Phys 4C Fall 2019

Chapter 2 Solutions to Exercises

2.31. **Finding the potential**

The line integral along the first path is (we'll suppress the z component of the argument)

$$\begin{aligned}\int_{(0,0)}^{(x_1,y_1)} \mathbf{E}\cdot d\mathbf{s} &= \int_0^{x_1} E_x(x,0)\,dx + \int_0^{y_1} E_y(x_1,y)\,dy \\ &= 0 + \int_0^{y_1} (3x_1^2 - 3y^2)\,dy = 3x_1^2 y_1 - y_1^3. \end{aligned} \tag{99}$$

The line integral along the second path is

$$\begin{aligned}\int_{(0,0)}^{(x_1,y_1)} \mathbf{E}\cdot d\mathbf{s} &= \int_0^{y_1} E_y(0,y)\,dy + \int_0^{x_1} E_x(x,y_1)\,dx \\ &= \int_0^{y_1} (0-3y^2)\,dy + \int_0^{x_1} 6xy_1\,dx = -y_1^3 + 3x_1^2 y_1. \end{aligned} \tag{100}$$

These two results are equal, as desired. The electric potential ϕ, if taken to be zero at $(0,0)$, is just the negative of our result, because we define ϕ by $\phi = -\int \mathbf{E}\cdot d\mathbf{s}$, or equivalently $\mathbf{E} = -\nabla\phi$. Hence $\phi(x,y) = y^3 - 3x^2 y$. The negative gradient of this is

$$-\nabla\phi = -\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (6xy,\, 3x^2 - 3y^2,\, 0), \tag{101}$$

which does indeed equal the given $\mathbf{E}$.

An alternative method of finding ϕ is to integrate the components of $\mathbf{E}$ to find the general form that ϕ must take. Since $-\partial\phi/\partial x$ equals $E_x = 6xy$, we see that $-\phi$ must take the form of $3x^2y + f(y,z)$, where $f(y,z)$ is an arbitrary function of y and z. Likewise, since $-\partial\phi/\partial y$ equals $E_y = 3x^2 - 3y^2$, we see that $-\phi$ must take the form of $3x^2y - y^3 + g(x,z)$. Finally, since $-\partial\phi/\partial z$ equals $E_z = 0$, we see that $-\phi$ must take the form of $0 + h(x,y)$, that is, ϕ is a function of only x and y. The only function consistent with all three of these forms is $-\phi = 3x^2y - y^3$ (plus a constant), in agreement with the above result.

2.42. **E and ϕ for a cylinder**

(a) Consider a coaxial cylinder with length ℓ and radius $r < a$. The charge contained inside is $\pi r^2 \ell \rho$. The area of the cylindrical part of the surface is $2\pi r\ell$, and since $\mathbf{E}$ is perpendicular to the surface by symmetry, the flux is $2\pi r\ell E$. So Gauss's law gives the internal electric field as

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \quad\Longrightarrow\quad 2\pi r\ell E = \frac{\pi r^2 \ell\rho}{\epsilon_0} \quad\Longrightarrow\quad E = \frac{\rho r}{2\epsilon_0} \qquad (\text{for } \; r < a). \quad (121)$$

We'll also need the external field for part (b). For this field, consider a cylinder of radius $r > a$. This contains a fixed amount of charge $\pi a^2 \ell\rho$, so Gauss's law gives

$$\int \mathbf{E}\cdot d\mathbf{a} = \frac{q}{\epsilon_0} \quad\Longrightarrow\quad 2\pi r\ell E = \frac{\pi a^2 \ell\rho}{\epsilon_0} \quad\Longrightarrow\quad E = \frac{\rho a^2}{2\epsilon_0 r} \qquad (\text{for } \; r > a). \quad (122)$$

This is the same as the field from a line of charge (namely $\lambda/2\pi\epsilon_0 r$) with linear density $\lambda = \pi a^2 \rho$. Note that the internal and external fields agree at $r = a$.

(b) If $\phi = 0$ at $r = 0$, then we have

$$\begin{aligned}
\text{For } r < a: \quad \phi(r) &= -\int_0^r E\,dr = -\int_0^r \frac{\rho r\,dr}{2\epsilon_0} = -\frac{\rho r^2}{4\epsilon_0}, \\
\text{For } r > a: \quad \phi(r) &= -\int_0^a E\,dr - \int_a^r E\,dr \\
&= -\frac{\rho a^2}{4\epsilon_0} - \int_a^r \frac{\rho a^2\,dr}{2\epsilon_0 r} = -\frac{\rho a^2}{4\epsilon_0} - \frac{\rho a^2}{2\epsilon_0}\ln(r/a). \quad (123)
\end{aligned}$$

This goes to $-\infty$ as $r \to \infty$. It also goes to $-\infty$ for any given value of r if $a \to 0$ while the charge per unit length ($\pi a^2 \rho$) is held constant.

2.55. **Hole in a disk**

(a) Slicing the disk into concentric rings, we find the potential at the center to be (with $\ell = 1\,\text{cm}$)

$$\phi = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int_{\ell}^{3\ell} \frac{2\pi r\sigma\, dr}{r} = \frac{\sigma\ell}{\epsilon_0}. \tag{155}$$

Plugging in the various quantities gives

$$\phi = \frac{\left(-\,10^{-5}\,\frac{\text{C}}{\text{m}^2}\right)(0.01\,\text{m})}{8.85\cdot 10^{-12}\,\frac{\text{s}^2\,\text{C}^2}{\text{kg}\,\text{m}^3}} = -11,300\ \text{V}. \tag{156}$$

(b) The electron's final kinetic energy at infinity equals the loss in potential energy. This loss has magnitude

$$(-e)\phi = (-1.6\cdot 10^{-19}\,\text{C})(-11,300\ \text{V}) = 1.81\cdot 10^{-15}\ \text{J}. \tag{157}$$

Since this is only about 2% of the electron's rest energy, namely $mc^2 = 8.2\cdot 10^{-14}$ J, a nonrelativistic calculation will suffice:

$$\frac{1}{2}mv^2 = 1.81\cdot 10^{-15}\,\text{J} \implies v = \left(\frac{2(1.81\cdot 10^{-15}\,\text{J})}{9.1\cdot 10^{-31}\,\text{kg}}\right)^{1/2} = 6.3\cdot 10^7\ \text{m/s}, \tag{158}$$

which is about 20% of the speed of light. This answer is very close to the answer obtained via the correct relativistic calculation: Conservation of energy gives

$$\gamma mc^2 = mc^2 + |\Delta U| \implies \gamma = 1 + \frac{1.81\cdot 10^{-15}\,\text{J}}{8.2\cdot 10^{-14}\,\text{J}} = 1.022. \tag{159}$$

Hence (with $\beta \equiv v/c$),

$$\beta = \sqrt{1 - 1/\gamma^2} = 0.206 \implies v = \beta c = 6.2\cdot 10^7\ \text{m/s}. \tag{160}$$

2.61. **Dipole field on the axes**

With the charges q and $-q$ located at $z = \ell/2$ and $-\ell/2$, consider a distant point on the positive z axis with $z = r$. The charge q is slightly closer than the charge $-q$ is to this point, so the upward field due to the charge q is slightly stronger than the downward field due to the charge $-q$. The net field will therefore point upward, and it has magnitude (with $k \equiv 1/4\pi\epsilon_0$)

$$\begin{aligned} E = \frac{kq}{(r-\ell/2)^2} - \frac{kq}{(r+\ell/2)^2} &= \frac{kq}{r^2}\left(\frac{1}{(1-\ell/2r)^2} - \frac{1}{(1+\ell/2r)^2}\right) \\ &\approx \frac{kq}{r^2}\left(\frac{1}{1-\ell/r} - \frac{1}{1+\ell/r}\right), \end{aligned} \tag{175}$$

where we have dropped terms of order ℓ^2/r^2. Using $1/(1 \pm \epsilon) \approx 1 \mp \epsilon$, we obtain

$$E \approx \frac{kq}{r^2}\left(\left(1+\frac{\ell}{r}\right) - \left(1-\frac{\ell}{r}\right)\right) = \frac{2kq\ell}{r^3}. \tag{176}$$

This field points in the positive $\hat{\mathbf{r}}$ direction, so it agrees with the result in Eq. (2.36),

$$\frac{kq\ell}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right), \tag{177}$$

when $\theta = 0$.

In the transverse direction, we have the situation shown in Fig. 47. The magnitudes of the two fields are equal. The horizontal components cancel, but the downward components add. The distances from the given point to the two charges are essentially equal to r, so the magnitudes of the fields are kq/r^2. The (negative) vertical components are obtained by multiplying by $\sin\beta$, which is approximately equal to $(\ell/2)/r$ in the small-angle approximation. The vertical field is therefore directed downward with magnitude

$$E \approx 2\left(\frac{kq}{r^2}\right)\frac{\ell/2}{r} = \frac{kq\ell}{r^3}. \tag{178}$$

This agrees with the result in Eq. (2.36) when $\theta = \pi/2$, because the $\hat{\boldsymbol{\theta}}$ vector points downward at the given point (in the direction of increasing θ, which is measured down from the vertical). This field is half as large as the field on the vertical axis, for a given value of r.

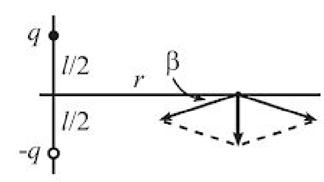

Figure 47

2.68. **E and ρ for a sphere**

As in the example in Section 2.10, the Cartesian components of the electric field are given by $E_x = (x/r)E_r$, and likewise for y and z.

Inside the sphere, the field is radial with $E_r = \rho r/3\epsilon_0$, so we quickly find the Cartesian components to be $(E_x, E_y, E_z) = (\rho/3\epsilon_0)(x, y, z)$. Equation (2.59) therefore gives $\text{div}\,\mathbf{E} = (\rho/3\epsilon_0)(1+1+1) = \rho/\epsilon_0$, as desired.

Outside the sphere, the field is radial with $E_r = \rho R^3/3\epsilon_0 r^2$. The Cartesian x component is

$$E_x = \frac{x}{r}\frac{\rho R^3}{3\epsilon_0 r^2} \propto \frac{x}{r^3} = \frac{x}{(x^2+y^2+z^2)^{3/2}}. \tag{198}$$

The constant of proportionality doesn't matter because the end result will be zero. The $\partial E_x/\partial x$ term in Eq. (2.59) is then

$$\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{x(-3/2)(2x)}{(x^2+y^2+z^2)^{5/2}} = \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}}, \tag{199}$$

with similar expressions for the $\partial E_y/\partial y$ and $\partial E_z/\partial z$ terms. The sum of all three terms is zero, because the coefficient of x^2 is $(-2+1+1)$, etc. This is consistent with $\text{div}\,\mathbf{E} = \rho/\epsilon_0$ because $\rho = 0$ outside the sphere.

2.69. E **and** ϕ **for a slab**

(a) At position x inside the slab, there is a slab with thickness $\ell - x$ to the right of x, which acts effectively like a sheet with surface charge density $\sigma_R = (\ell - x)\rho$. Likewise, to the left of x we effectively have a sheet with surface charge density $\sigma_L = (\ell + x)\rho$. Since the electric field from a sheet is $\sigma/2\epsilon_0$, the net field at position x inside the slab is

$$E = \frac{(\ell + x)\rho}{2\epsilon_0} - \frac{(\ell - x)\rho}{2\epsilon_0} = \frac{\rho x}{\epsilon_0}, \tag{200}$$

and it is directed away from the center plane (if ρ is positive). You can also quickly obtain this by using a Gaussian surface that extends a distance x on either side of the center plane.

Outside the slab, the slab acts effectively like a sheet with surface charge density $\rho(2\ell)$, so the field has magnitude $(2\rho\ell)/2\epsilon_0 = \rho\ell/\epsilon_0$ and is directed away from the slab. $E(x)$ is continuous at $x = \pm\ell$, as it should be since there are no surface charge densities in the setup.

(b) The potential relative to $x = 0$ is $\phi = -\int_0^x E\,dx$. Inside the slab this gives

$$\phi_{\text{in}}(x) = -\int_0^x \frac{\rho x}{\epsilon_0} = -\frac{\rho x^2}{2\epsilon_0}. \tag{201}$$

Outside the slab, we must continue the integral past $x = \pm\ell$. On the right side of the slab, where $x > \ell$, the potential is

$$\begin{aligned} \phi(x) &= -\int_0^\ell E_x\,dx - \int_\ell^x E_x\,dx = -\int_0^\ell \frac{\rho x}{\epsilon_0}\,dx - \int_\ell^x \frac{\rho\ell}{\epsilon_0}\,dx \\ &= -\frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell}{\epsilon_0}(x - \ell) = \frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell x}{\epsilon_0}. \end{aligned} \tag{202}$$

On the left side of the slab, where $x < -\ell$, you can show that the only change in ϕ is that there is a relative "+" sign between the terms (basically, just change ℓ to $-\ell$). So the potential outside the slab equals

$$\phi_{\text{out}}(x) = \frac{\rho\ell^2}{2\epsilon_0} - \frac{\rho\ell|x|}{\epsilon_0}. \tag{203}$$

From Eqs. (201) and (203) we see that $\phi(x)$ is continuous at the boundaries at $x = \pm\ell$, as it should be. Plots of $E(x)$ and $\phi(x)$ are shown in Fig. 54.

(c) For a single Cartesian direction, we have $\nabla \cdot \mathbf{E} = \partial E_x/\partial x$ and $\nabla^2\phi = \partial^2\phi/\partial x^2$. The following four relations are indeed all true:

$$\begin{aligned} \text{Inside}: &\quad \rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff \rho = \epsilon_0 \partial(\rho x/\epsilon_0)/\partial x, \\ \text{Outside}: &\quad \rho(x) = \epsilon_0 \nabla \cdot \mathbf{E} \iff 0 = \epsilon_0 \partial(\rho\ell/\epsilon_0)/\partial x, \\ \text{Inside}: &\quad \rho(x) = -\epsilon_0 \nabla^2\phi \iff \rho = -\epsilon_0 \partial^2(-\rho x^2/2\epsilon_0)/\partial x^2, \\ \text{Outside}: &\quad \rho(x) = -\epsilon_0 \nabla^2\phi \iff 0 = -\epsilon_0 \partial^2(\rho\ell^2/2\epsilon_0 \pm \rho\ell x/\epsilon_0)/\partial x^2. \end{aligned} \tag{204}$$

We also have $\mathbf{E} = -\nabla\phi$ both inside and outside, which is true by construction due to the line integrals we calculated in part (b).

2.75. **Curls and divergences**

In Cartesian coordinates,

$$
\begin{aligned}
(\nabla \times \mathbf{F}) &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right), \\
\nabla \cdot \mathbf{E} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \qquad (220)
\end{aligned}
$$

(a) If $\mathbf{F} = (x + y, -x + y, -2z)$ we quickly find $\nabla \times \mathbf{F} = (0, 0, -2)$ and $\nabla \cdot \mathbf{F} = 1 + 1 - 2 = 0$. Since the curl isn't zero, there is no associated potential ϕ.

(b) If $\mathbf{G} = (2y, 2x + 3z, 3y)$ then we find $\nabla \times \mathbf{G} = (0, 0, 0)$ and $\nabla \cdot \mathbf{G} = 0 + 0 + 0 = 0$. Since $\nabla \times \mathbf{G} = 0$ there exists a g such that $\mathbf{G} = \nabla g$. To determine g, we can compute the line integral of $\mathbf{G}$ from a fixed point, say $(0, 0, 0)$, to a general point (x_0, y_0, z_0) over any path. Using the path composed of the three segments in the x, then y, then z directions, we have

$$
\begin{aligned}
g(x_0, y_0, z_0) &= \int_{(0,0,0)}^{(x_0,y_0,z_0)} \mathbf{G} \cdot d\mathbf{s} \\
&= \int_0^{x_0} G_x(x, 0, 0)\, dx + \int_0^{y_0} G_y(x_0, y, 0)\, dy + \int_0^{z_0} G_z(x_0, y_0, z)\, dz \\
&= \int_0^{x_0} 0\, dx + \int_0^{y_0} 2x_0\, dy + \int_0^{z_0} 3y_0\, dz \\
&= 2x_0 y_0 + 3y_0 z_0. \qquad (221)
\end{aligned}
$$

Since (x_0, y_0, z_0) is a general point, we can drop the subscripts and write $g(x, y, z) = 2xy + 3yz$. You can quickly check that the gradient of g is indeed $\mathbf{G}$.

A quicker method of obtaining g is the following. The x component of $\nabla g = \mathbf{G}$ tells us that $\partial g/\partial x = 2y$. So g must be a function of the form $2xy + f_1(y, z)$. Similarly, the y component tells us that g must take the form $2xy + 3yz + f_2(x, z)$, and the z component tells us that g must take the form $3yz + f_3(x, y)$. You can quickly verify that the only function satisfying all three of these forms is $2xy + 3yz$ (plus a constant).

(c) If $\mathbf{H} = (x^2 - z^2, 2, 2xz)$ then we find $\nabla \times \mathbf{H} = (0, -4z, 0)$ and $\nabla \cdot \mathbf{H} = 2x + 0 + 2x = 4x$. Since the curl isn't zero, there is no associated potential ϕ.