Phys 4C Fall 2019

Chapter 2&5 Solutions to Exercises

2.76. **Zero curl**

We are given $E_x = 6xy$, $E_y = 3x^2 - 3y^2$, $E_z = 0$. So we have

$$\begin{aligned}(\nabla \times \mathbf{E})_x &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0, \\ (\nabla \times \mathbf{E})_y &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0, \\ (\nabla \times \mathbf{E})_z &= \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 6x - 6x = 0. \qquad (222)\end{aligned}$$

The divergence is

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 6y - 6y = 0. \qquad (223)$$

The zero here implies that the associated charge density is zero.

2.77. **Zero dipole curl**

The dipole **E** field in Eq. (2.36) has no angular ϕ dependence, and also no $\hat{\boldsymbol{\phi}}$ component. So we quickly see that only the $\hat{\boldsymbol{\phi}}$ component of the spherical-coordinate expression for $\nabla \times \mathbf{A}$ in Eq. (F.3) in Appendix F survives. Using the values of E_r and E_θ from Eq. (2.36) we have

$$\begin{aligned}\nabla \times \mathbf{E} &= \frac{1}{r}\left(\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial \theta}\right)\hat{\boldsymbol{\phi}} = \frac{q\ell}{4\pi\epsilon_0}\frac{1}{r}\left(\frac{\partial(\sin\theta/r^2)}{\partial r} - \frac{\partial(2\cos\theta/r^3)}{\partial \theta}\right)\hat{\boldsymbol{\phi}} \\ &= \frac{q\ell}{4\pi\epsilon_0}\frac{1}{r}\left(\frac{-2\sin\theta}{r^3} - \frac{-2\sin\theta}{r^3}\right)\hat{\boldsymbol{\phi}} = 0. \qquad (224)\end{aligned}$$

REMARK: Let's look at what's going on physically in the special case of $\theta = \pi/2$. Consider the circulation of the field around the loop shown in Fig. 63, which consists of radial and tangential segments. The tangential piece on the right is longer than the piece on the left, being proportional to r. If the field fell off like $1/r$, these effects would cancel in the line integral, and there would be no net circulation from the tangential parts. But for our dipole, the field falls off like $1/r^3$, so the contribution from the left piece dominates, yielding a net counterclockwise circulation from the tangential pieces. This has the correct sign to cancel with the clockwise circulation from the radial parts (which simply add; from Eq. (2.36) there is a very small positive E_r just above the $\theta = \pi/2$ line, and a very small negative E_r just below). So it's believable that things work out, although the above calculation is needed to show quantitatively that the curl is exactly zero.

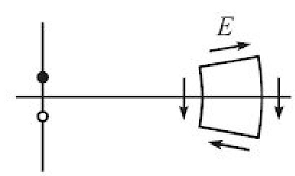

Figure 63

2.78. **Divergence of the curl**

(a) In Cartesian coordinates the divergence of the curl is

$$
\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{A}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \left(\frac{\partial^2 A_z}{\partial x\, \partial y} - \frac{\partial^2 A_y}{\partial x\, \partial z}\right) + \left(\frac{\partial^2 A_x}{\partial y\, \partial z} - \frac{\partial^2 A_z}{\partial y\, \partial x}\right) + \left(\frac{\partial^2 A_y}{\partial z\, \partial x} - \frac{\partial^2 A_x}{\partial z\, \partial y}\right) \\
&= 0. \qquad (225)
\end{aligned}
$$

We have used the fact that partial differentiation commutes, for any function with continuous derivatives.

(b) The derivation can be summed up by the relations,

$$\int_C \mathbf{A} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_V \nabla \cdot (\nabla \times \mathbf{A})\, dv. \qquad (226)$$

The first equality is the statement of Stokes' theorem, and the second is the statement of Gauss's theorem (the divergence theorem) applied to the vector "$\nabla \times \mathbf{A}$." The logic of the derivation is as follows. The line integral of $\mathbf{A}$ around the curve C in Fig. 2.52 is zero because the curve backtracks along itself. (We can make the two "circles" of C be arbitrarily close to each other, and they run in opposite directions.) Stokes' theorem then tells us that the surface integral of $\nabla \times \mathbf{A}$ over S is also zero. The surface S is essentially the same as the closed surface S' consisting of S plus the tiny area enclosed by C. So the surface integral of $\nabla \times \mathbf{A}$ over S' is zero. But S' encloses the volume V, so Gauss's theorem tells us that the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ over V is also zero. Since this result holds for any arbitrary volume V, the integrand $\nabla \cdot (\nabla \times \mathbf{A})$ must be identically zero, as we wanted to show.[2]

This logic here basically boils down to the mathematical fact that the boundary of a boundary is zero. More precisely, the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ equals (by Gauss) the surface integral of $\nabla \times \mathbf{A}$ over the boundary S' of the volume V, which in turn equals (by Stokes) the line integral of $\mathbf{A}$ over the boundary C of the boundary S' of the volume V. But S' has no boundary, so C doesn't exist. That is, C has zero length. The line integral over C is therefore zero, which means that the original volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ is also zero.

In view of this, there actually wasn't any need to pick the curve C to be of the specific stated form. We could have just picked a very tiny circle. The first step in the above derivation, namely that the line integral of $\mathbf{A}$ around the curve C is zero, still holds (but now simply because C has essentially no length), so the derivation proceeds in exactly the same way.

[2] If $\nabla \cdot (\nabla \times \mathbf{A})$ were different from zero at some point, then the integral over a small volume containing this point would be nonzero. This is true because we can pick the volume to be small enough so that $\nabla \cdot (\nabla \times \mathbf{A})$ is essentially constant, so there is no possibility of cancelation.

6.29. **Motion in a B field**

FIRST SOLUTION: The magnitude of the magnetic force is $F = qvB$, so the magnitude of the change in p during a short time dt is $dp = F\,dt = qvB\,dt$. The momentum itself is $p = \gamma m v$. Fig. 106(a) shows the $\mathbf{r}$ and $\mathbf{p}$ vectors at two nearby times. In Fig. 106(b) the angle θ is the same in the two triangles, because each $\mathbf{p}$ is perpendicular to the corresponding $\mathbf{r}$. So from similar triangles we have

$$\frac{|d\mathbf{r}|}{|\mathbf{r}|} = \frac{|d\mathbf{p}|}{|\mathbf{p}|} \implies \frac{v\,dt}{R} = \frac{qvB\,dt}{p} \implies R = \frac{p}{qB} = \frac{\gamma m v}{qB}. \qquad (417)$$

The time to complete one revolution is

$$t = \frac{2\pi R}{v} = \frac{2\pi}{v}\frac{\gamma m v}{qB} = \frac{2\pi \gamma m}{qB}. \qquad (418)$$

If $\mathbf{B}$ is uniform, then Eq. (417) actually *proves* that the particle travels in a circle, because it gives the radius of curvature at any point as $R = \gamma m v/qB$. Since v is constant (because the magnetic force is always perpendicular to the velocity), we see that R is constant, which means that the path is a circle.

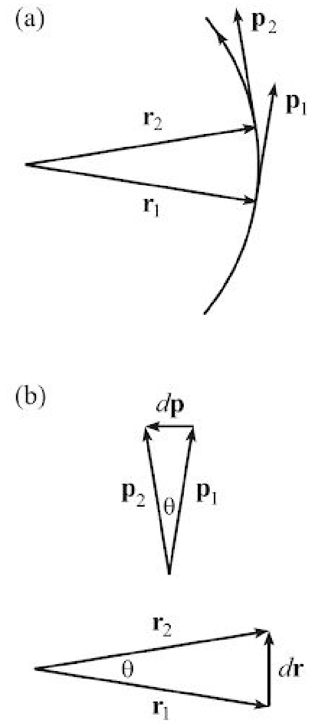

Figure 106

SECOND SOLUTION: We can also calculate R in a more mathematical way. The Lorentz-force law $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ combined with Newton's third law $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\frac{d(\gamma m \mathbf{v})}{dt} = q\mathbf{v} \times \mathbf{B} \implies \frac{d\mathbf{v}}{dt} = \frac{q}{\gamma m}\mathbf{v} \times \mathbf{B}. \qquad (419)$$

Note that we are in fact allowed to take the γ outside the derivative because we know that the speed v is constant.

Assume that $\mathbf{B}$ is uniform. Let the motion be in the x-y plane, with the magnetic field pointing in the z direction. Then $\mathbf{v} = (v_x, v_y, 0)$ and $\mathbf{B} = (0, 0, B)$. So $\mathbf{v} \times \mathbf{B} = B(v_y, -v_x, 0)$. The x and y components of Eq. (419) can then be written as

$$\frac{dv_x}{dt} = \frac{qB}{\gamma m}\,v_y \qquad \text{and} \qquad \frac{dv_y}{dt} = -\frac{qB}{\gamma m}\,v_x. \qquad (420)$$

Taking the derivative of the first of these equations, and then substituting in the value of dv_y/dt from the second, gives

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB}{\gamma m}\right)^2 v_x. \tag{421}$$

This is a simple-harmonic-oscillator type equation, for which the general solution takes the form,

$$v_x(t) = A\cos(\omega t + \phi), \qquad \text{where } \omega = \frac{qB}{\gamma m}. \tag{422}$$

The first of the equations in Eq. (420) then quickly gives $v_y(t) = -A\sin(\omega t + \phi)$. A and ϕ are arbitrary constants, determined by the initial conditions. However, if the momentum $p = \gamma m v$ is given, then v_x and v_y must each have an amplitude of $p/\gamma m$. Hence $A = p/\gamma m$.

The period is $2\pi/\omega = 2\pi\gamma m/qB$, in agreement with the result in part (a). Integrating $v_x(t)$ and $v_y(t)$ to find x and y gives (up to arbitrary additive constants, which only affect the position of the center of the circle)

$$\big(x(t), y(t)\big) = \frac{A}{\omega}\Big(\sin(\omega t + \phi), \cos(\omega t + \phi)\Big). \tag{423}$$

This describes a circle with radius $R = A/\omega = (p/\gamma m)/(qB/\gamma m) = p/qB$, in agreement with the result in part (a).

6.31. Field from three wires

At point P_1 at the center of the square, the magnetic fields due to wires A and C in Fig. 107 cancel. The field due to B is $\mu_0(2I)/2\pi(d/\sqrt{2}) = \sqrt{2}\,\mu_0 I/\pi d$, directed diagonally down to the left, as shown.

At point P_2 at the lower right-hand corner, the field due to B is half of the field at P_1, so it is $\mu_0 I/\sqrt{2}\,\pi d$, directed diagonally down to the left, as shown. The field due to A is $\mu_0 I/2\pi d$, directed upward. The field due to C is $\mu_0 I/2\pi d$, directed rightward. The sum of these two fields is $\mu_0 I/\sqrt{2}\,\pi d$, directed diagonally up to the right. The vector sum of all three fields is therefore zero at P_2.

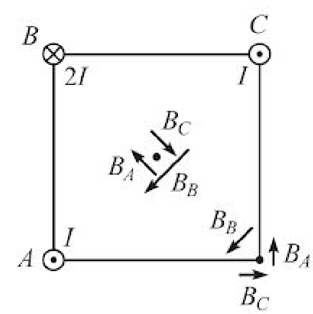

Figure 107

6.32. Oersted's experiment

The compass needle initially points in the direction of the earth's magnetic field, which has strength 0.2 gauss (in the horizontal direction). In Oersted's experiment, the wire runs parallel to the initial orientation of the needle. If the needle ends up rotated by 45°, the magnetic field from the wire must be 0.2 gauss in the perpendicular direction.

In other words, we have a current-carrying wire that produces a magnetic field of $2 \cdot 10^{-5}$ T at a distance of about 2 cm. Therefore,

$$B = \frac{\mu_0 I}{2\pi r} \implies I = \frac{2\pi r B}{\mu_0} = \frac{2\pi(0.02\,\text{m})(2\cdot 10^{-5}\,\text{T})}{4\pi\cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}} = 2\ \text{A}. \tag{426}$$

6.33. **Force between wires**

The magnetic field due to one of the wires in Fig. 5.1(b), at the location of the other, is

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(20\,\text{A})}{2\pi(0.05\,\text{m})} = 8 \cdot 10^{-5}\,\text{T}. \tag{427}$$

The force per unit length on each wire is then $IB = (20\,\text{A})(8\cdot 10^{-5}\,\text{T}) = 1.6\cdot 10^{-3}\,\text{N/m}$, and it is repulsive.

6.36. **Field at different radii**

The radius is 2 cm, so 1/4 of the cross-sectional area, and hence current (so 2000 A), is enclosed within $r = 1$ cm. The current enclosed in both the $r = 2$ cm and $r = 3$ cm cases is 8000 A. So we have

$$\begin{aligned} B_1 &= \frac{\mu_0 I_1}{2\pi r_1} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(2000\,\text{A})}{2\pi(0.01\,\text{m})} = 0.04\,\text{T}, \\ B_2 &= \frac{\mu_0 I_2}{2\pi r_2} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(8000\,\text{A})}{2\pi(0.02\,\text{m})} = 0.08\,\text{T}, \\ B_3 &= \frac{\mu_0 I_3}{2\pi r_3} = \frac{\left(4\pi \cdot 10^{-7}\,\frac{\text{kg m}}{\text{C}^2}\right)(8000\,\text{A})}{2\pi(0.03\,\text{m})} = 0.0533\,\text{T}. \end{aligned} \tag{435}$$

These fields are 400, 800, and 533 gauss, respectively.

6.39. **Constant magnitude of B**

If I_r is the current inside radius r, then Ampere's law gives

$$B \cdot 2\pi r = \mu_0 I_r \implies B = \frac{\mu_0 I_r}{2\pi r}. \tag{438}$$

If we want B to be independent of r, then we need I_r to be proportional to r. I_r is found by integrating the current density $J(r)$:

$$I_r = \int J\, da = \int_0^r J(r') \cdot (2\pi r'\, dr'). \tag{439}$$

It is easiest to guess and check the form of $J(r')$. If $J(r')$ is proportional to $1/r'$, then it takes the form of $J(r') = \alpha/r'$, so

$$I_r = \int_0^r (\alpha/r')(2\pi r'\, dr') = 2\pi\alpha r, \tag{440}$$

as desired. The field is then

$$B = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0(2\pi\alpha r)}{2\pi r} = \mu_0\alpha. \tag{441}$$

The above "$1/r$" result for the current density is the same result that holds for the charge density in the case of the electric field due to a charged cylinder or sphere. In both of these cases the electric field is independent of r if the density ρ is proportional to $1/r$.

Note that even though the current density diverges at $r = 0$, the actual current does not. There is a finite amount of current in any cross section with radius r, and it is given (by construction) by $I_r = 2\pi\alpha r$. Any ring (at any radius) with thickness dr contains the same amount of current, $dI = 2\pi\alpha\, dr$.

We can also solve this exercise by using the differential form of Ampere's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Since $\mathbf{B}$ points tangentially and has a uniform value, it can be written as $\mathbf{B} = B_0\hat{\boldsymbol{\theta}}$. Equation F.2 in Appendix F then gives

$$\nabla \times \mathbf{B} = \frac{1}{r}\frac{\partial(rB_0)}{\partial r}\hat{\mathbf{z}} = \frac{B_0}{r}\hat{\mathbf{z}}. \tag{442}$$

Setting this equal to $\mu_0 J\hat{\mathbf{z}}$ gives $J = B_0/(\mu_0 r)$, consistent with the $1/r$ dependence we found above. The factor of B_0/μ_0 here equals the α from above.

6.41. **Integral of A, flux of B**

Using Stokes' theorem, along with $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\int_C \mathbf{A}\cdot d\mathbf{s} = \int_S \nabla \times \mathbf{A}\cdot d\mathbf{a} = \int_S \mathbf{B}\cdot d\mathbf{a} = \Phi, \tag{444}$$

as desired. This relation is similar to Ampere's law because the differential form of that law, $\mu_0\mathbf{J} = \nabla \times \mathbf{B}$, takes the same form as the above $\mathbf{B} = \nabla \times \mathbf{A}$ relation.

6.42. **Finding the vector potential**

Since $\mathbf{B} = \nabla \times \mathbf{A}$, we want

$$\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \qquad \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, \qquad \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0. \tag{445}$$

From inspection, a few choices for $\mathbf{A}$ that satisfy these equations are $\mathbf{A} = (0, B_0 x, 0)$, or $(-B_0 y, 0, 0)$, or $(-B_0 y/2, B_0 x/2, 0)$. In general, any vector of the form $(-ay, bx, 0)$ works if $a + b = B_0$. And even more generally, adding on any vector with zero curl also works.

6.43. **Vector potential inside a wire**

Since area is proportional to r^2, the current contained within a radius r is $I_r = Ir^2/r_0^2$. The magnitude of the magnetic field at radius r is then

$$B(r) = \frac{\mu_0 I_r}{2\pi r} = \frac{\mu_0 I r}{2\pi r_0^2}, \tag{446}$$

and it points in the positive $\hat{\boldsymbol{\theta}}$ direction. The $\hat{\boldsymbol{\theta}}$ vector equals $(-y/r, x/r, 0)$ because this vector has length 1 and has zero dot product with the radial vector $(x, y, 0)$. So the Cartesian components of $\mathbf{B}$ are

$$B_x = -\frac{y}{r}B = -\frac{\mu_0 I y}{2\pi r_0^2}, \qquad \text{and} \qquad B_y = \frac{x}{r}B = \frac{\mu_0 I x}{2\pi r_0^2}. \tag{447}$$

The magnetic field associated with the potential $\mathbf{A} = A_0\hat{\mathbf{z}}(x^2 + y^2)$ is

$$\mathbf{B} = \nabla \times \mathbf{A} = \hat{\mathbf{x}}\frac{\partial A_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial A_z}{\partial x} = 2A_0 y\hat{\mathbf{x}} - 2A_0 x\hat{\mathbf{y}}. \tag{448}$$

This agrees with the $\mathbf{B}$ in Eq. (447) if $A_0 = -\mu_0 I/4\pi r_0^2$.

Alternatively, in cylindrical coordinates we have $\mathbf{A} = A_0\hat{\mathbf{z}}r^2$. From Eq. (F.2) in Appendix F the associated magnetic field is $\mathbf{B} = \nabla \times \mathbf{A} = -(\partial A_z/\partial r)\hat{\boldsymbol{\theta}} = -2A_0 r\hat{\boldsymbol{\theta}}$. Comparing this with the B in Eq. (446), which points in the positive $\hat{\boldsymbol{\theta}}$ direction, we find $A_0 = -\mu_0 I/4\pi r_0^2$, as above.

Since A_0 is negative, $\mathbf{A}$ points in the direction opposite to the current (which points in the positive $\hat{\mathbf{z}}$ direction). You might be wondering how this can be, in view of the fact that Eq. (6.44) seems to say that $\mathbf{A}$ points in the same direction as $\mathbf{J}$. The answer is that we can add an arbitrary constant to the $\mathbf{A}$ in Eq. (6.44), and it will still yield the same value of $\mathbf{B} = \nabla \times \mathbf{A}$. Adding on a sufficiently large vector pointing in the negative $\hat{\mathbf{z}}$ direction will make $\mathbf{A}$ point opposite to $\mathbf{J}$.