7.24. **Pulling a frame**

We could solve this exercise piecemeal, but let's instead derive a single expression for the force, which will take care of all the questions. Let ℓ be the total perimeter of the rectangle, and let b be the length of the side that sweeps through the field. The current in the frame is $I = \mathcal{E}/R$, where $\mathcal{E} = d\Phi/dt = Bbv$, and where $R = \rho\ell/A = \rho\ell/\pi r^2$. So $I = Bbv/(\rho\ell/\pi r^2) = Bbv\pi r^2/\rho\ell$. The force on the trailing side of the frame is $F = IBb$, and you can show with Lenz's law and the right-hand rule that this force is directed to the left; that is, it is a drag force. The force required to balance the magnetic drag force therefore equals

$$F = \frac{B^2 b^2 v \pi r^2}{\rho\ell}\,. \tag{491}$$

(You can check that this does indeed have units of force.) Since we are ignoring the inertia of the frame, the applied force must be exactly equal to the magnetic force, in magnitude. (If $m = 0$, then $\sum \mathbf{F} = m\mathbf{a}$ implies that $\sum \mathbf{F} = 0$.) For any particular F that you pick, Eq. (491) can be solved for the velocity v that the frame will have. We can now answer the various questions.

Eq. (491) implies that twice the force means twice the velocity. So a force of 2 N will pull the frame out in half the time, or 0.5 sec.

Keeping everything else the same, doubling ρ means halving F (there is half as much current). So a brass frame will be pulled out in 1 sec by a force of 0.5 N.

Doubling the radius increases F by a factor $2^2 = 4$ (there is four times as much current). So a 1 cm aluminum frame will be pulled out in 1 sec by a force of 4 N. (We effectively have four of the original frames stacked on top of each other, each of which requires 1 N.)

7.25. **Sliding loop**

In Fig. 130 the y axis points into the page. We've arbitrarily chosen the current in the wire to flow in the negative y direction (out of the page), but the sign doesn't matter since all we care about is the magnitude of the emf. At the leading edge of the square loop, the magnitude of B is $\mu_0 I/2\pi r$, where $r = \sqrt{h^2 + (b/2)^2}$. Only the z component matters in the flux, and this brings in a factor of $(b/2)/r$. So

$$B_z = \frac{\mu_0 I}{2\pi r}\frac{b/2}{r} = \frac{\mu_0 I b}{4\pi(h^2 + b^2/4)}\,. \tag{492}$$

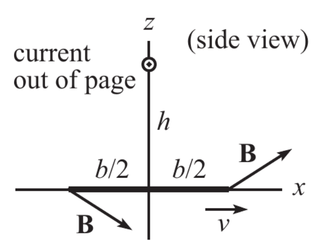

Figure 130

At the trailing edge, B_z has the opposite sign. If the loop moves a small distance $v\,dt$, there is additional positive flux through a thin rectangle with area $b(v\,dt)$ at the leading edge, and also less negative flux through a similar rectangle at the trailing edge. Both of these effects cause the upward flux to increase. Therefore,

$$\mathcal{E} = \frac{d\Phi}{dt} = 2\frac{b(v\,dt)B_z}{dt} = 2bvB_z = \frac{\mu_0 I b^2 v}{2\pi(h^2 + b^2/4)}\,. \tag{493}$$

The flux is increasing upward. So for our choice of direction of the current in the wire, the induced emf is clockwise when viewed from above, because that creates a downward field inside the loop which opposes the change in flux. For $h = 0$ (or in general for $h \ll b$) $\mathcal{E}$ reduces to $2\mu_0 Iv/\pi$. This is independent of b because the field at the leading and trailing edges decreases with b, while the length of the thin rectangles at these edges increases with b.

You can show that our result for $\mathcal{E}$ has the correct units, either by working them out explicitly, or by noting that $\mathcal{E}$ has the units of B (which are the same as $\mu_0 I/2\pi r$) times length squared divided by time, which correctly gives flux per time.

7.26. **Sliding bar**

(a) Let v be the instantaneous velocity of the bar. The area of the circuit increases at a rate $b(v\,dt)/dt = bv$, so the induced emf is $\mathcal{E} = d\Phi/dt = Bbv$. The current is therefore $I = \mathcal{E}/R = Bbv/R$. The general expression for the force on piece of wire (the bar in our setup) is $F = IBb$, which yields B^2b^2v/R here. So $F = ma$ gives (including the minus sign because the force opposes the motion, as you can check with Lenz's law and the right-hand rule)

$$-F = m\frac{dv}{dt} \implies -\frac{B^2b^2v}{R} = m\frac{dv}{dt} \implies -\int_0^t \frac{B^2b^2}{mR}\,dt' = \int_{v_0}^{v} \frac{dv'}{v'}$$
$$\implies -\frac{B^2b^2t}{mR} = \ln\left(\frac{v}{v_0}\right) \implies v = v_0 e^{-t/T}, \qquad \text{where } T \equiv \frac{mR}{B^2b^2}\,. \tag{494}$$

(You can check that T has units of time.) We see that the velocity decreases exponentially, so technically the rod never stops moving (in an ideal world). This exponential decay of v is a familiar result for forces that are proportional to (the negative of) v.

(b) The total distance traveled in the limit $t \to \infty$ is

$$x = \int_0^\infty v\,dt = \int_0^\infty v_0 e^{-t/T}\,dt = -v_0 T e^{-t/T}\Big|_0^\infty = v_0 T = \frac{v_0 mR}{B^2b^2}\,. \tag{495}$$

So the rod travels a finite distance in an infinite time.

(c) The initial kinetic energy of the rod is $mv_0^2/2$. This must eventually show up as heat in the resistor, so let's check this. The instantaneous power dissipated in the resistor is $P = I^2R$, where I is given above as $I = Bbv/R = (Bbv_0/R)e^{-t/T}$. The total energy loss in the resistor is therefore

$$\begin{aligned}\int_0^\infty I^2R\,dt &= \frac{B^2b^2v_0^2}{R}\int_0^\infty e^{-2t/T}\,dt = -\frac{B^2b^2v_0^2}{R}\frac{T}{2}\,e^{-2t/T}\Big|_0^\infty \\ &= \frac{B^2b^2v_0^2}{R}\frac{T}{2} = \frac{B^2b^2v_0^2}{R}\frac{mR}{2B^2b^2} = \frac{1}{2}mv_0^2.\end{aligned} \tag{496}$$

7.27. Ring in a solenoid

(a) The magnetic field inside the solenoid is $B(t) = \mu_0 n I(t) = \mu_0 n I_0 \cos\omega t$. Faraday's law applied to the given ring yields

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\pi r^2 \frac{dB}{dt} = \pi r^2 \mu_0 n I_0 \omega \sin\omega t. \tag{497}$$

With the given positive direction of I, the right-hand rule gives the positive direction of B as upward, and then also gives the positive direction of $\mathcal{E}$ as counterclockwise when viewed from above (as for I). The current in the loop is $I_{\text{loop}}(t) = \mathcal{E}/R = (\pi r^2 \mu_0 n I_0 \omega / R) \sin\omega t$.

(b) The force on a little piece of the ring is $F(t) = I_{\text{loop}}(t)\, d\boldsymbol{l} \times \mathbf{B}$. With positive I counterclockwise and positive B upward, this force is radial and equals

$$F(t) = \frac{\pi r^2 \mu_0 n I_0 \omega}{R} \sin\omega t \cdot dl \cdot \mu_0 n I_0 \cos\omega t = \frac{\pi r^2 \mu_0^2 n^2 I_0^2 \omega\, dl}{R} \sin\omega t \cos\omega t. \tag{498}$$

The force is radially outward if this quantity is positive, inward if it is negative. Since $\sin\omega t \cos\omega t = (1/2)\sin(2\omega t)$ we see that the force is maximum outward when $\omega t = \pi/4$ (plus multiples of π), and maximum inward when $\omega t = 3\pi/4$ (plus multiples of π).

(c) Since the force lies in the horizontal plane, it serves only to stretch/shrink the ring (negligibly, if the ring is rigid).

7.35. M for two rings

From Eq. (6.53), the magnetic field along the axis of a ring of radius a, a distance b from the center, is $B = \mu_0 I a^2/2(a^2 + b^2)^{3/2}$. For $b \gg a$ this can be approximated as $B = \mu_0 I a^2/2b^3$. In this limit we can also neglect the variation of B over the interior of the other ring. The flux through the other ring is therefore $\Phi = \pi a^2 B = \mu_0 \pi I a^4/2b^3$. The mutual inductance is then $\Phi/I = \mu_0 \pi a^4/2b^3$.

7.37. Flux through two rings

Figure 137 shows a side view of the field due to the inner ring. (The dots are the intersections of the rings with the plane of the paper.) The key point here is that the flux through the outer ring comes not only from the field lines pointing upward in the interior of the inner ring, but also from the field lines pointing downward in the region between the rings. The latter flux partially cancels the former flux. The larger the outer ring is, the larger this canceling effect is, and so the smaller the net flux is. The field lines within the dotted curves yield a net flux of zero through the outer ring, so it is only the lines in the central region that contribute to the net flux. The larger the outer ring is, the smaller this central region is.

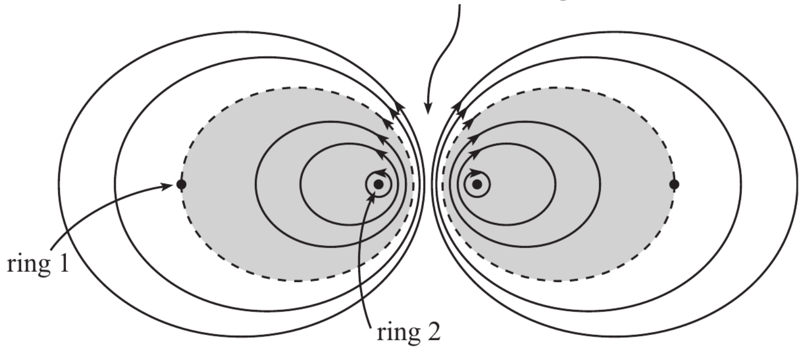

Figure 137

7.39. **Small** L

One way to wind resistance wire into a "non-inductive" coil is shown in Fig. 138. Of course, the inductance is not exactly zero. The residual inductance is approximately that of the long, narrow "hair-pin" configuration shown in Fig. 139. Technically, if the wire is infinitely thin, then the self-inductance is actually infinite, due to the issue discussed at the end of Section 7.8. But real wires have thickness, so the self-inductance of the hair pin will indeed be small.

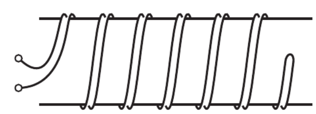

Figure 138

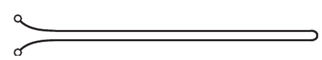

Figure 139

Note that the configuration in Fig. 138 effectively consists of two solenoids with currents in opposite directions. So there is essentially no B field inside the cylinder. However, this is actually irrelevant, because the area relevant to the flux is *not* the cross-sectional areas of all the circular loops. Rather, the area spanned by the wire in Fig. 138 is the hair-pin area in Fig. 139, which wraps around the surface of the cylinder. This area has nothing to do with the inside of the cylinder.

7.41. **Opening a switch**

After the switch has been closed a while, the currents are steady and the inductor is irrelevant. So 10 V is the initial voltage across each of the branches of the circuit. The initial currents across the 150 Ω and 50 Ω resistors are therefore 0.067 A and 0.2 A, respectively. They are both directed downward. Initially both A and B are at 10 V with respect to ground.

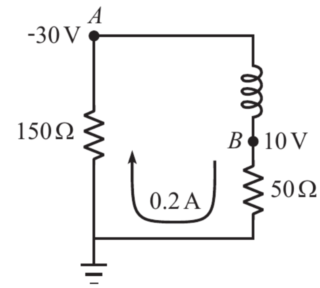

Figure 140

Right after the switch is opened, we have the circuit shown in Fig. 140. The current through the inductor cannot change abruptly (otherwise there would be an infinite $d\Phi/dt$ and hence infinite $\mathcal{E}$, which would cause the current to not change abruptly after all). Therefore, the current through the circuit is 0.2 A in the clockwise direction. The current *does* change abruptly in the 150 Ω resistor; it goes from 0.067 A downward to 0.2 A upward. The potential at B with respect to ground is still $V_B = (0.2\,\text{A})(50\,\Omega) = 10\,\text{V}$, but the potential of A is now $V_A = -(0.2\,\text{A})(150\,\Omega) = -30\,\text{V}$.

The circuit in Fig. 140 is a simple RL circuit, so as time goes on, the current equals $I(t) = I_0 e^{-(R/L)t}$, where $I_0 = 0.2\,\text{A}$ and where the time constant L/R equals $(0.1\,\text{H})/(200\,\Omega) = 5 \cdot 10^{-4}\,\text{s} = 0.5$ millisec. The potentials at A and B are proportional to I, so they decrease like $e^{-(R/L)t}$. After 0.5 millisec they have decreased by a factor $1/e = 0.37$, and after 1 millisec by $1/e^2 = 0.14$. After 5 millisec the factor is $1/e^{10} = 4.5 \cdot 10^{-5}$, which is negligible. The plots are shown in Fig. 141. We have only plotted up to $t = 2$ millisec, because the curves are essentially zero after that. Note the discontinuity in V_A.

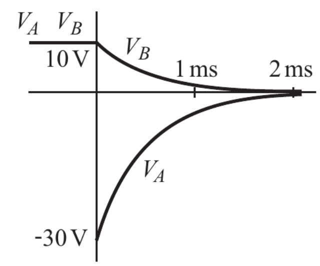

Figure 141

7.42. RL **circuit**

From Eq. (7.69) the current is $I(t) = I_0\big(1 - e^{-(R/L)t}\big)$, where $I_0 = \mathcal{E}_0/R$. In the problem at hand,

$$I_0 = \frac{\mathcal{E}_0}{R} = \frac{12\,\text{V}}{0.01\,\Omega} = 1200\,\text{A} \qquad \text{and} \qquad \frac{R}{L} = \frac{0.01\,\Omega}{0.5 \cdot 10^{-3}\,\text{H}} = 20\,\text{s}^{-1}. \tag{516}$$

So the time scale is $L/R = 0.05\,\text{s}$. The current reaches a value of $(0.9)I_0$ when

$$e^{-(R/L)t} = 0.1 \quad \Longrightarrow \quad (20\,\text{s}^{-1})t = \ln 10 \quad \Longrightarrow \quad t = 0.115\,\text{s}. \tag{517}$$

At this time, the current is $I = (0.9)(1200) = 1080\,\text{A}$, so the energy stored in the magnetic field is

$$\frac{1}{2}LI^2 = \frac{1}{2}(0.5 \cdot 10^{-3}\,\text{H})(1080\,\text{A})^2 = 292\,\text{J}. \tag{518}$$

The instantaneous power delivered by the battery is $\mathcal{E}_0 I$, but since I is changing we must perform an integral to find the energy delivered by the battery between $t = 0$ and $t = 0.115\,\text{s}$:

$$\begin{aligned}
\int_0^{t=0.115\,\text{s}} \mathcal{E}_0 I(t')\,dt' &= \mathcal{E}_0 I_0 \int_0^t \big(1 - e^{-(R/L)t'}\big)\,dt' \\
&= \mathcal{E}_0 I_0 \left(t' + \frac{L}{R}e^{-(R/L)t'}\right)\Bigg|_0^t \\
&= \mathcal{E}_0 I_0 \left(t + \frac{L}{R}e^{-(R/L)t} - \frac{L}{R}\right) \\
&= (12\,\text{V})(1200\,\text{A})\Big(0.115\,\text{s} + (0.05\,\text{s})(0.1) - (0.05\,\text{s})\Big) \\
&= 1008\ \text{J}.
\end{aligned} \tag{519}$$

From conservation of energy, apparently $1008\,\text{J} - 292\,\text{J} = 716\,\text{J}$ has been dissipated in the resistor. The task of Problem 7.15 is to show that the energy delivered by the battery does indeed equal the energy stored in the magnetic field plus the energy dissipated in the resistor, at any general time t.