6.37. **Off-center hole**

The given setup is equivalent to the superposition of a complete solid rod with current flowing into the page plus a smaller rod (where the hole is) with current flowing out of the page. If the two current densities are equal and opposite, then there will be zero current in the hole, in agreement with the given setup. Given the ratio of the areas of the two circular cross sections, currents of 1200 A into the page and 300 A out of the page will yield the given 900 A into the page. The large rod produces zero field on its axis, so the desired field is due entirely to the smaller rod with 300 A coming out of the page. The magnitude of the field is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(300\,\text{A})}{2\pi(0.02\,\text{m})} = 0.003\ \text{T}, \tag{436}$$

or 30 gauss, and it points to the left. A more remarkable fact (see Exercise 6.38) is that the field is 30 gauss pointing to the left not only at P but everywhere inside the cylindrical hole.

6.40. **The pinch effect**

If the conduction electrons are forced closer to the axis, there will be uncompensated negative charge near the axis. This will generate an inward radial electric field E that pushes outward on the electrons, preventing further constriction when the outward electric force balances the inward magnetic force, that is, when $eE = evB \Longrightarrow E = vB$.

The magnetic field at radius r is $B_r = \mu_0 I_r/2\pi r$, where I_r is the current contained within radius r. Assuming no redistribution of the charge, I_r is given by $I_r = \pi r^2 J$, where $J = nev$ is the current density (n is the number of electrons per unit volume, and v is the drift velocity). The B field is therefore $B_r = \mu_0(\pi r^2 nev)/2\pi r = \mu_0 rnev/2$.

Suppose that the cloud of electrons at radius r is squeezed inward by a small distance Δr. The cylinder of radius r will now contain, per unit length, an excess of negative charge in the amount of $\Delta\lambda = (ne)(2\pi r\,\Delta r)$; this is the volume charge density times the cross-sectional area. This causes an inward electric field equal to $E_r = \Delta\lambda/2\pi\epsilon_0 r = ne\,\Delta r/\epsilon_0$. The condition for equilibrium in then (using $\mu_0\epsilon_0 = 1/c^2$)

$$E_r = vB_r \quad\Longrightarrow\quad \frac{ne\,\Delta r}{\epsilon_0} = v\frac{\mu_0 rnev}{2} \quad\Longrightarrow\quad \frac{\Delta r}{r} = \frac{\mu_0\epsilon_0 v^2}{2} = \frac{v^2}{2c^2}\,. \tag{443}$$

In solid conductors we always have $v/c \ll 1$. In metal conduction, v/c is seldom much greater than 10^{-10}, so $(\Delta r)/r \approx 10^{-20}$ is too small to detect. In highly ionized gases, however, the "pinch effect," as it is called, can be not only detectable but important.

If the effect were large enough to measure, a Hall probe in the spirit of Fig. 6.33 could be used, with one lead connected to the axis (by drilling a thin tube in the rod), and the other lead connected to the surface of the rod. If $v \approx 10^{-3}$ m/s and $B \approx 1$ T, the resulting $E \approx 10^{-3}$ V/m would be large enough to generate a measurable voltage difference.

6.44. **Line integral along the axis**

The magnetic field on the axis is $B_z = \mu_0 I b^2/2(b^2+z^2)^{3/2}$, so the given line integral is (using the integral table in Appendix K)

$$\int_{-\infty}^{\infty} B_z\,dz = \frac{\mu_0 I b^2}{2}\int_{-\infty}^{\infty}\frac{dz}{(b^2+z^2)^{3/2}} = \frac{\mu_0 I b^2}{2}\,\frac{z}{b^2(b^2+z^2)^{1/2}}\bigg|_{-\infty}^{\infty} = \frac{\mu_0 I b^2}{2}\,\frac{2}{b^2} = \mu_0 I, \tag{449}$$

as desired. If you want, you can derive this integral with a trig substitution, $z = b\tan\theta$.

To see why the integral along the axis should indeed be equal to $\mu_0 I$, consider the closed path shown in Fig. 110, which involves a semicircle touching the points $z = \pm r$. Assume that $r \gg b$. Along the z axis, B_z behaves like $1/z^3$ for $z \gg b$. And $|\mathbf{B}|$ also behaves like $1/r^3$ along the (large) semicircle. Accepting that this is true (see below), then since the length of the semicircle is proportional to r, the line integral along the semicircle is at least as small (in order of magnitude) as $r/r^3 = 1/r^2$, which goes to zero as $r\to\infty$. We can therefore ignore the return semicircular path. So the line integral along the whole loop (which encloses a current I) equals the line integral along the z axis, in the $r\to\infty$ limit.

Let's now argue why $|\mathbf{B}|$ behaves like $1/r^3$ for large r. Consider the point at the "side" of the semicircle in Fig. 110. In order of magnitude, the field at this point, due to the ring, is the same as the field due to a square with side b. But the field due to the square has contributions from two opposite sides (the sides perpendicular to the $\hat{\mathbf{r}}$ vector) that nearly cancel, because the current moves in opposite directions along these sides. The Biot-Savart law says that each side gives a contribution of order $1/r^2$. Taking the difference of these contributions is essentially the same as taking a derivative, and the derivative of $1/r^2$ is proportional to $1/r^3$, as desired. Additionally, the two sides parallel to the $\hat{\mathbf{r}}$ vector also happen to produce a contribution of order $1/r^3$; see Problem 6.14. At points in between the axis and the "side" point on the semicircle, there will be various angles that come into play. But these simply bring in factors of order 1 that morph the $1/z^3$ result on the axis to the $1/r^3$ result at the side point, so they don't change the general $1/r^3$ result.

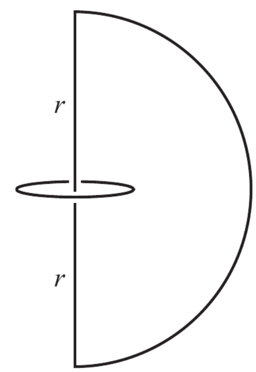

Figure 110

6.45. **Field from an infinite wire**

Consider a small piece of the wire at angle θ, subtending an angle $d\theta$, as shown in Fig. 111. If r is the distance from a given point P to the small piece, then Fig. 112 shows that the length of the piece is $dl = r\,d\theta/\cos\theta$. But r equals $b/\cos\theta$, so $dl = b\,d\theta/\cos^2\theta$. (This can also be obtained by taking the differential of $l = b\tan\theta$.) In the Biot-Savart law, the cross product between $d\boldsymbol{l}$ and $\hat{\mathbf{r}}$ brings in a factor of $\sin\phi$, which is the same as $\cos\theta$. If the current points rightward, then we have (with $\hat{\mathbf{z}}$ pointing out of the page)

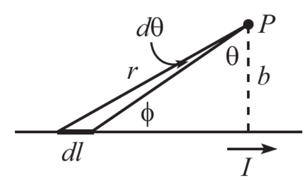

Figure 111

$$\begin{aligned}\mathbf{B} &= \frac{\mu_0 I}{4\pi}\int\frac{d\boldsymbol{l}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{(b\,d\theta/\cos^2)\cos\theta}{(b/\cos\theta)^2}\hat{\mathbf{z}}\\ &= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi b}\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta = \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi b}\sin\theta\bigg|_{-\pi/2}^{\pi/2} = \hat{\mathbf{z}}\frac{\mu_0 I}{2\pi b}\,, \end{aligned} \tag{450}$$

which agrees with the standard result obtained more much quickly via Ampere's law.

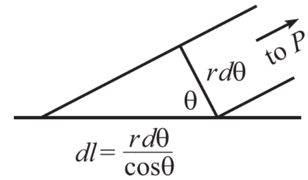

Figure 112

6.47. **Field at the center of an orbit**

The time for one revolution is $t = 2\pi r/v$, so the average current is $I = e/t = ev/2\pi r$. From Eq. (6.54) the field at the center of the orbit is therefore

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ev}{4\pi r^2} = \frac{\left(4\pi \cdot 10^{-7} \, \frac{\text{kg m}}{\text{C}^2}\right)(1.6 \cdot 10^{-19}\,\text{C})(0.01 \cdot 3 \cdot 10^8\,\text{m/s})}{4\pi (10^{-10}\,\text{m})^2} = 4.8\ \text{T}. \quad (451)$$

6.48. **Fields from two rings**

The Biot-Savart law is $d\mathbf{B} = (\mu_0/4\pi) I\, d\boldsymbol{l} \times \hat{\mathbf{r}}/r^2$. Consider corresponding pieces of the two rings that subtend the same angle $d\theta$. The dl for the larger piece is twice the dl for the smaller piece. And the I for the larger ring is also twice the I for the smaller ring, because I is proportional to the speed of the ring, which in turn is proportional the radius, because the ω's are the same. These two powers of 2 in the numerator cancel the two powers of 2 in the r^2 in the denominator, so the fields at the centers of the two rings are the same. This reasoning works for any ratio of ring sizes, of course. In terms of the various parameters, you can show that the field at the center is $B = \mu_0 \lambda \omega/2$, which is independent of r, as we just showed.

6.49. **Field at the center of a disk**

Consider a ring with radius r and thickness dr. The effective linear charge density along the ring is $d\lambda = \sigma\, dr$. The speed of all points on the ring is $v = \omega r$, so the current in the ring is $dI = (d\lambda)v = (\sigma\, dr)(\omega r)$. From the Biot-Savart law, a small piece of the ring with length dl produces a $d\mathbf{B}$ field at the center that points perpendicular to the ring and has magnitude $(\mu_0/4\pi) I\, dl/r^2$. Integrating over the whole ring turns the dl into $2\pi r$, so the field at the center due to the ring is $(\mu_0/4\pi)(\sigma\omega r\, dr)(2\pi r)/r^2 = \mu_0 \sigma\omega\, dr/2$. Integrating over r (that is, integrating over all the rings in the disk) turns the dr into an R, so the field at the center equals $\mu_0\sigma\omega R/2$. It points perpendicular to the disk, with the direction determined by the righthand rule.

6.50. **Hairpin field**

Each of the two straight segments contributes half the field of an infinite wire. (The contributions do indeed add and not cancel.) The semicircle contributes half the field of an entire ring at the center, which is given by Eq. (6.54). The desired field therefore points out of the page and has magnitude

$$B = 2 \cdot \frac{1}{2}\frac{\mu_0 I}{2\pi r} + \frac{1}{2}\frac{\mu_0 I}{2r} = \left(\frac{1}{2\pi} + \frac{1}{4}\right)\frac{\mu_0 I}{r} = (0.409)\frac{\mu_0 I}{r}\,. \quad (452)$$

6.55. **Helmholtz coils**

Let the symmetry axis of the setup be the z axis, and let the centers of the rings be located at $z = \pm b/2$. If the currents in the rings are equal and point in the same direction, then from Eq. (6.53) the field along the axis at position z is given by

$$B_z(z) \propto \frac{1}{[a^2 + (z + b/2)^2]^{3/2}} + \frac{1}{[a^2 + (z - b/2)^2]^{3/2}} . \tag{459}$$

If we expand this function in a Taylor series around $z = 0$, the first derivative and all other odd derivatives are zero at $z = 0$, because $B_z(z)$ is an even function of z, due to the symmetry of the setup. So the function will be most uniform near $z = 0$ if the second derivative is zero there. The deviations will then be of order z^4. That is, the Taylor series will look like $B_z(z) = B_z(0) + Cz^4 + \cdots$. Differentiating the first term in Eq. (459) twice and evaluating the result at $z = 0$ yields

$$3\frac{5(z + b/2)^2 - [a^2 + (z + b/2)^2]}{[a^2 + (z + b/2)^2]^{7/2}}\bigg|_{z=0} = \frac{3(b^2 - a^2)}{[a^2 + b^2/4]^{7/2}} . \tag{460}$$

The second derivative of the second term in Eq. (459) simply involves replacing $b/2$ with $-b/2$, so we end up with the same result, because there are no odd powers of b in Eq. (460). We therefore see that the second derivative is zero at $z = 0$ if $a = b$. You can show that if $a = b$, the field halfway from $z = 0$ to the plane of each ring (that is, at $z = \pm b/4$) is only 0.4% smaller than the field at $z = 0$. And at $z = \pm b/8$ the field is only 0.03% smaller. Two coils arranged with $a = b$ are called Helmholtz coils.

A continuity argument shows why there must exist a point where the second derivative of $B_z(z)$ equals zero. If the rings are far apart (for example, if $b = 4a$), then the plot of B_z consists of two bumps, as shown in Fig. 119. But if the rings are close together (for example, if $b = a/4$), then they act effectively like one ring with twice the current, so there is just one bump. The second derivative at $z = 0$ is positive in the former case, and negative in the latter, so somewhere in between it must be zero.

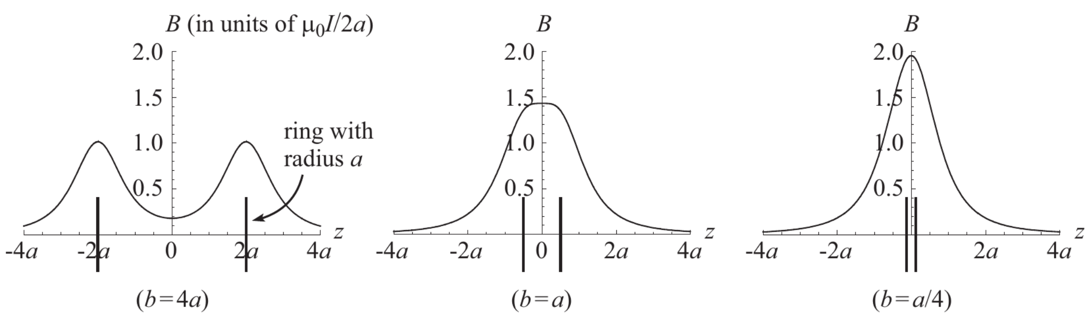

Figure 119

6.57. **A rotating cylinder**

Eq. (6.57) gives the magnetic field inside an infinite solenoid as $B = \mu_0 n I$, where n is the number of turns per unit length. The surface current density (per unit length) is $\mathcal{J} = nI$, so we can write the field as $B = \mu_0 \mathcal{J}$.

What is the current density in our rotating cylinder? The amount of charge that passes a given segment of length ℓ on the cylinder in a time dt is $dq = \sigma\ell(v\,dt)$. The current per unit length (that is, the surface current density) is therefore $\mathcal{J} = (1/\ell)(dq/dt) = \sigma v$. In terms of the angular frequency, $\mathcal{J}$ equals $\sigma\omega R$.

To find the field inside the rotating cylinder, we simply need to replace the current density $\mathcal{J} = nI$ in the original solenoid formula with the present current density $\mathcal{J} = \sigma\omega R$. This yields a field of $B = \mu_0\sigma\omega R$.

6.73. **Hall voltage**

Our strategy will be to find the current density, then the drift velocity, then the transverse field, then the transverse (Hall) voltage. The current density is

$$J = \frac{I}{A} = \frac{V/R}{A} = \frac{V/(\rho L/A)}{A} = \frac{V}{\rho L} = \frac{1\,\text{V}}{(0.016\,\text{ohm-m})(0.005\,\text{m})} = 1.25 \cdot 10^4\ \text{A/m}^2. \tag{486}$$

The drift velocity is then

$$v = \frac{J}{ne} = \frac{1.25 \cdot 10^4\ \text{C/s}\,\text{m}^2}{(2 \cdot 10^{21}\ \text{m}^{-3})(1.6 \cdot 10^{-19}\ \text{C})} = 39\,\text{m/s}. \tag{487}$$

The induced electric field is $E_\text{t} = vB = (39\,\text{m/s})(0.1\,\text{T}) = 3.9\,\text{V/m}$. The Hall voltage across the ribbon of width 0.002 m is therefore $(3.9\,\text{V/m})(0.002\,\text{m}) = 7.8 \cdot 10^{-3}\,\text{V}$, or 7.8 millivolts. Symbolically, the Hall voltage equals $VBw/\rho Lne$, where w is the width.