Phys 4C Fall 2019

Chapter 4 Solutions to Exercises

4.28. **Parallel resistors**

The two loop equations are

$$\begin{aligned} \mathcal{E} - (I_1 - I_2)R_1 &= 0, \\ -(I_2 - I_1)R_1 - I_2 R_2 &= 0. \end{aligned} \tag{352}$$

Adding these two equations quickly gives $I_2 = \mathcal{E}/R_2$ (which corresponds to the loop around the whole circuit). Either equation then gives

$$I_1 = \frac{\mathcal{E}(R_1 + R_2)}{R_1 R_2} \equiv \frac{\mathcal{E}}{R_{\text{eff}}}, \qquad \text{where} \qquad R_{\text{eff}} \equiv \frac{R_1 R_2}{R_1 + R_2}. \tag{353}$$

The current through the battery is I_1, so $\mathcal{E} = I_1 R_{\text{eff}}$ tells us that the effective resistance is R_{eff}.

4.29. **Keeping the same resistance**

We have an R_1 resistor in series with the parallel combination of R_1 and $(R_1 + R_0)$. So we want

$$\begin{aligned} R_1 + \frac{R_1(R_1 + R_0)}{R_1 + (R_1 + R_0)} = R_0 \quad &\Longrightarrow \quad (2R_1^2 + R_1 R_0) + (R_1^2 + R_1 R_0) = 2R_1 R_0 + R_0^2 \\ &\Longrightarrow \quad 3R_1^2 = R_0^2 \quad \Longrightarrow \quad R_1 = \frac{R_0}{\sqrt{3}}. \end{aligned} \tag{354}$$

4.30. **Automobile battery**

If the voltage drop across the $0.5\,\Omega$ resistor is $9.8\,\text{V}$, then the current in the circuit is $I = V/R = (9.8\,\text{V})/(0.5\,\Omega) = 19.6\,\text{A}$. The voltage drop across the internal resistor is then $R_\text{i}(19.6\,\text{A})$. But we know that this voltage drop is $12.3\,\text{V} - 9.8\,\text{V} = 2.5\,\text{V}$. Therefore, $2.5\,\text{V} = R_\text{i}(19.6\,\text{A}) \Longrightarrow R_\text{i} = 0.128\,\Omega$.

4.35. **Resistances in a cube**

(a) In Fig. 91 the three vertices adjacent to A (which are labeled as "a") are all at the same potential (by symmetry under rotations around the AB diagonal), so we can collapse them to one point. (Equivalently, if we connect them with resistance-less wires, no current will flow in these wires.) Likewise for the three vertices adjacent to B (which are labeled as "b"). So the circuit is equivalent to the second setup shown in Fig. 91 (the number of lines is still 12), which can be simplified as indicated. The equivalent resistance is therefore $5R/6$.

Alternatively, we can work in terms of currents. The input current I_0 gets divided evenly, by symmetry, into three $I_0/3$ currents. It then divides into six $I_0/6$ currents, and then converges to three $I_0/3$ currents. The total potential drop across any of the possible paths from A to B is given by $V = (I_0/3)R + (I_0/6)R + (I_0/3)R = (5/6)I_0R$. The effective resistance is then $V/I_0 = 5R/6$.

(b) In Fig. 92 there are four vertices (labeled as "c") that lie in the plane that is equidistant from A and B. These vertices are all at the same potential (halfway between V_A and V_B), so we can collapse them to a point. (In the second setup shown, there are only 10 lines because 2 of the original 12 lines were collapsed). The circuit can then be simplified as shown, and the equivalent resistance is $3R/4$.

(c) From symmetry, the two points marked as a in Fig. 93 are at the same potential, so we can collapse them to a point. Likewise for the two b's. The circuit can then be simplified as shown, and the equivalent resistance is $7R/12$. As expected, this is smaller than the answer to part (b), which in turn is smaller than the answer to part (a).

Note that the sum of the effective resistances across all 12 resistors is $12(7R/12) = 7R = (8-1)R$, where the 8 here is the number of corners in the cube. This is a special case of the general result in Problem 4.9.

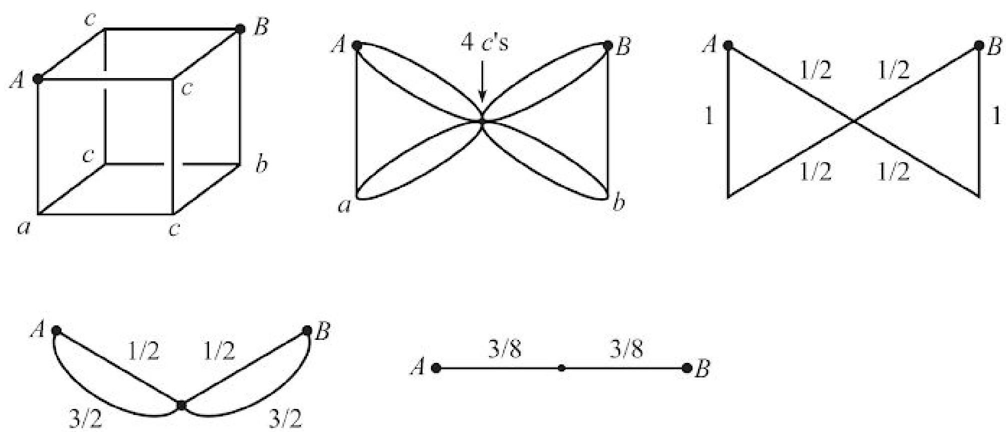

Figure 92

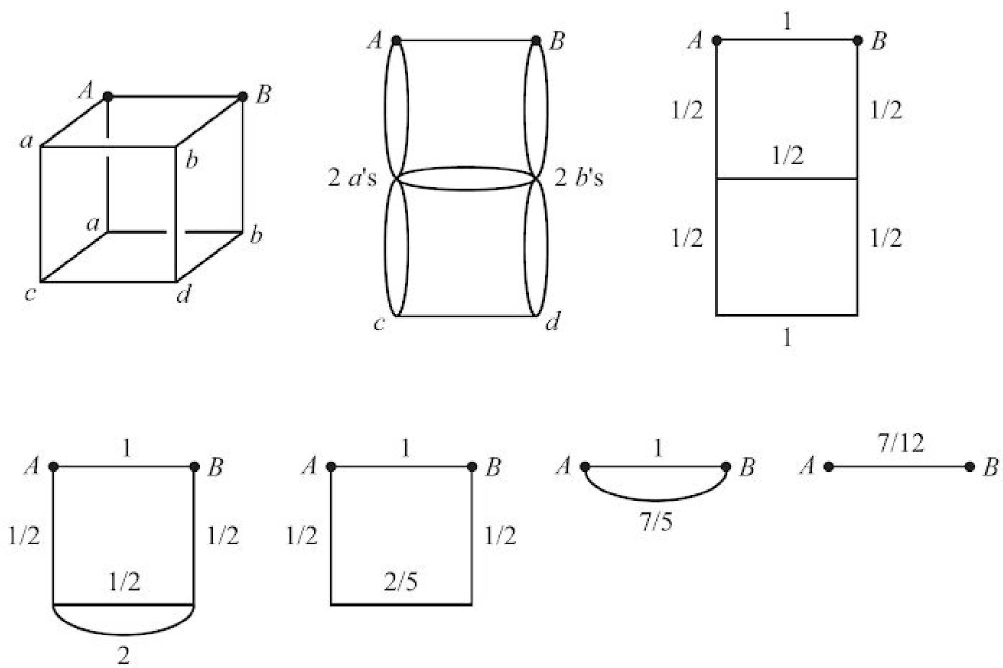

Figure 93

4.38. **Two light bulbs**

(a) The power dissipated takes the form of V^2/R. Both bulbs have the same voltage drop V, so if Bulb 1 is twice as bright as Bulb 2, it must have half the R. Bulb 2's resistance is therefore larger by a factor of 2. (The larger resistor is dimmer.)

(b) The power dissipated also takes the form of I^2R. Both bulbs now have the same current I, so if Bulb 2 has twice the resistance, as we found in part (a), then it is twice as bright – the opposite of the case in part (a). (The larger resistor is brighter.) Note that in part (a) we used the expression $P = V^2/R$ because both bulbs (in parallel) had the same V, whereas now we are using the expression $P = I^2R$ because both bulbs (in series) have the same I.

We can also compare the total power dissipated in each case. If the resistances are R and $2R$, then in part (a) the total power dissipated is $V^2/R+V^2/2R = 3V^2/2R$. In part (b) the total power is $I^2R + I^2(2R) = 3I^2R$, where $I = V/3R$. So the power is $V^2/3R$. This is 2/9 of the power in part (a). In units of V^2/R, the powers in part (a) are 1 and 1/2, while in part (b) they are 1/9 and 2/9.

4.39. **Maximum power**

The R_i and R resistors are in series, so the current in the circuit is $I = \mathcal{E}/(R + R_\text{i})$. The power dissipated in the R resistor is therefore $P = I^2R = \mathcal{E}^2R/(R+R_\text{i})^2$. Taking the derivative with respect to R and setting the result equal to zero gives

$$0 = \frac{(R + R_\text{i})^2 \cdot 1 - R \cdot 2(R + R_\text{i})}{(R + R_\text{i})^4} = \frac{R_\text{i} - R}{(R + R_\text{i})^3} \implies R = R_\text{i}. \tag{365}$$

This is indeed a maximum, because $dP/dR > 0$ for $R < R_\text{i}$, and $dP/dR < 0$ for $R > R_\text{i}$. Equivalently, the second derivative is negative at $R = R_\text{i}$, as you can check.

It makes sense that a maximum exists for some finite value of R, because $P = 0$ both at $R = 0$ (because $P = I^2R$, with I finite and R zero) and at $R = \infty$ (because $P = V^2/R$, with V finite and R infinite).

Consider a different question, "Given a fixed external resistance R, what value of the internal resistance R_i yields the maximum power delivered to the external resistor R?" In view of the above expression for the power, the answer is simply $R_\text{i} = 0$. This makes sense; we want the largest possible current passing through the given external resistor.

4.47. **Discharging a capacitor**

From Eq. (4.44) the current is $I(t) = (V_0/R)e^{-t/RC}$. The power dissipated in the resistor is $P = I^2R = (V_0^2/R)e^{-2t/RC}$, so the total energy dissipated is

$$E = \int_0^\infty P\,dt = \int_0^\infty \frac{V_0^2}{R}e^{-2t/RC}\,dt = -\frac{V_0^2}{R}\frac{RC}{2}e^{-2t/RC}\bigg|_0^\infty = \frac{1}{2}CV_0^2, \tag{374}$$

which is the initial energy in the capacitor, as desired.

Suppose we have a 1 microfarad capacitor charged to 100 volts. The initial charge is $Q = CV_0 = (10^{-6}\,\text{F})(100\,\text{V}) = 10^{-4}\,\text{C}$. From Eq. (4.43) the charge decreases according to $Q(t) = Q_0 e^{-t/t_0}$, where $t_0 = RC$ is the time constant. Since the charge of an electron is $1.6 \cdot 10^{-19}$ C, we will have roughly one electron left when $1.6 \cdot 10^{-19}\,\text{C} = (10^{-4}\,\text{C})e^{-t/t_0} \Longrightarrow t = -t_0 \ln(1.6 \cdot 10^{-15}) = 34t_0$. So if the time constant were, say, 1 second (which would mean $R = 10^6\,\Omega$ here), we would be down to roughly one electron in a little over half a minute. For a $1\,\text{k}\Omega$ resistor, the time would be 0.034 s.

4.48. **Charging a capacitor**

The total work done by the battery is $Q_\text{f}\mathcal{E}$, where Q_f is the final charge on the capacitor. This is true because the battery transfers a charge Q_f through a constant potential difference of $\mathcal{E}$.

The final energy of the capacitor is $Q_\text{f}\phi/2 = Q_\text{f}\mathcal{E}/2$, because the final potential ϕ across the capacitor equals the voltage $\mathcal{E}$ across the battery. (There is no current flowing after a long time, so there is no voltage drop across the resistor.)

The energy dissipated in the resistor is the integral of the power, that is, $\int RI^2\,dt$. From the solution to Problem 4.17, we have $I(t) = (\mathcal{E}/R)e^{-t/RC}$. Therefore,

$$\int_0^\infty RI^2\,dt = R\frac{\mathcal{E}^2}{R^2}\int_0^\infty e^{-2t/RC}\,dt = -\frac{\mathcal{E}^2}{R}\frac{RC}{2}e^{-2t/RC}\bigg|_0^\infty = \frac{\mathcal{E}^2}{R}\frac{RC}{2} = \frac{C\mathcal{E}^2}{2} = \frac{Q_\text{f}\mathcal{E}}{2}, \tag{375}$$

where we have used $Q_\text{f} = C\mathcal{E}$. The conservation-of-energy statement is then

$$W_\text{battery} = U_\text{capacitor} + E_\text{resistor} \quad \Longrightarrow \quad Q_\text{f}\mathcal{E} = \frac{Q_\text{f}\mathcal{E}}{2} + \frac{Q_\text{f}\mathcal{E}}{2}, \tag{376}$$

which is indeed true.

Remark: It is also possible to use the general formulas for $Q(t)$ and $I(t)$ from Problem 4.17 to show that energy is conserved at all times (not just $t \to \infty$), as we know it must be. But we can show this in a quicker manner by demonstrating that the conservation-of-energy statement is equivalent to the Kirchhoff loop equation, $\mathcal{E} - Q/C - RI = 0$. We can do this either by differentiating the former to obtain the latter, or by integrating the latter to obtain the former. Let's take the first of these routes. The conservation-of-energy statement at any intermediate time is

$$\mathcal{E}Q(t) = \frac{Q(t)^2}{2C} + \int_0^t RI^2\,dt. \tag{377}$$

Differentiating with respect to t gives (using $dQ/dt = I$ and canceling a factor of I)

$$\mathcal{E}\frac{dQ}{dt} = \frac{Q}{C}\frac{dQ}{dt} + RI^2 \quad \Longrightarrow \quad \mathcal{E} = \frac{Q}{C} + RI, \tag{378}$$

which is the Kirchhoff loop equation, as desired. If you want to go in the reverse direction, just multiply by I and then integrate with respect to t (using the fact that $Q = 0$ at $t = 0$).