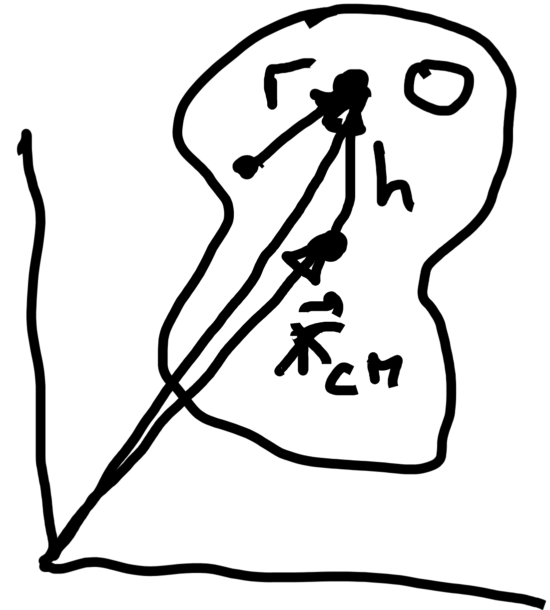
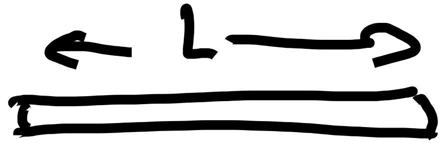
$$I = \int d^3r \, \rho(r) \, r^2$$



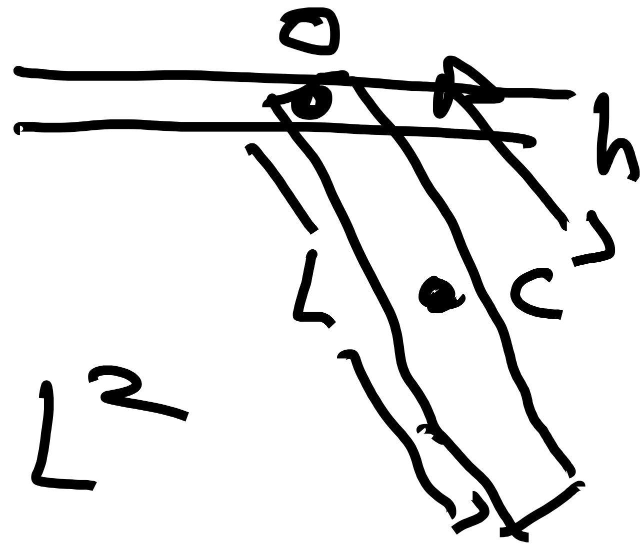
$$I = I_{cm} + m h^2$$

Measure g

Uniform rod



$$h = \frac{1}{2} L$$



$$I_{cm} = \frac{1}{12} m L^2$$

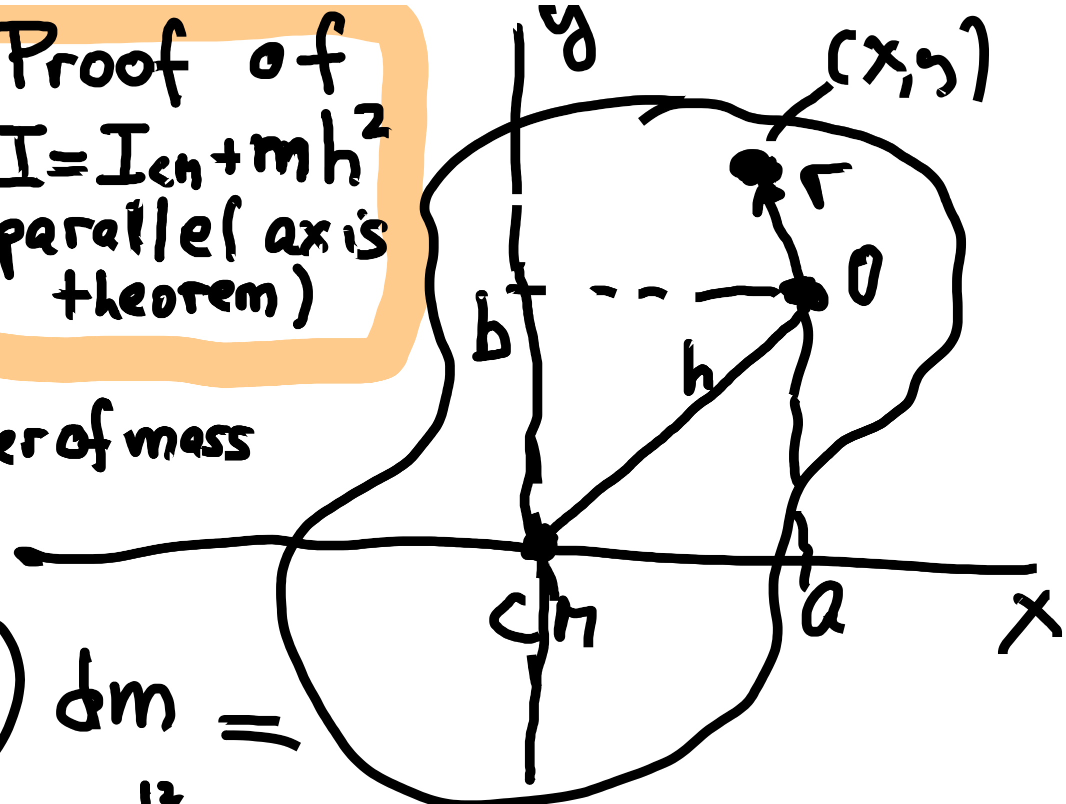
$$I = \frac{1}{12} m L^2 + m \frac{L^2}{4} = \frac{1}{3} m L^2$$

Proof of $I = I_{cm} + mh^2$ (parallel axis theorem)

$$I = \int r^2 \, dm$$

$$\vec{r} = (x-a,\ y-b)$$

origin of coord system at center of mass



$$I = \int \left((x-a)^2 + (y-b)^2 \right) dm =$$

$$= \underbrace{\int (x^2+y^2)\, dm}_{I_{cm}} + \underbrace{\int \overbrace{(a^2+b^2)}^{h^2} dm}_{h^2 m} = I_{cm} + h^2 m$$

$$\underbrace{-2a \cancel{\int x\, dm}}_{= x_{cm} = 0} - 2b \cancel{\int y\, dm}$$

$$x_{cm} = \frac{\int x\, dm}{m}$$

15-5 Damped Simple Harmonic Motion

Learning Objectives

15.38 Describe the motion of a damped simple harmonic oscillator and sketch a graph of the oscillator's position as a function of time.

15.39 For any particular time, calculate the position of a damped simple harmonic oscillator.

15.40 Determine the amplitude of a damped simple harmonic oscillator at any given time.

15.41 Calculate the angular frequency of a damped simple harmonic oscillator in terms of the spring constant, the damping constant, and the mass, and approximate the angular frequency when the damping constant is small.

15.42 Apply the equation giving the (approximate) total energy of a damped simple harmonic oscillator as a function of time.

15-5 Damped Simple Harmonic Motion

- When an external force reduces the motion of an oscillator, its motion is **damped**
- Assume the liquid exerts a **damping force** proportional to velocity (accurate for slow motion)

$$F_d = -bv, \quad \textbf{Eq. (15-39)}$$

- *b* is a damping constant, depends on the vane and the viscosity of the fluid

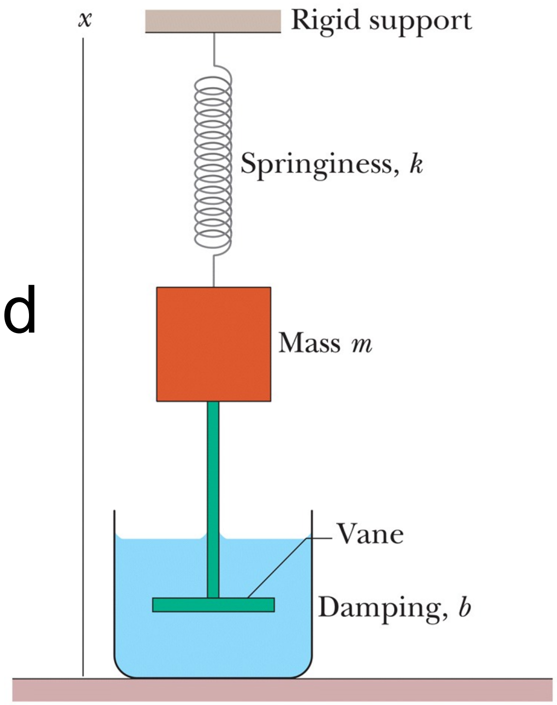


Figure 15-16

15-5 Damped Simple Harmonic Motion

- We use Newton's second law and rearrange to find:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \qquad \textbf{Eq. (15-41)}$$

- The solution to this differential equation is:

$$x(t) = x_m e^{-bt/2m}\cos(\omega' t + \phi), \qquad \textbf{Eq. (15-42)}$$

- With angular frequency:

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \qquad \textbf{Eq. (15-43)}$$

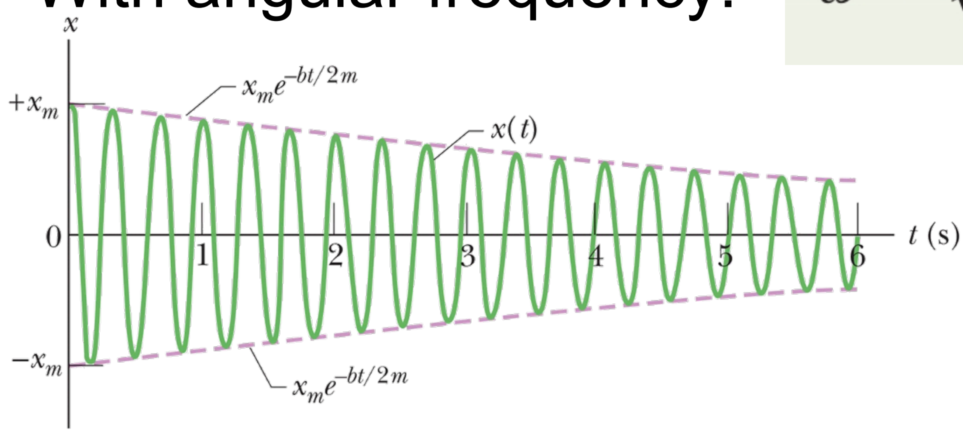


Figure 15-17

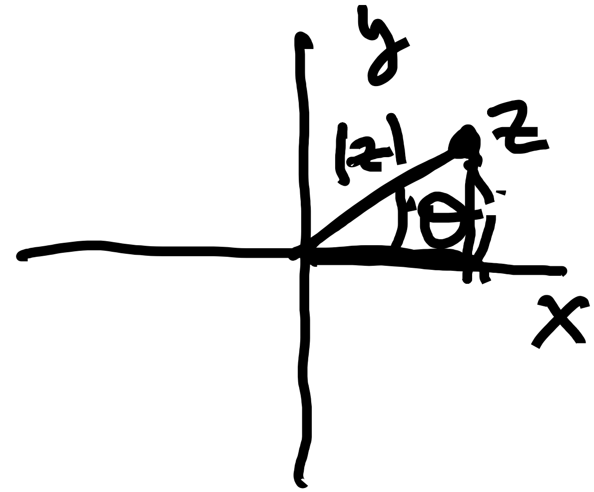
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = \underline{|z|\cos\theta} + \underline{i}\,|z|\sin\theta$$

$$|e^{i\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$z = x + iy = |z|e^{i\theta}$$

$$\tan\theta = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{y}{x}$$



$$z = x + iy$$

$$i = \sqrt{-1}$$

$$z = |z|e^{i\theta}$$

$$|z| = \sqrt{x^2 + y^2}$$

$$m\ddot{x} = -kx - b\dot{x}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x = -\frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}}$$

Complex exponentials

$$x = x_m e^{\Omega t + \phi}; \quad \dot{x} = x_m \Omega e^{\Omega t}; \quad \ddot{x} = x_m \Omega^2 e^{\Omega t}$$

$$m\Omega^2 + b\Omega + k = 0 \Rightarrow \Omega^2 + \frac{b}{m}\Omega + \frac{k}{m} = 0$$

$$\Omega = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \omega^2} = -\frac{b}{2m} \pm i\underbrace{\sqrt{\omega^2 - \left(\frac{b}{2m}\right)^2}}_{\omega'}$$

$$x(t) = x_m e^{-\frac{b}{2m}t} e^{i\omega' t + \phi}$$

$$ax^2 + bx + c = 0$$

$$\mathrm{Re}\, x(t) = x(t) = x_m e^{-\frac{b}{2m}t} \cos(\omega' t + \phi)$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

15-5 Damped Simple Harmonic Motion

- If the damping constant is small, $\omega' \approx \omega$
- For small damping we find mechanical energy by substituting our new, decreasing amplitude:

$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m},$$ **Eq. (15-44)**

Checkpoint 6

Here are three sets of values for the spring constant, damping constant, and mass for the damped oscillator of Fig. 15-16. Rank the sets according to the time required for the mechanical energy to decrease to one-fourth of its initial value, greatest first.

Set 1	$2k_0$	b_0	m_0
Set 2	k_0	$6b_0$	$4m_0$
Set 3	$3k_0$	$3b_0$	m_0

Answer: 1,2,3

15-6 Forced Oscillations and Resonance

Learning Objectives

15.43 Distinguish between natural angular frequency and driving angular frequency.

15.44 For a forced oscillator, sketch a graph of the oscillation amplitude versus the ratio of the driving angular frequency to the natural angular frequency, identify the approximate location of resonance, and indicate the effect of increasing the damping.

15.45 For a given natural angular frequency, identify the approximate driving angular frequency that gives resonance.

15-6 Forced Oscillations and Resonance

- Forced, or driven, oscillations are subject to a periodic applied force
- A forced oscillator oscillates at the angular frequency of its driving force:

$$x(t) = x_m \cos(\omega_d t + \phi), \quad \textbf{Eq. (15-45)}$$

- The displacement amplitude is a complicated function of ω and ω_0
- The velocity amplitude of the oscillations is greatest when:

$$\omega_d = \omega \quad \textbf{Eq. (15-46)}$$

15-6 Forced Oscillations and Resonance

- This condition is called **resonance**
- This is also approximately when the displacement amplitude is largest
- Resonance has important implications for the stability of structures
- Forced oscillations at resonant frequency may result in rupture or collapse

Xm

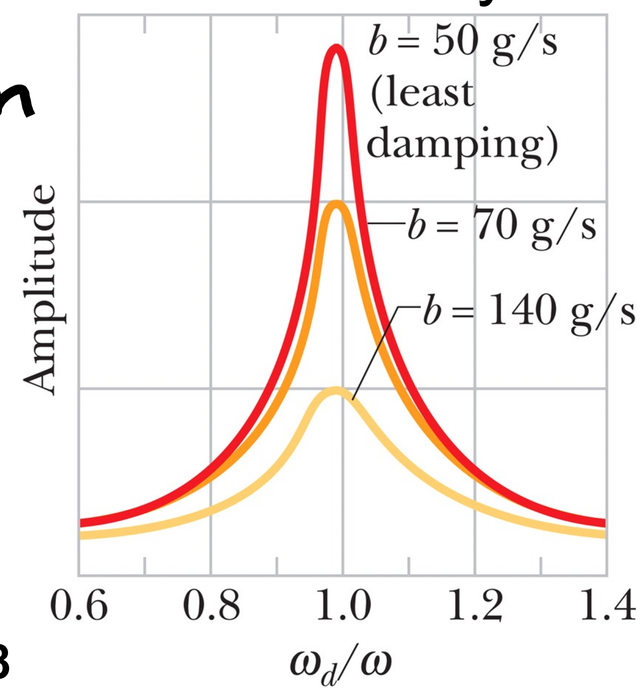


Figure 15-18

$$m\ddot{x} + b\dot{x} + kx = f\cos\omega_d t \quad \Big| \; x = x_m\cos(\omega_d t + \phi)$$

$$\to f e^{i\omega_d t}$$

$$x = x_m e^{i(\omega_d t + \phi)}$$

$$\dot{x} = x_m i\omega_d e^{i(\omega_d t + \phi)}$$

$$\ddot{x} = -x_m \omega_d^2 e^{i(\omega_d t + \phi)}$$

$$-m\omega_d^2 x_m e^{i\phi} + ib\,\omega_d x_m e^{i\phi} + k x_m e^{i\phi} = f$$

solve for x_m, ϕ

$$x_m\left(\omega^2 + i\frac{b}{m}\omega_d - \omega_d^2\right)e^{i\phi} = f/m \quad \Big| \; \frac{k}{m} = \omega^2$$

$$x_m = \frac{f/m}{\omega^2 - \omega_d^2 + i\frac{b}{m}\omega_d}\, e^{-i\phi}$$

x_m is real so $x_m = |x_m|$

$|e^{-i\phi}| = 1$

$$x_m = \frac{f/m}{\sqrt{(\omega^2 - \omega_d^2)^2 + b^2\omega_d^2/m^2}}$$

$$e^{-i\phi} = \frac{x_m}{F/m}\left(\omega^2 - \omega_d^2 + i\frac{b}{m}\omega_d\right)$$

$$\tan\phi = -\frac{\mathrm{Im}(e^{-i\phi})}{\mathrm{Re}(e^{-i\phi})} = \frac{-\frac{b}{m}\omega_d}{\omega^2 - \omega_d^2}$$

$$\boxed{\tan\phi = \frac{\frac{b}{m}\omega_d}{\omega_d^2 - \omega^2}}$$

15 Summary

Frequency

- 1 Hz = 1 cycle per second

Period

$$T = \frac{1}{f}. \quad \textbf{Eq. (15-2)}$$

The Linear Oscillator

$$\omega = \sqrt{\frac{k}{m}} \quad \textbf{Eq. (15-12)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \textbf{Eq. (15-13)}$$

Simple Harmonic Motion

- Find *v* and *a* by

$$x(t) = x_m \cos(\omega t + \phi) \quad \textbf{Eq. (15-3)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f. \quad \textbf{Eq. (15-5)}$$

Energy

- Mechanical energy remains constant as *K* and *U* change
- K = ½ mv^2, U = ½ kx^2

15 Summary

Pendulums

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$ **Eq. (15-23)**

$$T = 2\pi\sqrt{\frac{L}{g}}$$ **Eq. (15-28)**

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$ **Eq. (15-29)**

Simple Harmonic Motion and Uniform Circular Motion

- SHM is the projection of UCM onto the diameter of the circle in which the UCM occurs

Damped Harmonic Motion

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi),$$

Eq. (15-42)

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}.$$ **Eq. (15-43)**

Forced Oscillations and Resonance

- The velocity amplitude is greatest when the driving force is related to the natural frequency by:

$$\omega_d = \omega$$ **Eq. (15-46)**