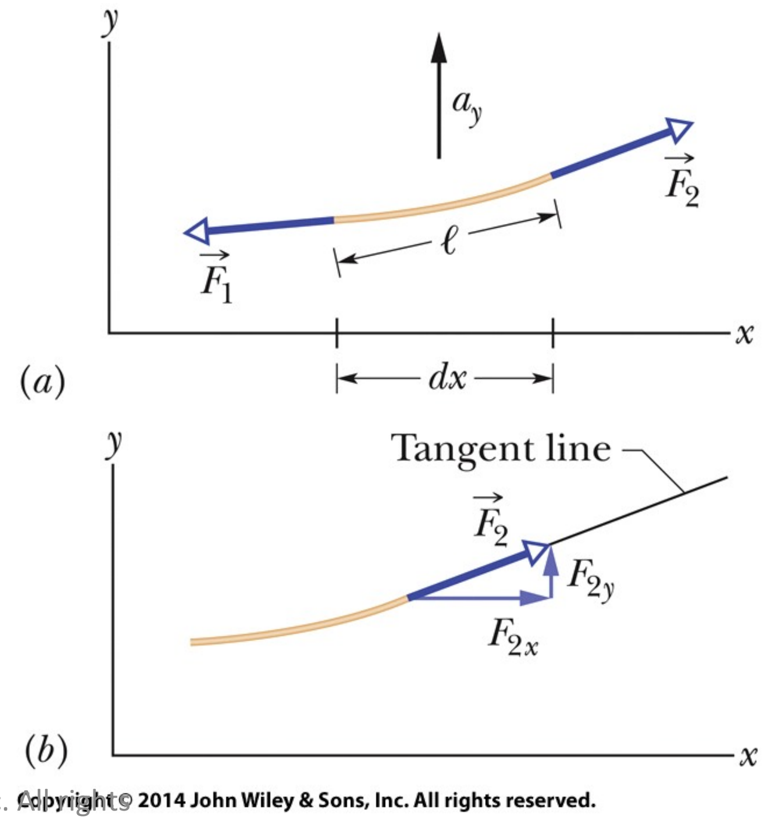
16-4 The Wave Equation

By applying Newton's second law to the element's motion, we can derive a general differential equation, called the wave equation, that governs the travel of waves of any type.

(a) A string element as a sinusoidal transverse wave travels on a stretched string. Forces $\vec{F}_1$ and $\vec{F}_2$ act at the left and right ends, producing acceleration with a vertical component a_y .

(b) The force at the element's right end is directed along a tangent to the element's right side.



16-4 The Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \quad \text{(wave equation).}$$

- This is the general differential equation that governs the travel of waves of all types. Here the waves travel along an *x* axis and oscillate parallel to the *y* axis, and they move with speed *v*, in either the positive x direction or the negative *x* direction.

Wave equation

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}}$$

$y(x,t) =$

$= y_m \sin(kx - \omega t)$

$$\frac{\partial^2 y}{\partial x^2} = -k^2 y_m \sin(kx - \omega t)$$

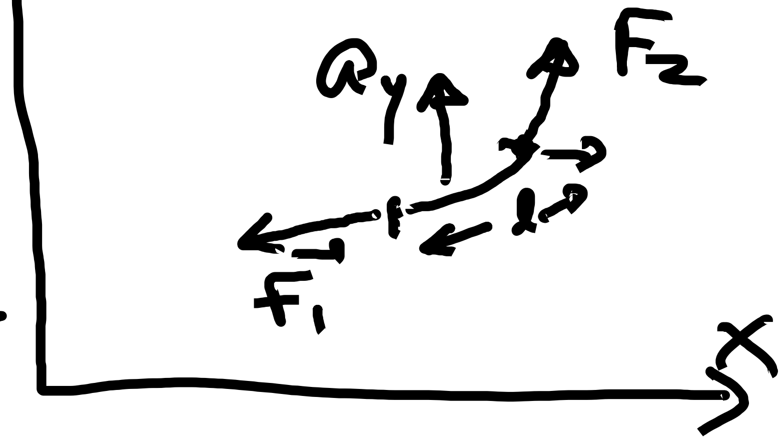
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y_m \sin(kx - \omega t)$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2}$$

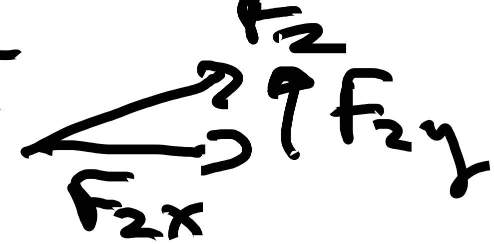
$$\boxed{v = \frac{\omega}{k}}$$

derivation of wave equation

$$S = \frac{dy}{dx}$$



$$v = \sqrt{\frac{\tau}{\mu}}$$



$$F_{2x} \approx \tau = |\vec{F}_2|$$

$$F_{2,y} - F_{1,y} = dm \cdot a_y = \mu \, dx \, a_y$$

$$y(x,t)$$

$$\frac{F_{2y}}{F_{2x}} = S_2 \implies F_{2y} \approx S_2 \cdot \tau$$

$$F_{2y} - F_{1y} \approx (S_2 - S_1)\tau = \mu \, dx \cdot \frac{d^2y}{dt^2}$$

$$\frac{S_2 - S_1}{dx} = \frac{dS}{dx} = \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{\tau}\frac{\partial^2 y}{\partial t^2}} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

16-5 Interference of Waves

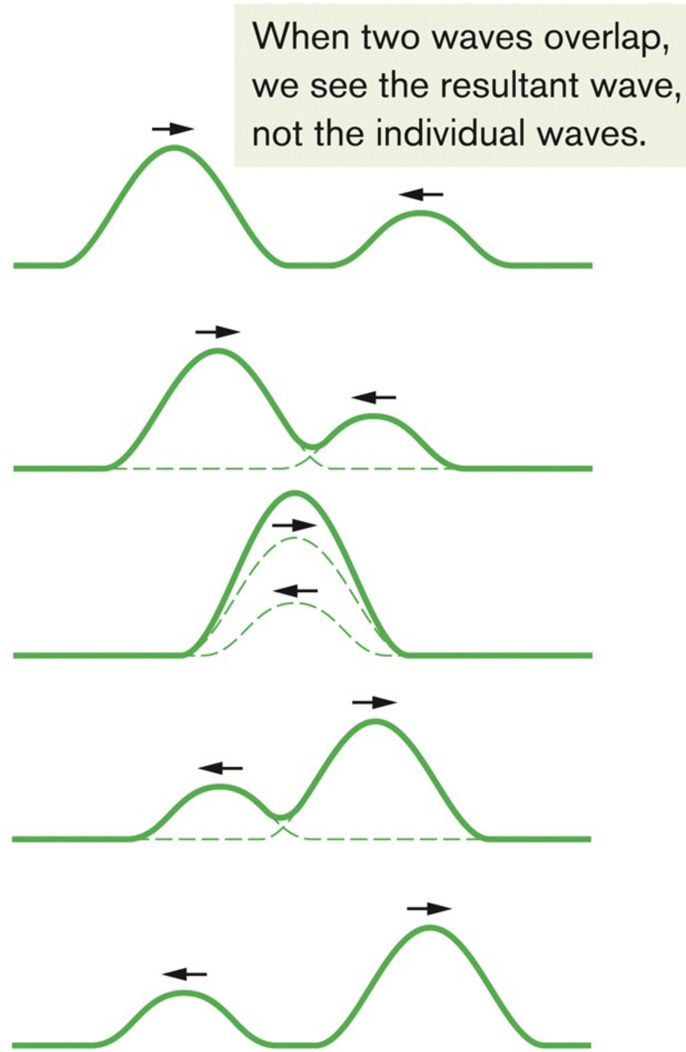
Principle of Superposition of waves

Let $y_1(x, t)$ and $y_2(x, t)$ be the displacements that the string would experience if each wave traveled alone. The displacement of the string when the waves overlap is then the algebraic sum

$$y'(x, t) = y_1(x, t) + y_2(x, t).$$

This summation of displacements along the string means that

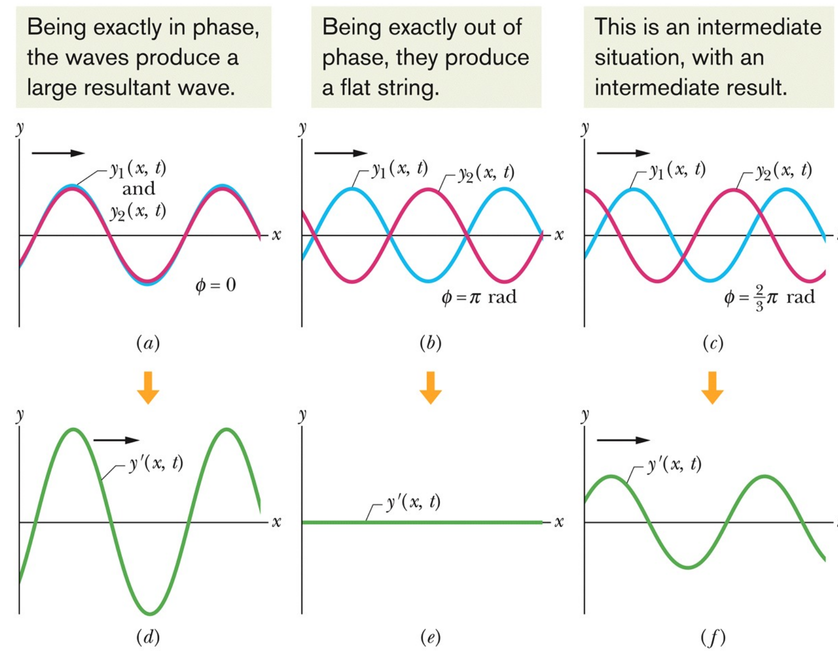
- Overlapping waves algebraically add to produce a **resultant wave** (or **net wave**).
- Overlapping waves do not in any way alter the travel of each other.



16-5 Interference of Waves

Constructive and Destructive Interference

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{\left[2y_m \cos \tfrac{1}{2}\phi\right]}_{\text{Magnitude gives amplitude}} \underbrace{\sin\left(kx - \omega t + \tfrac{1}{2}\phi\right)}_{\text{Oscillating term}}$$



Two identical sinusoidal waves, $y_1(x, t)$ and $y_2(x, t)$, travel along a string in the positive direction of an *x* axis. They interfere to give a resultant wave $y'(x, t)$. The resultant wave is what is actually seen on the string. The phase difference Φ between the two interfering waves is (a) *0* rad or 0^o, (b) π rad or 180^o, and (c) $2/3\,\pi$ rad or 120^o. The corresponding resultant waves are shown in (d), (e), and (f).

$$y_1(x,t) = y_m \sin(\ x - \omega t)$$

$$y_2(x,t) = y_m \sin(kx - \omega t + \phi)$$

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}$$

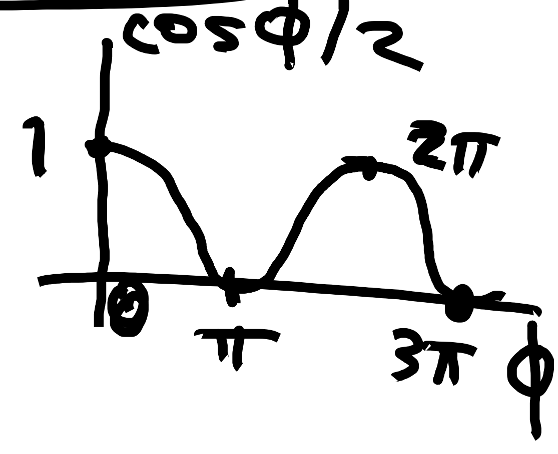
$$\boxed{y(x,t) = 2\cos\frac{\phi}{2}\,\sin\left(kx - \omega t + \phi/2\right)}$$

constructive interference

$$\phi = 0, 2\pi, \ldots = 2\pi n$$

destructive:

$$\phi = \pi, 3\pi, \ldots = 2\pi\left(n + \tfrac{1}{2}\right) = \pi(2n+1)$$

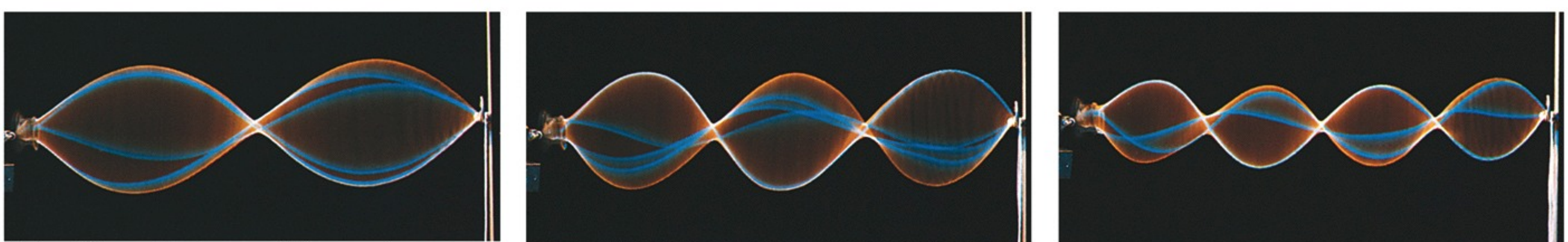


16-7 Standing Waves and Resonance

Standing Waves

•The interference of two identical sinusoidal waves moving in opposite directions produces standing waves. For a string with fixed ends, the standing wave is given by

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \sin kx]}_{\substack{\text{Magnitude}\\ \text{gives}\\ \text{amplitude}\\ \text{at position } x}} \underbrace{\cos \omega t}_{\substack{\text{Oscillating}\\ \text{term}}}$$



Richard Megna/Fundamental Photographs

Stroboscopic photographs reveal (imperfect) standing wave patterns on a string being made to oscillate by an oscillator at the left end. The patterns occur at certain frequencies of oscillation.

$$y_1(x,t) = y_m \sin(x - \omega t)$$

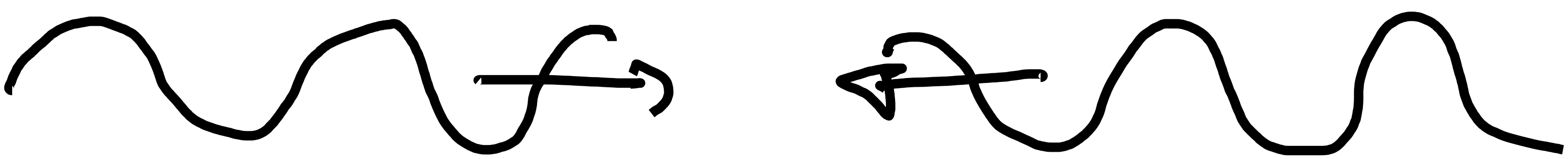
$$y_2(x,t) = y_m \sin(kx + \omega t)$$

$$y(x,t) = y_1(x,t) + y_2(x,t)$$

$$\sin\alpha + \sin\beta = 2 \sin\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2}$$

$$\boxed{y(x,t) = 2y_m \sin(kx) \cos(\omega t)}$$

Standing waves = superposition of waves traveling in opposite direction



$$\lambda f = v$$

$$\boxed{f_n = \frac{v}{\lambda} = n\frac{v}{2L}}$$

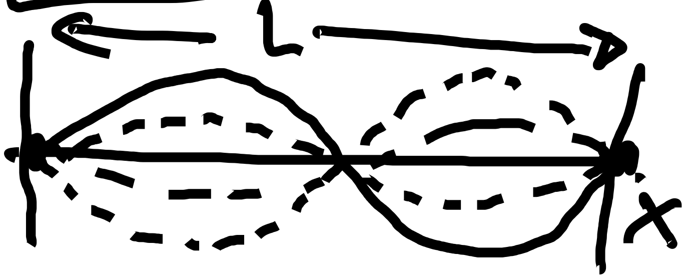
Standing waves

$$v = \sqrt{\frac{\tau}{\mu}}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\boxed{y(x,t) = 2y_m \sin(kx)\cos(\omega t)}$$

standing wave

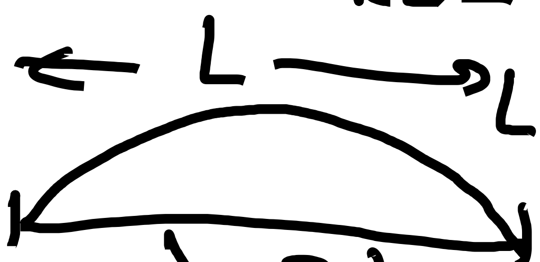


$$\sin(kx) = 0 \text{ for } x = 0,\ x = L$$

$$kL = n\pi \qquad n = 1, 2, \ldots$$

$$n\lambda = 2L$$

$$\boxed{n\cdot\frac{\lambda}{2} = L}$$



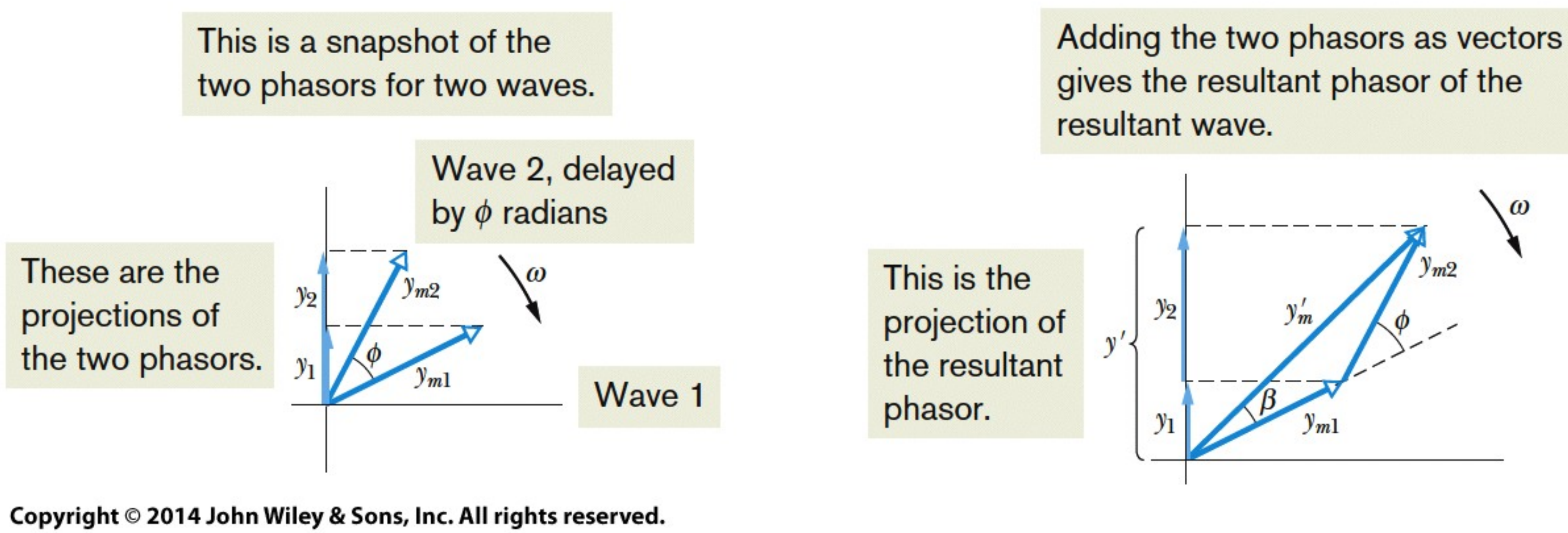
$$Lk = \frac{2\cancel{\pi}}{\lambda}L = n\cancel{\pi}$$

$$\lambda = 2L$$

$$\lambda = 2L/n$$

16-6 Phasors

A phasor is a vector that rotates around its tail, which is pivoted at the origin of a coordinate system. The magnitude of the vector is equal to the amplitude y_m of the wave that it represents.



(a) (b)

(a) A second phasor, also of angular speed *ω* but of magnitude y_{m2} and rotating at a constant angle *β* from the first phasor, represents a second wave, with a phase constant *Φ*. (b) The resultant wave is represented by the vector sum y'_m of the two phasors.

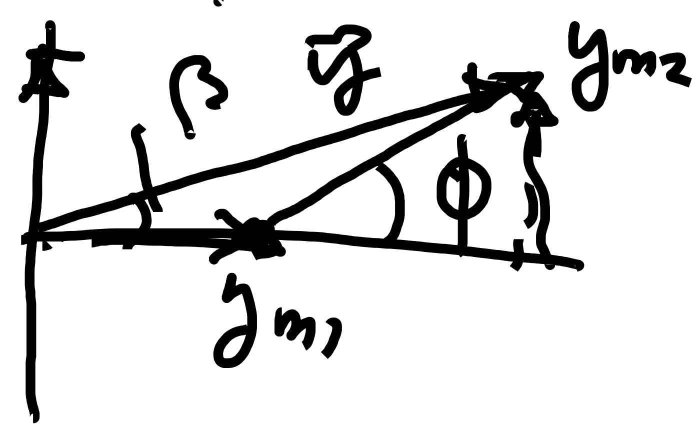
Phasors:

$y_{m1} = y_{m2}$

$$y_1(x,t) = y_{m1} \sin(kx - \omega t)$$

$$y_2(x,t) = y_{m2} \sin(kx - \omega t + \phi)$$

$$y(x,t) = y_1 + y_2 = y_m \sin(kx - \omega t + \beta)$$



$$(\tilde{y}_m)_x = y_{m1} + y_{m2} \cos\phi$$

$$(\tilde{y}_m)_y = y_{m2} \sin\phi$$

$$y_m = \sqrt{(y_{m1} + y_{m2}\cos\phi)^2 + y_{m2}^2 \sin^2\phi}$$

2

$$\boxed{y_m = \sqrt{y_{m1}^2 + y_{m2}^2 + 2 y_{m1} y_{m2} \cos\phi}}$$

$$\boxed{\tan\beta = \frac{(y_m)_y}{(y_m)_x} = \frac{y_{m2}\sin\phi}{y_{m1} + y_{m2}\cos\phi}}$$