds. Almost all scaling flow lines end on stable fixed points, located at $\sigma_{xx} = 0$ for $\sigma_{xy} = N$, with N integer. These fixed points represent quantum Hall plateaus and (for $N = 0$) the zero magnetic field localised phase. The experimental result that (with some idealisation) the Hall conductance is quantised and dissipation vanishing in the low temperature limit, for almost almost all filling factors, has its correspondence in the fact that nearly all flow is towards these points. The fact that flow on the $\sigma_{xy} = 0$ axis is towards a fixed point with $\sigma_{xx} = 0$ is the representation in Figure 3 of the implications of Figure 2 for two-dimensional systems without a magnetic field. A discrete set of exceptional flow lines end at unstable fixed points, which have σ_{xx} non-zero and $\sigma_{xy} = N + 1/2$. These fixed points represent the plateau transitions. Flow in their vicinity has one unstable direction, leading to the adjacent stable fixed points on either side. It is principally the nature of this flow from the unstable fixed point that is probed in studies of the quantum Hall plateau transition as a critical point.

3 The plateau transitions as quantum critical points

The quantum Hall plateau transition is one of the best examples we have at present of a quantum critical point in a disordered system. Viewing it

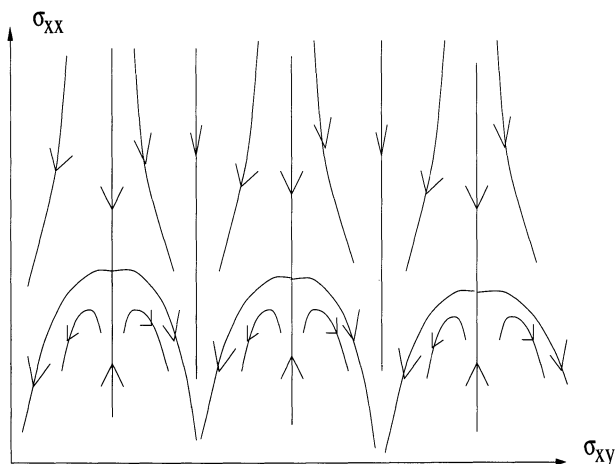


Fig. 3. The scaling flow diagram for the IQHE.

in this way brings certain additional expectations to those that follow from the scaling flow of Figure 4. In this section we summarise the ideas involved, which are reviewed in the article by Sondhi *et al.* [7].

Consider a system that is close to a plateau transition, with, for example, magnetic field strength as the control parameter that tunes the system through the transition: let ΔB be the deviation of this control parameter from its critical value. We expect the correlation length, ξ , to be finite away from the critical point, and to diverge as the critical point is approached, with a critical exponent denoted by ν (but not to be confused with the filling factor)

$$\xi \sim |\Delta B|^{-\nu}. \quad (3)$$

This correlation length corresponds in a single-particle description to the localisation length at the Fermi energy, and is the scale at which flow leaves the vicinity of the unstable fixed point and reaches that of the stable fixed point. The interacting quantum system is also expected to have a characteristic correlation time, τ , and this too will diverge as the critical point is approached. The dynamical scaling exponent z relates the diverging spatial and temporal scales *via*

$$\tau \sim \xi^z. \quad (4)$$

Correspondingly, the characteristic energy scale shrinks as the critical point is approached, with the dependence

$$\hbar/\tau \sim \xi^{-z} \sim |\Delta B|^{\nu z}. \quad (5)$$

At the critical point itself, this energy scale vanishes. If some other energy scale remains non-zero, set for example by temperature, T , or by the

frequency, ω , at which the system is probed, then the transition is rounded: because of this, the plateau transition is a zero temperature phase transition. Close to the critical point, scaling theory constrains the dependence of physical quantities on ΔB and external scales. The situation is particularly simple for the components of the conductivity tensor, which in a two-dimensional system have a magnitude fixed by e^2/h [15]: they should be given by scaling functions, the arguments of which can be chosen to be ratios of the various energy scales, so that

$$\sigma_{ij} = F_{ij} \left(\frac{|\Delta B|^{\nu z}}{T}, \frac{\omega}{T}, \dots \right). \quad (6)$$

From this, one expects the width, ΔB^* , of the transition to scale, for example, with temperature at $\omega = 0$ as $\Delta B^* \sim T^{-1/(\nu z)}$.

A number of experiments have probed this and other aspects of scaling behaviour, determining the width in magnetic field of the Shubnikov - de Haas peak in the dissipative resistivity, or the width of the riser between two plateaus in the Hall resistivity. Wei *et al.* find power-law scaling of ΔB^* with T over about one and a half decades, and obtain the value $\nu z \approx 2.4$ [16]; Engel *et al.* demonstrate that ΔB^* is independent of ω for $\omega < T$ and find dependence consistent with $\Delta B^* \sim \omega^{-1/(\nu z)}$ and the same value of νz for $\omega > T$ [17]. Determination of ν and z separately requires different approaches. One, employed by Koch *et al.* [18], is to work at small T and ω using mesoscopic samples, so that broadening of the plateau transition is a consequence of finite sample size, rather than of an external energy scale. In this way $\nu \approx 2.3$ is obtained [18], implying $z \approx 1$. An alternative is to work at small T and ω in a macroscopic sample, and to use finite electric field strength, \mathcal{E} , to broaden the transition. Since $e\mathcal{E}\xi$ sets an energy scale, one expects the transition width to satisfy $\mathcal{E}\xi \sim (\Delta B^*)^{\nu z}$, and hence $\Delta B^* \sim \mathcal{E}^{1/(\nu[z+1])}$; in combination with temperature scaling, this allows ν and z to be determined separately. By this route Wei *et al.* obtain $\nu \approx 2.3$ and $z \approx 1$ [19].

4 Single particle models

To arrive at a satisfactory scaling theory of the plateau transition as a quantum phase transition, beginning from a microscopic description, would necessarily involve a treatment of the many electron system with interactions and disorder. While some progress (which we summarise in Sect.6) has been made in this direction, the single-particle localisation problem provides a useful and very much simpler starting point. Even in this case, the obstacles to analytic progress are formidable. The size of the relevant coupling constant is the value of σ_{xx} (or, strictly, its inverse) at the unstable fixed points of the scaling flow diagram of Figure 4; since this is $\mathcal{O}(1)$

(and, in fact, the fixed point is invisible in perturbation theory), a non-perturbative approach is presumably required. So far most known quantitative results have been obtained from numerical simulations, which we outline in Section 5. In the present section we introduce models that have been studied numerically, and describe a semiclassical picture of the transition.

The Hamiltonian for a particle moving in two dimensions with a uniform magnetic field and random scalar potential should provide a rather accurate description of the experimental system, apart from the neglect of electron-electron interactions. It is characterised by two energy scales and two length scales, with in each case one scale set by the magnetic field and one by the disorder. The energy scales are the cyclotron energy and the amplitude of fluctuations in the random potential (if necessary, averaged over a cyclotron orbit). The IQHE occurs only when the first of these is the larger; a natural but limited simplification is to take it to be much larger, in which case inter-Landau level scattering is suppressed and the potential fluctuations establish the only energy scale of importance. The length scales are the magnetic length and the correlation length of the disorder, and varying their ratio provides some scope for theoretical simplification, as we shall explain. Experimentally, both limits for the ratio can be realised: disorder on atomic length scales is presumably dominant in MOSFETs, while in heterostructures the length scale of the potential experienced by electrons is set by their separation from remote ionised donors, and this may be larger than the magnetic length.

A semiclassical limit for the localisation problem is reached if the potential due to disorder is smooth on the scale of the magnetic length. This limit has the advantage that it can be used to make the existence of a delocalisation transition intuitively plausible [20], and to construct a simplified model for the transition, known as the network model [21]. If the potential is smooth, then the local density of states at any given point in the system will consist of a ladder of Landau levels, displaced in energy by the local value of the scalar potential. As a function of position in the system, the displaced Landau levels form a series of energy surfaces, which are copies of the potential energy, $V(x, y)$, itself, having energies $V(x, y) + (N + 1/2)\hbar\omega_c$. Suppose one Landau level, and for simplicity the lowest, is partially occupied, so that the filling factor is $0 < \nu < 1$. This value of the filling factor arises, for a smooth potential, from a spatial average over some regions in which the local filling factor is $\nu_{\text{local}} = 1$, (those places at which, with chemical potential μ , the potential satisfies $V(x, y) + \hbar\omega_c/2 < \mu$) and others in which the local filling factor is $\nu_{\text{local}} = 0$ (because at these places $V(x, y) + \hbar\omega_c/2 > \mu$). As illustrated in Figure 5, for small average filling factors, there will be a percolating region with $\nu_{\text{local}} = 0$, dotted with isolated, finite “lakes”, in which $\nu_{\text{local}} = 1$. By contrast, for average filling factors close to 1, a region with $\nu_{\text{local}} = 1$ will percolate, and this “sea” will

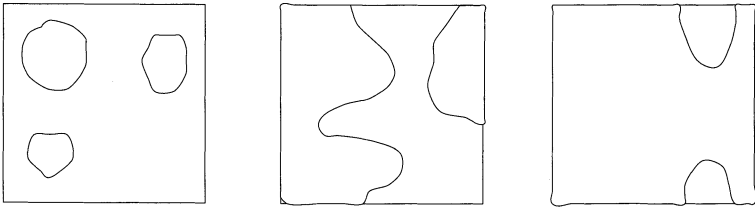


Fig. 4. Snapshots of a quantum Hall system with smooth random potential at three successive values of the average filling factor. Shaded regions have local filling factor $\nu_{\text{local}} = 1$, and unshaded regions have $\nu_{\text{local}} = 0$.

contain isolated “islands” in which $\nu_{\text{local}} = 0$. A transition between these two situations must occur at an intermediate value of ν . (In particular, if the random potential distribution is symmetric under $V(x, y) \rightarrow -V(x, y)$, the critical point is at $\nu = 1/2$.)

To connect this geometrical picture with the nature of eigenstates in the system, recall that states at the chemical potential lie on the boundary between the regions in which $\nu_{\text{local}} = 0$ and those in which $\nu_{\text{local}} = 1$, so that one has a Fermi surface in real space. There are two components to the classical dynamics of electrons on the Fermi surface, and they have widely-separated time scales in the smooth potential we are considering. The fast component involves cyclotron motion around a guiding centre: when quantised, it contributes $(N + 1/2)\hbar\omega_c$ to the total energy. The slow component involves drift of the guiding centre in the local electric field that arises from the gradient of the potential $V(x, y)$. Since this gradient is almost constant on the scale of the magnetic length, the guiding centre drift is analogous to the Hall current that flows when a uniform electric field is applied to an ideal system, and therefore carries the guiding centres along contours of constant potential. If one imagines quantising this classical guiding centre drift, say by a Bohr-Sommerfeld procedure, then eigenstates result which have their probability density concentrated in strips lying around contours of the potential, with width set by the magnetic length. States in the low-energy tail of a Landau level are associated with contours that encircle minima in the potential, while states in the high-energy tail belong to contours around maxima of the potential. At a critical point between these two energies, the characteristic size of contours diverges, and one has the possibility of extended states.

An obvious factor which complicates the simple association of eigenstates with closed contour lines is the possibility of tunneling near saddle-points in the potential, between disjoint pieces of a given energy contour. Equally, once tunneling is allowed for, there may be more than one path by which electrons can travel between two points, and interference effects

can become important. Away from the critical point, tunneling and interference are unimportant provided the potential is sufficiently smooth, but as the critical point is approached their influence always dominates. It is for this reason that the delocalisation transition is not in the same universality class as classical percolation. The network model [21] provides a simple way to incorporate these quantum effects. In this model, portions of a given equipotential are represented by links, which carry probability flux in one direction, corresponding to that of guiding centre drift. The wavefunction is caricatured by a complex current amplitude, defined on links. On traversing a link, a particle acquires an Aharonov-Bohm phase: if z_i and z_j are amplitudes at opposite ends of the link k , $z_j = e^{i\phi_k} z_i$. Tunneling at saddlepoints of the random potential is included in the model at nodes, where two incoming and two outgoing links meet. The amplitudes (say, z_1 and z_2) on the outgoing links are related to those on incoming links (z_3 and z_4) by a scattering matrix, which must be unitary for current conservation and can be made real by a suitable choice of gauge. Then

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \quad (7)$$

where the single real parameter β characterises the node. The model as a whole is built by connecting together these two elements, links and nodes, to form a lattice. The simplest choice is a square lattice: thinking of this as a chess board, the black squares represent regions in which $\nu_{\text{local}} = 1$, while for the white squares $\nu_{\text{local}} = 0$. Guiding centre drift, and the direction of links of the model, is (for a one sense of the magnetic field) clockwise around the white squares and anticlockwise around the black squares. Randomness is introduced into the model by taking the link phases, ϕ_k , to be independent random variables, for simplicity uniformly distributed between 0 and 2π . Variation of the node parameter from $\beta = 0$ to $\beta = \pi/2$ corresponds to sweeping the Fermi energy through a Landau level, and a delocalisation transition occurs at $\beta = \pi/2$. This transition has been studied numerically, using the approach described in Section 5. In addition, the model itself has been mapped onto other descriptions of the problem, notably a supersymmetric quantum spin chain in $1 + 1$ dimensions [22].

5 Numerical studies

The quantitative information available on the delocalisation transition in models of the IQHE without interactions comes from numerical simulations, reviewed by Huckestein [6]. The most important results of these calculations are as follows. First, there is universality, in the sense that most choices of model lead to the same results (and those choices that do not are plausibly argued to be plagued by a slow crossover, preventing one from

reaching asymptotic behaviour in the available system sizes [23]). Second, it is confirmed that the scaling flow diagram of Figure 4 is correct (at least for a non-interacting system), in particular, in the sense that the localisation length is divergent only at one energy within a disorder-broadened Landau level. Third, a value $\nu = 2.3 \pm 0.1$ is obtained for the localisation length exponent. This value is in agreement with experimental results, as summarised in Section 3, a fact that raises several questions, which we touch on in Section 6.

A necessary first step in numerical calculations on the delocalisation transition is to discretise the problem. There are several ways of doing this. One is to project the Hamiltonian onto the subspace spanned by states from a single Landau level. A second is to study the network model, described in the previous section, and a third is to treat a tight-binding model, with a magnetic field introduced by including Peierls phases in the hopping matrix elements.

Given a suitable model, there are two approaches to simulations. The most direct is simply to diagonalise the Hamiltonian for a square sample, and use an appropriate criterion to distinguish localised and extended states. The method has the potential disadvantage that, in a system of finite size, all states having a localisation length larger than the sample size will appear extended. For that reason it is important to examine the fraction of apparently extended states as a function of system size, L : at the plateau transition this fraction tends to zero as $L^{-1/\nu}$. Bhatt *et al.* have used this method extensively [24], introducing boundary conditions which include phase shifts, so that Chern numbers can be defined for each state. Extended states are identified as those with non-zero Chern number. An alternative approach is to study transmission properties of systems that are in the shape of long cylinders. This transmission problem (or a calculation of the Green function) has computational advantages, since there exist for it algorithms [25] which are much less demanding on computer memory than matrix diagonalisation. Since the geometry is quasi-one dimensional, all states are localised along the length of the sample for any finite radius. Because of this, transmission amplitudes, for example, decay exponentially with sample length. The computational procedure is to calculate the mean decay rate (or, technically, the smallest Lyapunov exponent). The inverse of this is the localisation length, $\xi(E, M)$ which in general depends on the circumference, M , of the cylinder as well as the energy, E , under consideration. At energies for which states in the two-dimensional system are localised, the localisation length in the quasi-one dimensional system tends to the bulk localisation length, $\xi(E)$, as the circumference is taken to infinity, while at the critical energy, the quasi-one dimensional localisation length remains proportional to the cylinder circumference for arbitrarily large values of M . Finite size scaling theory provides a framework for analysing data from these

calculations: one expects

$$\xi(M, E) = Mf(\xi(E)/M) \quad (8)$$

where $f(x)$ is a function of the single scaling variable $x = \xi(E)/M$, rather than of $\xi(E)$ and M separately. Calculations of this kind for the network model [21, 26] and for a Hamiltonian projected onto the lowest Landau level [27] both lead to the same scaling function, and to the value for ν quoted above.

6 Discussion and outlook

A number of important open questions remain.

Recent experiments by Shahar and collaborators have produced several intriguing results which are not fully understood. Examining the plateau transition in the lowest Landau level, they find a reflection symmetry in the current-voltage characteristics which they interpret in terms of charge-flux duality [28]. Following properties to higher magnetic fields, and therefore into the insulating phase, they find the Hall resistance to be nearly quantised, even far from the transition [29]. Finally, and disturbingly, in the samples they study, the transition apparently retains a finite width ΔB^* , even in the low-temperature limit [30], in serious contradiction to the scaling ideas of Section 3.

On the theoretical side, one of the important problems is to understand better the effect of interactions on the plateau transition. In view of the apparent agreement between the value of ν determined from experiment and that from simulations of models without electron-electron interactions, it is initially tempting to think that interactions might be irrelevant in the renormalisation group sense. In fact, this is not directly tenable, since in a non-interacting system with a finite density of states at the mobility edge, the dynamical exponent necessarily takes the value $z = 2$. A Hartree-Fock study of interacting electrons in a Landau level with disorder [31], and a numerical calculation of the scaling dimension of interaction strength at the non-interacting fixed point [32], both suggest that it may be possible to attribute $z = 1$ to interaction effects, whilst retaining the value of ν found in the non-interacting system.

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