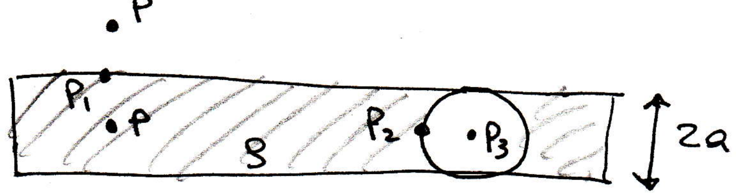
Physics 4C — Final exam solutions — Fall 2019

Problem 1



Electric field of slab: $\sigma = \rho \cdot 2a$ is the charge per unit area

$\boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{z} = \frac{\rho a}{\varepsilon_0}}$ for $z > a$. $\boxed{\vec{E} = \frac{\rho z}{\varepsilon_0}\hat{z}}$ for $0 < z < a$

Electric field of sphere:

$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$ for $r > a$, $\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{a^3}$ for $r \leq a$. Take $q = \frac{4}{3}\pi a^3 \times (-\rho)$

$\Rightarrow \boxed{\vec{E} = -\frac{\rho}{3\varepsilon_0} a^3 \frac{\hat{r}}{r^2}}$ for $r > a$, $\boxed{\vec{E} = -\frac{\rho}{3\varepsilon_0}\vec{r}}$ for $r \leq a$

Electric potential: $\boxed{\phi_B - \phi_A = -\int_A^B \vec{E}\cdot d\vec{s}}$. $\boxed{\phi(P) = 0}$

$$\phi(P_1) = -\int_P^{P_1} \vec{E}\cdot d\vec{s} = -\frac{\rho a^2}{2\varepsilon_0} \quad ; \quad \phi(P') = \phi(P_1) - \int_{P_1}^{P'} \vec{E}\cdot d\vec{s} = -\frac{\rho a^2}{2\varepsilon_0} - \frac{\rho a^2}{\varepsilon_0}$$

$$\Rightarrow \phi(P') = -\frac{3}{2}\frac{\rho a^2}{\varepsilon_0} = -3\,V \Rightarrow \boxed{\frac{\rho a^2}{\varepsilon_0} = 2\,V}$$

$$\Rightarrow \boxed{\phi(P_1) = -1\,V} \text{ (a)}$$

(b) $$\phi(P_2) = \phi(P) - \int_\infty^a dr \frac{\rho}{3\varepsilon_0}\frac{a^3}{r^2} = \phi(P) - \frac{\rho a^2}{3\varepsilon_0} = -\frac{2}{3}\,V$$

$$\Rightarrow \boxed{\phi(P_2) = -0.66\,V}$$

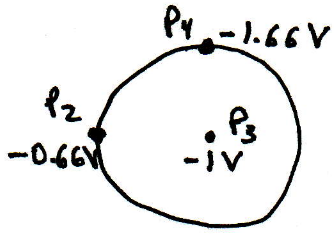
(c) $$\phi(P_3) = \phi(P_2) + \int_a^0 dr \frac{\rho}{3\varepsilon_0} r = \phi(P_2) - \frac{\rho a^2}{6\varepsilon_0} = \phi(P_2) - \frac{1}{3}\,V$$

$$\Rightarrow \boxed{\phi(P_3) = -1\,V}$$

(d) $\phi(P_4) = \phi(P_2) + [\phi(P_1) - \phi(P)] = -0.66\,V - 1\,V$

$\Rightarrow \boxed{\phi(P_4) = -1.66\,V}$

(e) So we have:



negative charge goes where potential is high

$\Rightarrow$ $\boxed{\text{negative charge goes to } P_2}$

positive charge goes where potential is low

$\Rightarrow$ $\boxed{\text{positive charge goes to } P_4}$

Problem 2

(a) Immediately after S is closed, C is a short, L has ∞ resistance

$$\Rightarrow \boxed{I_{R_1} = \frac{\mathcal{E}}{R}, \quad I_{R_2} = 0}$$

(b) Long time after S is closed, L is a short, C has ∞ resistance

$$\Rightarrow \boxed{I_{R_1} = 0, \quad I_{R_2} = \frac{\mathcal{E}}{R}}$$

(c) Difference in potential across capacitor is $\mathcal{E}$ $(I_{R_1} = 0) \Rightarrow$

$$\boxed{Q = C\mathcal{E}}$$

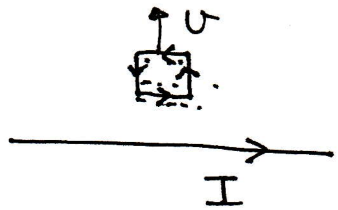
(d) Right after S is ~~clo~~ opened again, I_{R_2} cannot change $\Rightarrow$

$$\boxed{I_{R_1} = I_{R_2} = \frac{\mathcal{E}}{R}}$$

(e) After S is opened we have an RLC circuit. The condition for the charge to oscillate in sign is ω real, with

$$\omega = \sqrt{\omega_0^2 - \frac{R^2}{4L^2}} \Rightarrow \omega_0^2 = \frac{1}{LC} > \frac{R^2}{4L^2} \Rightarrow \boxed{\frac{R^2 C}{4L} < 1}$$

Problem 3



B-field from wire : $B = \frac{\mu_0 I}{2\pi z}$ points out of paper

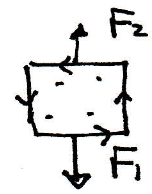
$\Rightarrow$ induced current goes $\boxed{\text{counterclockwise}}$

$$\mathcal{E} = -\frac{d\phi}{dt} \; . \quad \phi = \frac{\mu_0 I}{2\pi} a \int_z^{z+a} \frac{dz}{z} = \frac{\mu_0 I}{2\pi} a \ln\left(1 + \frac{a}{z}\right)$$

$$\Rightarrow \mathcal{E} = -\frac{\mu_0 I a}{2\pi} \frac{1}{1+\frac{a}{z}} \left(-\frac{a}{z^2}\right) \frac{dz}{dt} . \quad \text{Now } \frac{dz}{dt} = v, \text{ and } i = \frac{\mathcal{E}}{R} \Rightarrow$$

$$\Rightarrow \boxed{i = \frac{\mu_0 I}{2\pi R} \frac{a^2}{z^2} \frac{1}{1+\frac{a}{z}} \cdot v} \quad (a)$$

(b) $$\boxed{P = i^2 R = \frac{\mu_0^2 I^2}{4\pi^2 R} \frac{a^4}{z^4} \frac{1}{\left(1+\frac{a}{z}\right)^2} v^2}$$



(c) Magnetic force on wire : $\vec{F} = i \vec{l} \times \vec{B}$

Net force in z direction we have to apply:

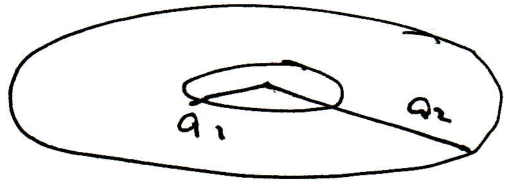
$$F_{net} = F_1 - F_2 = i a \left[B(z) - B(z+a)\right] = i a \frac{\mu_0 I}{2\pi}\left(\frac{1}{z} - \frac{1}{z+a}\right) = \frac{i a^2 \mu_0 I}{2\pi z^2} \frac{1}{1+\frac{a}{z}}$$

$$\Rightarrow F_{net} = \frac{\mu_0 I}{2\pi R} \frac{a^2}{z^2} \frac{1}{1+\frac{a}{z}} \cdot v \times \frac{a^2 \mu_0 I}{2\pi z^2} \frac{1}{1+\frac{a}{z}} \Rightarrow$$

$$\Rightarrow \boxed{F_{net} = \frac{\mu_0^2 I^2}{4\pi^2 R} \frac{a^4}{z^4} \frac{1}{\left(1+\frac{a}{z}\right)^2} v}$$

(d) $\boxed{F_{net} \cdot v = P}$ correct, the work done per unit time in moving the loop = power dissipated in loop.

Problem 4



$R = \frac{\rho l}{A}$ = resistance

$a_2/a_1 = 10$

$M = M_{12} = \frac{\Phi_{12}}{I_2}$; Φ_{12} = flux through 1 due to current in 2.

$\Phi_{12} = \mu_0 \frac{I_2}{2a_2} \pi a_1^2$ since $a_2 >> a_1 \Rightarrow$ $\boxed{M = \mu_0 \frac{\pi}{2} \frac{a_1^2}{a_2}}$ (a)

(b) When $I(t)$ circulates in ~~outer~~ inner ring, we have in outer ring

$\mathcal{E}_2 = -M \frac{\partial I}{\partial t}$, $\quad I_2 = \frac{\mathcal{E}_2}{R_2} \Rightarrow$ $\boxed{I_2 = \frac{M}{R_2} \frac{\partial I}{\partial t}}$

If instead $I(t)$ circulates in outer ring, we have in inner ring

$\mathcal{E}_1 = -M \frac{\partial I}{\partial t}$, $\quad I_1 = \frac{\mathcal{E}_1}{R_1} \Rightarrow$ $\boxed{I_1 = \frac{M}{R_1} \frac{\partial I}{\partial t}}$

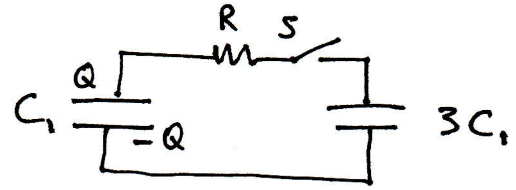
therefore, $\boxed{I_1(t) = I_2(t) \frac{R_2}{R_1}} = I_2(t) \cdot \frac{2\pi a_2}{2\pi a_1} = I_2(t) \frac{a_2}{a_1}$

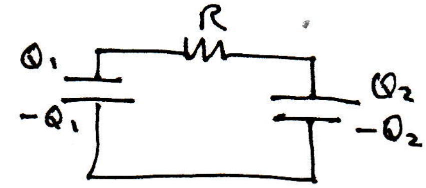
Since $I_2(t = 1s) = 3mA \Rightarrow$ $\boxed{I_1(t = 1s) = 30\,mA}$ (b)

(c) Since $I(t) = I_0 t^2/\tau^2 \Rightarrow \frac{\partial I}{\partial t} = 2 I_0 \frac{t}{\tau^2}$ linear in $t \Rightarrow$

$\boxed{I_1(t = 2s) = 60\,mA}$

Problem 5





(a) When S is closed, potential diff across both capacitors is same

$$\Rightarrow \quad \frac{Q_1}{C_1} = \frac{Q_2}{C_2} = \frac{Q_2}{3C_1} \Rightarrow 3Q_1 = Q_2 \quad , \text{ and } Q_1 + Q_2 = Q \Rightarrow$$

$$\Rightarrow \quad 4Q_1 = Q \Rightarrow \boxed{Q_1 = Q/4 \quad , \quad Q_2 = \frac{3}{4} Q}$$

(b) $$\boxed{\frac{Q_1}{C_1} - IR - \frac{Q_2}{C_2} = 0} \quad , \text{ and } Q_1(t) = Q - Q_2(t), \text{ and } I(t) = \frac{dQ_2}{dt}$$

$$\Rightarrow \quad \frac{Q - Q_2(t)}{C_1} - \frac{dQ_2}{dt} R - \frac{Q_2}{3C_1} = 0 \Rightarrow$$

$$\Rightarrow \quad \boxed{3RC_1 \frac{dQ_2}{dt} + 4Q_2 = 3Q} \quad (c)$$

(d) Homogeneous eq: $\frac{dQ_2}{dt} = -\frac{4}{3RC_1} Q_2 \Rightarrow Q_2(t) = A e^{-\frac{4}{3RC_1}t}$

Particular solution: $4Q_2 = 3Q \Rightarrow Q_2 = 3/4\, Q$

General solution: $Q_2(t) = \frac{3}{4} Q + A e^{-\frac{4}{3RC_1}t}$. Initial condition is

$$Q_2(t) = 0 \Rightarrow A = -\frac{3}{4} Q \Rightarrow$$

$$\boxed{Q_2(t) = \frac{3}{4} Q \left(1 - e^{-\frac{4}{3RC_1}t}\right)}$$

satisfies $Q_2(0) = 0$, $Q_2(\infty) = \frac{3}{4} Q$

(e) The energy dissipated is the difference in initial and final energies.

$$U_{in} = \frac{Q^2}{2C_1} \; ; \; U_{fin} = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} = \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{6C_1} \Rightarrow$$

$$U_{fin} = \frac{Q^2}{32C_1} + \frac{\overset{3}{\cancel{9}}\, Q^2}{16 \cdot \underset{2}{\cancel{6}}\, C_1} = \frac{Q^2}{8C_1} \Rightarrow U_{in} - U_{fin} = \frac{3}{8}\frac{Q^2}{C_1}$$

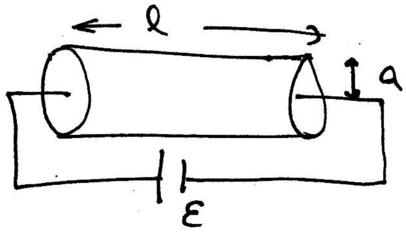
So energy dissipated = $\boxed{W = \frac{3}{8}\frac{Q^2}{C_1}}$

Check: we should have $W = \int_0^\infty dt\, I(t)^2 R$

$$I(t) = \frac{dQ_2}{dt} = \frac{\cancel{3}}{\cancel{4}} Q \cdot \frac{\cancel{4}}{\cancel{3}RC_1} e^{-\frac{4}{3RC_1}t} = \frac{Q}{RC_1} e^{-\frac{4}{3RC_1}t} \Rightarrow$$

$$\Rightarrow W = \frac{Q^2}{R^2C_1^2} R \int_0^\infty dt\, e^{-\frac{8}{3RC_1}t} = \frac{Q^2}{\cancel{R^2}C_1^{\cancel{2}}} \cancel{R} \cdot \frac{3\cancel{R}\cancel{C_1}}{8} = \frac{3}{8}\frac{Q^2}{C_1} \longleftarrow$$

Problem 6



$A = \pi a^2 =$ cross-sectional area

Resistance is $R = \frac{\rho \cdot \ell}{A}$, current is $I = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}}{\rho \ell} \cdot A$

Current density $\boxed{J = \frac{I}{A} = \frac{\mathcal{E}}{\rho \ell}}$

(b) $P = I_{r_0}^2 \cdot R_{r_0}$ in the region $r < r_0$, with

$$I_{r_0} = J \cdot \pi r_0^2 = \frac{\mathcal{E}}{\rho \ell} \pi r_0^2 \; ; \; R_{r_0} = \frac{\rho \cdot \ell}{\pi r_0^2} \Rightarrow$$

$$P = \frac{\mathcal{E}^2}{\cancel{\rho^2} \ell^{\cancel{2}}} \pi^{\cancel{2}} r_0^{\cancel{4}\,2} \cdot \frac{\cancel{\rho}\,\cancel{\ell}}{\cancel{\pi} \cancel{r_0^2}} = \boxed{\frac{\mathcal{E}^2 \pi r_0^2}{\rho \ell} = P}$$

(c) $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ is Poynting vector.

$\boxed{E = \mathcal{E}/\ell}$ points along axis of cylinder.

We get B at r_0 from Ampere law: $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \Rightarrow$

$$B \cdot 2\pi r_0 = \mu_0 \cdot J \cdot \pi r_0^2 = \mu_0 \frac{\mathcal{E}}{\rho \ell} \pi r_0^2 \Rightarrow \boxed{B = \mu_0 \frac{\mathcal{E}}{2 \rho \ell} r_0}$$

E and B are perpendicular $\Rightarrow$

$$\boxed{S = \frac{E B}{\mu_0} = \frac{\mathcal{E}^2}{2 \rho \ell^2} r_0}$$ it points towards axis of cylinder.

(d) The energy influx is $\oint \vec{S} \cdot d\vec{a} = S \cdot 2\pi r_0 \cdot \ell \Rightarrow$

$$\oint \vec{S} \cdot d\vec{a} = \frac{\mathcal{E}^2}{\cancel{2} \rho \ell^2} r_0 \cdot \cancel{2} \pi r_0 \ell = \boxed{\frac{\mathcal{E}^2 \pi r_0^2}{\rho \ell}}$$

So we find $\boxed{\oint \vec{S} \cdot d\vec{a} = P}$ by conservation of energy.

Problem 7

$$\vec{E} = E_0 \hat{x} \cos(kx+\omega t) + E_0 \hat{y} \sin(kz+\omega t)$$

(a) Wave propagates in $-\hat{z}$ direction.

$\vec{E} \times \vec{B}$ should be in direction of propagation

$E_0 = B_0 c$ for electromagnetic wave $\Rightarrow$

$$\boxed{\vec{B} = -\frac{E_0}{c} \hat{y} \cos(kx+\omega t) + \frac{E_0}{c} \hat{x} \sin(kz+\omega t)}$$

(b) Faraday's law: $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

$$\vec{\nabla} \times \vec{E} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_0 \cos & E_0 \sin & 0 \end{pmatrix} = -\hat{x} \frac{\partial}{\partial z} E_0 \sin(kz+\omega t) + \hat{y} \frac{\partial}{\partial z} E_0 \cos(kz+\omega t)$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{E} = -E_0 k \hat{x} \cos(kz+\omega t) - E_0 k \hat{y} \sin(kz+\omega t)}$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = +\frac{E_0}{c} \omega \hat{y} \sin(kz+\omega t) + \frac{E_0}{c} \omega \hat{x} \cos(kz+\omega t)}$$

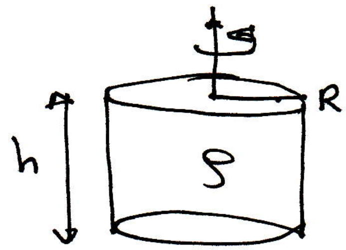
$E_0 k = E_0 \frac{\omega}{c}$ since $\omega / k = c$ for em wave $\Rightarrow$ it is satisfied.

(c) $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ points in $-\hat{z}$ direction

$$\vec{E} \times \vec{B} = \left[-\frac{E_0^2}{c} \cos^2(kz+\omega t) - \frac{E_0^2}{c} \sin^2(kz+\omega t) \right] \hat{z}$$

$$\Rightarrow \boxed{\vec{S} = -\frac{E_0^2}{c \mu_0} \hat{z}}$$

Problem 8



(a) Consider a ring of height dh and radius r

$dq = \rho \cdot 2\pi r\, dr\, dh.$ Period: $\omega = 2\pi f = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega}{2\pi}$

current: $di = \frac{dq}{T} = \rho \cdot 2\pi r\, dr\, dh \frac{\omega}{2\pi}$

magnetic moment: $dm = di \cdot \pi r^2 = \rho \pi r^3 dr\, dh\, \omega$

Integrate to find total magnetic moment:

$$m = \int dm = \frac{\rho \pi \omega h R^4}{4} \Rightarrow \boxed{\vec{m} = \frac{\rho \pi \omega h R^4}{4} \hat{z}} \quad (a)$$

(b) Angular momentum:

$$L = I\omega = \frac{MR^2}{2}\omega = \frac{\rho_m \cdot \pi R^2 h \cdot R^2}{2}\omega = \frac{\rho_m \pi \omega h R^4}{2}$$

$$\Rightarrow \boxed{\vec{L} = \frac{\rho_m \pi \omega h R^4}{2} \hat{z}}$$

(c) $\boxed{\vec{m} = \frac{\rho}{2\rho_m} \vec{L}}$

which agrees with the general formula $\boxed{\vec{m} = \frac{q}{2M} \vec{L}}$

since $q = \rho V$, $M = \rho_m V$, with V the volume.