On two-dimensional magnetohydrodynamic turbulence

By A. POUQUET

Centre National de la Recherche Scientifique,
Observatoire de Nice, France

It is shown that two-dimensional MHD turbulence is in certain respects closer to three-dimensional than to two-dimensional hydrodynamic turbulence. A second-order closure indicates that:

(i) at zero viscosity and magnetic diffusivity, a singularity appears at a finite time;

(ii) there is an energy cascade to small scales and an inverse cascade of squared magnetic potential, in agreement with a conjecture of Fyfe & Montgomery (1976);

(iii) small-scale magnetic energy acts like a negative eddy viscosity on large-scale magnetic fields;

(iv) upon injection of magnetic energy, a stationary state is obtained which has zero magnetic energy for a positive magnetic diffusivity λ (anti-dynamo theorem); however, this stationary state is preceded by a very long non-zero magnetic energy plateau which probably extends to infinite times as $\lambda \to 0$.

It is suggested that direct numerical simulation of the two-dimensional MHD equations with high resolution (a 512^2 or 1024^2 grid) could lead to a better understanding of the small-scale structure of fully developed turbulence, especially questions of intermittency and geometry.

1. Introduction

The study of two-dimensional turbulence at large Reynolds number appears to be relevant to atmospheric dynamics, at least for the large scales, and as such is important to meteorology (Desbois 1975). It is possible that certain large-scale features of solar dynamics can be modelled by the two-dimensional magnetohydrodynamic (MHD) equations (Weiss 1966; Krause & Rüdiger 1975). It is also generally believed that, in the presence of a strong external magnetic field, the motions become two-dimensional (Kolesnikov & Tsinober 1972) but it has been shown by Alemany *et al.* (1978) that this case is actually very different from two-dimensional turbulence (see also Moffatt 1967).

It was hoped for a time that two-dimensional non-magnetic turbulence could be taken as a model of three-dimensional turbulence, but it was soon realized that they differ in several ways (Lee 1952; Kraichnan 1966; Batchelor 1969; Krause & Rüdiger 1974). Indeed, the presence of an extra invariant in two dimensions, the vertical vorticity ω, modifies drastically the dynamics of the turbulence. In three dimensions, it takes a finite time of the order of the large-scale eddy turnover time for large-scale excitation to be transferred to the smallest scales available to the system; velocity gradients presumably become infinite and a finite dissipation of energy occurs in the limit of zero viscosity (Brissaud *et al.* 1973; Frisch, Sulem & Nelkin 1978). However,

in two-dimensional turbulence this time becomes infinite with decreasing scale, so that no singularity appears (in a finite time) in a two-dimensional flow (Pouquet *et al.* 1975). Moreover, intermittency in the small scales of three-dimensional turbulence modifies the energy spectrum (Kolmogorov 1962; Frisch *et al.* 1978), whereas in two dimensions it probably does not modify the enstrophy spectrum (Kraichnan 1975).

The situation is quite different in two-dimensional MHD turbulence: the Lorentz force relaxes the vorticity constraint, so that one does not know *a priori* if two-dimensional MHD turbulence will be closer to two-dimensional or to three-dimensional non-magnetic turbulence.

For an incompressible fluid, the two-dimensional MHD equations may be written in terms of the vertical vorticity ω and the magnetic potential a:

$$D\omega/Dt = (\partial/\partial t + \mathbf{v}.\nabla)\,\omega = \mathbf{b}.\nabla j + \nu\nabla^2\omega, \tag{1.1}$$

$$Da/Dt = (\partial/\partial t + \mathbf{v}.\nabla)\,a = \lambda\nabla^2 a, \tag{1.2}$$

$$\mathbf{b} = \mathrm{curl}\,(a\mathbf{e}_3), \quad \nabla.\mathbf{v} = 0, \quad \nabla.\mathbf{b} = 0.$$

Here $\mathbf{v}$ is the velocity ($\omega\mathbf{e}_3 = \mathrm{curl}\,\mathbf{v}$), the magnetic induction field $\mathbf{b}$ is normalized by $(\rho\mu_0)^{\frac{1}{2}}$ (where ρ is the density and μ_0 the permeability), and $j = \mathrm{curl}\,\mathbf{b}$ is the current. It follows from (1.2) that the magnetic potential is carried along like a passive scalar, at least as long as the reaction of the Lorentz force on the velocity field can be neglected. When $\nu = \lambda = 0$ (ν being the kinematic viscosity and λ the magnetic diffusivity), (1.1) and (1.2) possess three quadratic invariants: the total energy

$$\int (v^2 + b^2)\, d^2\mathbf{r},$$

the variance of the magnetic potential

$$\int a^2\, d^2\mathbf{r}$$

and the cross-helicity

$$\int \mathbf{v}.\mathbf{b}\, d^2\mathbf{r}.$$

It is likely that a seed magnetic field will grow in time through line stretching by velocity gradients as in three dimensions. But what happens if the magnetic energy is of the same order as the kinetic energy? The Lorentz force may react on the velocity field in such a way as to prevent further growth of the magnetic field. This might make two-dimensional MHD turbulence resemble two-dimensional non-magnetic turbulence. Another possibility is that the Lorentz force (possibly in combination with the pressure force) would enhance the velocity gradients, leading to further growth of the magnetic field. Because of the nonlinearity of the equations, the latter case would produce catastrophic growth of both magnetic fields and velocity gradients (and possibly of velocities themselves); at zero viscosity and zero magnetic diffusivity, a singularity would occur in a finite time. We could then have a problem more akin to three-dimensional turbulence. It is the purpose of this paper to show that this is the most likely situation.

Various tools are available to tackle this question. First, there are mathematical proofs that two-dimensional non-magnetic ideal (inviscid) flows have no singularities

at any finite time (Wolibner 1933; Frisch & Bardos 1975; see also Rose & Sulem 1978 for a review). This result, which relies heavily on vorticity conservation, does not seem to be generalizable to two-dimensional MHD flows: at present the best result concerns regularity for a finite time (Sulem 1977). One can also obtain an upper limit for the exponent of the inertial range of two-dimensional MHD turbulence by following a technique of Sulem & Frisch (1975) but this does not shed any light on the question of singularities (see appendix).

Second, a numerical study can be attempted (Tappert & Hardin 1971; Fyfe, Montgomery & Joyce 1977; Orszag & Tang 1978; § 5 below). In order to get a better hold on the physics of two-dimensional MHD, in this paper we shall resort to soluble stochastic models (or second-order closure) of the statistical problem, assuming homogeneity and isotropy. Although such techniques have their shortcomings (see § 5), they have provided very valuable information on both two- and three-dimensional turbulence: singularities, inertial ranges and direct or inverse cascades (see Rose & Sulem 1978 for a review).

The outline of the paper is as follows: § 2 is concerned with singularities; § 3 deals with inertial ranges, particularly the inverse cascade of magnetic potential conjectured by Fyfe & Montgomery (1976), this cascade being reinterpreted in terms of a negative eddy viscosity; § 4 discusses the ultimate fate of the magnetic energy when the momentum equation is subject to random forcing; § 5 summarizes the main results and discusses several perspectives, in particular intermittency and direct numerical simulations.

2. Singularities in two-dimensional MHD using second-order closure

It is a straightforward matter to write down a quasi-normal approximation to the two-dimensional MHD equations which, for Gaussian initial conditions, is exact to order t^2. However, it is known that this can be extended further in time by suitable transformation. In particular, Markovianization ensures realizability. In this paper, we use the eddy-damped quasi-normal Markovian (EDQNM) approximation (Orszag 1976) and a simplified version of the Markovian random coupling (MRC) model (Frisch, Lesieur & Brissaud 1974). The same method has been applied in Pouquet, Frisch & Léorat (1976) to three-dimensional MHD turbulence, and we refer the reader to this paper for details. The EDQNM approximation allows one to close the equations at the level of second-order moments. If we denote by E_k^V and E_k^M the kinetic and magnetic energy spectra, we have in the absence of $\mathbf{v}$, $\mathbf{b}$ correlation (i.e. zero cross-helicity; see Pouquet *et al.* 1976, p. 323, for a discussion of this point)

$$\frac{\partial E_k^V}{\partial t} + 2\nu k^2 E_k^V = \int_{\Delta_k} \frac{dp\,dq}{2\pi} \theta_{kpq} \{k^2 p^{-1} q^{-1} b_2(k,p,q)\,[kE_p^V E_q^V - pE_q^V E_k^V]$$
$$+ (q^2 - p^2)\,d_2(k,p,q)\,k^{-1}q^{-1}[pE_k^V E_q^M - kE_p^M E_q^M]\} + F_k^V, \quad (2.1)$$

$$\frac{\partial E_k^M}{\partial t} + 2\lambda k^2 E_k^M = \int_{\Delta_k} \frac{dp\,dq}{2\pi} \theta_{kpq} kp^{-2}q^{-1} d_2(k,p,q)$$
$$\times \{(q^2 - k^2)\,pE_k^M E_q^M + k^3 E_p^V E_q^M - p^3 E_q^V E_k^M\} + F_k^M, \quad (2.2)$$

$$b_2(k,p,q) = 2k^{-4}\sin\alpha(k^2 - q^2)\,(p^2 - q^2), \quad d_2(k,p,q) = 2pk^{-1}\sin\alpha, \quad (2.3)$$

$$\mu_k = (\nu + \lambda)\, k^2 + C_S \left(\int_0^k q^2 (E_q^V + E_q^M)\, dq \right)^{\frac{1}{2}} + \frac{k}{3^{\frac{1}{2}}} \left(2 \int_0^k E_q^M\, dq \right)^{\frac{1}{2}},$$

$$\mu_{kpq} = \mu_k + \mu_p + \mu_q,$$

$$\theta_{kpq} = [1 - \exp(-t\mu_{kpq})]/\mu_{kpq} \quad \text{(EDQNM approximation)} \quad \text{or} \quad \theta_{kpq} \equiv \theta_0$$
$$\text{(MRC model),} \tag{2.4}$$

where Δ_k is a region of the p, q plane such that k, p and q can form a triangle in which α is the angle opposite to the side k, while F_k^V and F_k^M are the kinetic and magnetic energy injection spectra. These equations have been derived independently by D. Montgomery (1976, private communication). The eddy-damping rate μ_k is determined, in the EDQNM framework, on phenomenological grounds (see §2 of Pouquet *et al.* 1976). It can be verified that the nonlinear terms of (2.1) and (2.2) conserve the total energy

$$E^T = \int_0^\infty (E_k^V + E_k^M)\, dk \tag{2.5}$$

and the variance of the magnetic potential

$$E^a = \int_0^\infty k^{-2} E_k^M\, dk. \tag{2.6}$$

The study of singularities arising from (2.1) and (2.2) for $\nu = \lambda = 0$ and $F_k^V = F_k^M = 0$ is somewhat simplified if one uses the MRC model, in which $\theta \equiv \theta_0$. In the non-magnetic case, it is then possible to prove that the kinetic enstrophy

$$\Omega^V = \int_0^\infty k^2 E_k^V\, dk \tag{2.7}$$

blows up at a finite time whereas in two dimensions it remains constant and the kinetic 'palinstrophy'

$$P^V = \int_0^\infty k^4 E_k^V\, dk \tag{2.8}$$

grows at most exponentially. In the two-dimensional MHD case we obtain from (2.1) and (2.2) after some algebra

$$d\Omega^V/dt = \theta_0 \pi(-E^M(P^V - P^M) - (\Omega^M)^2), \tag{2.7}$$

$$d\Omega^M/dt \geqslant \theta_0 \pi(E^M(P^V - P^M) + 3\Omega^V\Omega^M). \tag{2.8}$$

It is impossible to conclude from this pair of equations for four unknowns that the enstrophies will blow up. However, one can make some interesting observations. Notice that the kinetic enstrophy may grow only if $P^M > P^V$, so that there is an excess of magnetic excitation in the small scales. Next we can eliminate the palin-strophies by adding (2.7) and (2.8):

$$d(\Omega^V + \Omega^M)/dt \geqslant \theta_0 \pi \Omega^M (3\Omega^V - \Omega^M). \tag{2.9}$$

In two-dimensional MHD, as in three-dimensional MHD, there is an Alfvén effect which tends to bring small-scale kinetic and magnetic excitation into equipartition and which is particularly important for small scales where the Alfvén time becomes less than the turnover time (Pouquet *et al.* 1976, §3). It is therefore plausible to assume that the kinetic and magnetic enstrophies are of the same order of magnitude:

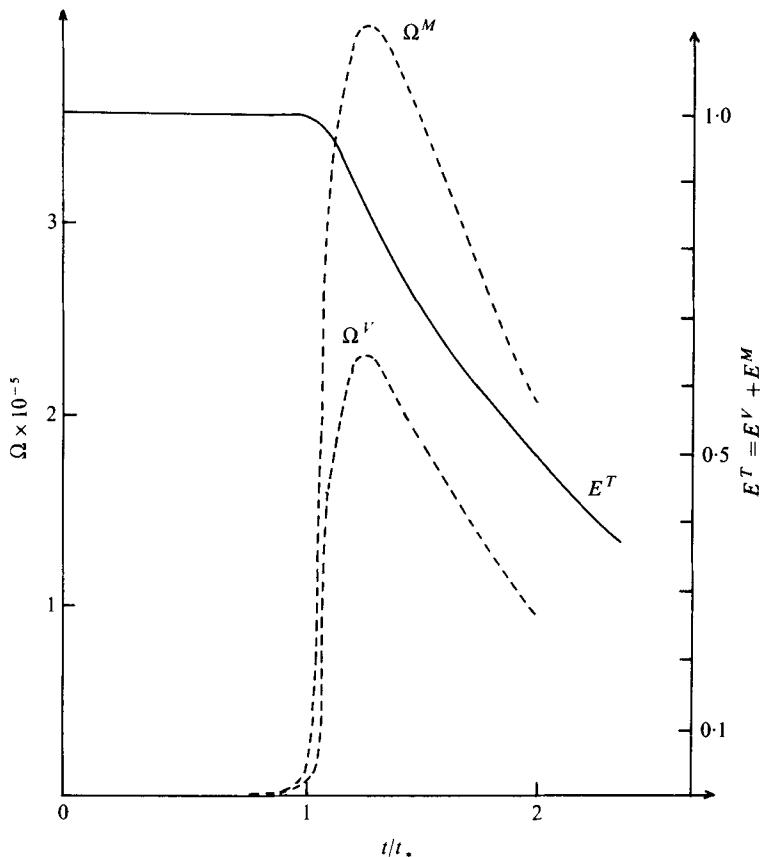$$\Omega^M = \psi_1 \Omega^V, \tag{2.10}$$

FIGURE 1. Variation of the total energy $E^T = E^V + E^M$ (solid line) and total kinetic and magnetic enstrophies (dashed lines) in two-dimensional MHD turbulence under the EDQNM closure. Initial energy and integral scale of order unity. Kinetic and magnetic Reynolds numbers based on integral scale $R = 3 \times 10^7$. Notice the onset of energy dissipation after a time t_* of the order of a few large-eddy turnover times.

where ψ_1 is a numerical constant. It then follows that both enstrophies will blow up in a finite time (depending on θ_0 and on ψ_1). In the absence of a rigorous analytic argument, we resorted to a numerical integration of the EDQNM equations (2.1) and (2.2). The numerical technique is described in Pouquet *et al.* (1976) and in Léorat (1975). The quadratic invariants (total energy and variance of the magnetic potential) are conserved by the numerical scheme to within round-off errors. Throughout this paper, the magnetic Prandtl number $Pr_m = \nu/\lambda$ is set equal to unity. The initial energy spectrum is given by

$$\mathscr{E}_k = k^3 \exp\left(-1\cdot5k^2\right), \quad \int_{k_{\min}}^{k_{\max}} \mathscr{E}_k \, dk = 1, \tag{2.11}$$

with $k_{\min} = 2^{-2}$ and $k_{\max} = 2^{14}$ respectively the minimum and maximum wave-numbers. The Reynolds number is $R = 3 \times 10^7$. The spectral equations are integrated in time in the absence of forcing ($F_k^V = F_k^M = 0$). Figure 1 shows the time evolution of the kinetic and magnetic enstrophies (dashed lines) and of the total energy (solid line). Both enstrophies increase sharply around $t = t_*$, the time at which the energy
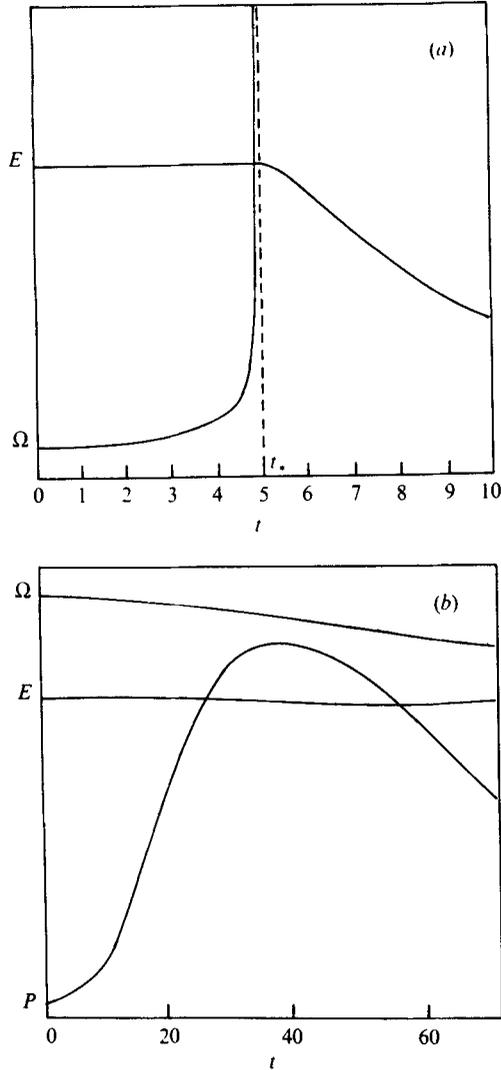
FIGURE 2. Non-magnetic turbulence in (a) three and (b) two dimensions. The total energy E, enstrophy Ω and palinstrophy P are shown. These figures have been extracted from Pouquet *et al.* (1975) for comparison with figure 1.

dissipation, previously $O(R^{-1})$, becomes $O(1)$. The 'catastrophe' time t_* is of the same order as the large-eddy turnover time. For the MRC model ($\theta \equiv \theta_0$) similar results are obtained. Observe that $\Omega^M < 3\Omega^V$ in agreement with the condition (2.9) for enstrophy growth. Comparing figure 1 with similar figures for two- and three-dimensional non-magnetic turbulence (see figure 2, taken from Pouquet *et al.* 1975), one may say that two-dimensional MHD turbulence behaves like three-dimensional non-magnetic turbulence. Numerical results based on both the EDQNM and the MRC closure therefore indicate that there are singularities at a finite time in two-dimensional MHD.

3. Inverse cascade of magnetic potential

Since the magnetic potential is carried along like a passive scalar [equation (1.2)], its moments are conserved in the absence of magnetic diffusivity. Fyfe & Montgomery (1976), in a study of the absolute equilibrium ensemble of two-dimensional MHD, have been led to conjecture that for the variance of the magnetic potential there exists an inverse cascade similar to the inverse cascade of energy in two-dimensional non-magnetic turbulence. We now present a simple argument to support the existence of such an inverse cascade. We shall show that small-scale magnetic energy acts like a negative viscosity on large-scale magnetic fields. The analysis parallels Kraichnan's (1976) study of eddy viscosities in two and three dimensions and was in fact suggested by R. H. Kraichnan. The eddy viscosities can be obtained by sorting out various non-local effects on the EDQNM equations (2.1) and (2.2). For details see Pouquet *et al.* (1976, §3) and Kraichnan (1976). The rates of change of E_k^V and E_k^M due to interactions with wavenumbers p and/or $q > k_m \gg k$ may be expressed as

$$\partial E_k^V / \partial t = -(\nu^{Vv} + \nu^{Vm}) k^2 E_k^V, \tag{3.1}$$

$$\partial E_k^M / \partial t = -(\nu^{Mv} + \nu^{Mm}) k^2 E_k^M, \tag{3.2}$$

where
$$\left. \begin{aligned} \nu^{Vv} &= \frac{\pi}{4} \int_{k_m}^{\infty} \theta_{kpp} \frac{\partial}{\partial p} (p E_p^V) \, dp, \quad \nu^{Vm} = \frac{\pi}{4} \int_{k_m}^{\infty} \theta_{kpp} \left(3 E_p^M - p \frac{\partial E_p^M}{\partial p} \right), \\ \nu^{Mv} &= \pi \int_{k_m}^{\infty} \theta_{kpp} E_p^V \, dp, \qquad \nu^{Mm} = \pi \int_{k_m}^{\infty} \theta_{kpp} (-E_p^M) \, dp. \end{aligned} \right\} \tag{3.3}$$

The following interesting points emerge.

(i) The presence of two eddy diffusivities ν^{Mv} and ν^{Mm} acting on the magnetic field may appear surprising at first sight since in the induction equation there is only one term: $\mathbf{v} \times \mathbf{b}$. In fact, we must distinguish the eddy viscosity due directly to small-scale kinetic turbulence from that due to the kinetic turbulence generated, through the Lorentz force, by the small-scale magnetic turbulence.

(ii) The negative eddy viscosity ν^{Vv} is just the non-magnetic one (Kraichnan 1976; cf. also Krause & Rüdiger 1974). It will be negative if E_p^V decreases faster than p^{-1} at large p, but does not depend on the molecular viscosity (at high Reynolds number). The total kinetic eddy viscosity $\nu^{Vv} + \nu^{Vm}$ becomes positive as soon as the small-scale magnetic energy spectrum exceeds the kinetic spectrum by a factor ψ_0 ($= \frac{1}{9}$ for a $-\frac{3}{2}$ range). The total kinetic eddy viscosity will therefore usually be positive, allowing kinetic energy to drain to the small scales.

(iii) The viscosity ν^{Mv} (the effect of small-scale kinetic energy on large-scale magnetic fields) is positive. Krause & Rüdiger (1975) have also found such a positive eddy viscosity. The viscosity ν^{Mm} is negative and the total magnetic viscosity $\nu^{Mv} + \nu^{Mm}$ is negative as soon as the small-scale magnetic energy exceeds the small-scale kinetic energy. So the small-scale turbulence can destabilize large-scale magnetic fields in much the same way as small-scale helicity does in three-dimensional MHD (Pouquet *et al.* 1976). A negative magnetic viscosity can be obtained directly from the primitive equations by a simple phenomenological argument (Frisch 1976, private communication; see also Pouquet *et al.* 1976, §3). Let there be given initially a random homogeneous small-scale magnetic field (with current j and potential a), a deterministic large-scale magnetic field (with potential A) and no kinetic turbulence at all. Then

$$\partial \mathbf{v} / \partial t \approx j \nabla A \quad \text{hence} \quad \mathbf{v} \approx \theta j \nabla A, \tag{3.4}$$

where θ is a coherence time of the small-scale velocity field. If one introduces the flux of magnetic potential $\boldsymbol{\phi}_a = \mathbf{v}a$, where

$$\partial a/\partial t = -(\mathbf{v}.\nabla)a = -\operatorname{div}(\mathbf{v}a) = -\operatorname{div}\boldsymbol{\phi}_a, \qquad (3.5)$$

the main result is that $\boldsymbol{\phi}_a$ is in the *same* direction (not the opposite direction) as the gradient ∇A. Indeed, if $\langle . \rangle_{ss}$ denotes averaging over small scales, one obtains the following for the time evolution of the large-scale magnetic potential from the expression for $\mathbf{v}$ found above:

$$\langle\boldsymbol{\phi}_a\rangle_{ss} = \theta\nabla A\langle ja\rangle_{ss} = \theta\nabla A\langle b^2\rangle_{ss}.$$

Hence the 'negative' diffusion equation is

$$\partial A/\partial t = -\operatorname{div}\langle\boldsymbol{\phi}_a\rangle_{ss} \approx -\theta\langle b^2\rangle_{ss}\nabla^2 A. \qquad (3.6)$$

The effect of small-scale magnetic excitation on the large-scale magnetic potential A is thus to reinforce the gradient of A, thereby destabilizing the large scales of A (negative diffusion coefficient).

Remark. For simplicity, the pressure has not been included in (3.4); when this is done, an additional factor of $\frac{1}{2}$ appears in (3.6) but the proof becomes somewhat more technical. It is not worth giving a more detailed derivation since this would essentially duplicate the closure calculation.

Using the EDQNM closure, it is checked by numerical integration of (2.1) and (2.2) that there is indeed an inverse cascade. Kinetic and magnetic energy are injected into a narrow band around $k = 1$ at a constant rate. If no magnetic energy were injected, the inverse cascade could not take place since $\langle a^2\rangle$ is then not increasing. Initial spectra of kinetic and magnetic energy are given by (2.11). In this calculation, the Reynolds number is $R = 4500$ and the minimum (maximum) wavenumber is taken to be $k_{\min} = 2^{-9}$ ($k_{\max} = 2^7$). As time elapses, the total energy saturates (with a slight excess of magnetic energy) whereas the variance of the magnetic potential

$$E^a = \int_0^\infty E_k^a\,dk = \int_0^\infty k^{-2}E_k^M\,dk$$

increases linearly. Figure 3 shows the spectrum E_k^a at three different times. The power law is in good agreement with the prediction from a Kolmogorov-type dimensional analysis, namely a $-\frac{7}{3}$ exponent. The inverse cascade is quasi-stationary, a wavenumber $k_{\min}(t)$ being reached in a time proportional to $(t^3\eta)^{-\frac{1}{4}}$, where η is the injection rate of magnetic potential. Together with this inverse cascade, there is a direct cascade of energy (kinetic plus magnetic) towards the small scales with a $-\frac{3}{2}$ spectrum, just as in three-dimensional MHD. Figure 4 shows the energy flux

$$\Pi(k) = -\int_0^k (E_p^V + E_p^M)\,dp \qquad (3.7)$$

and the flux of squared magnetic potential

$$Y(k) = -\int_0^k p^{-2}E_p^M\,dp \qquad (3.8)$$

for a run in which $R = 10^5$ ($k_{\min} = 2^{-5}$ and $k_{\max} = 2^{11}$). $\Pi(k)$ is positive and constant over roughly two decades, indicating the existence of energy transfer to the small
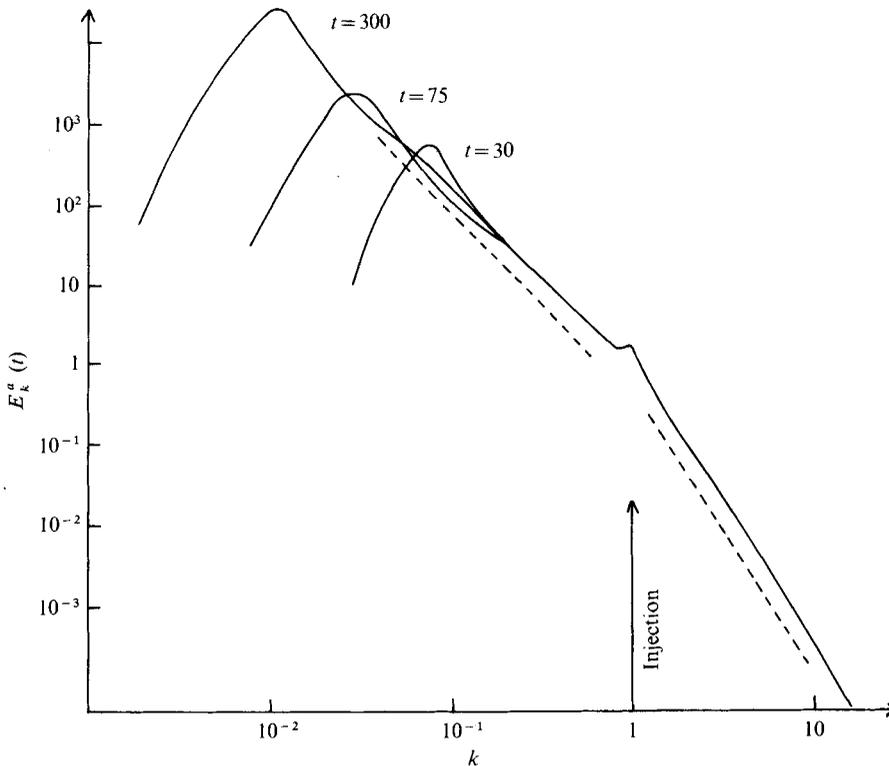
FIGURE 3. Direct energy cascade (kinetic plus magnetic) and inverse cascade of magnetic potential for three different times. Kinetic and magnetic energy are injected into a narrow band around $k = 1$. Reynolds numbers $R = 4500$.

scales. $Y(k)$ is null in this wavenumber range but is negative (and constant over one decade) at small wavenumbers, indicating the existence of an inverse cascade of magnetic potential. The existence of simultaneous transfer of energy to the small scales and squared magnetic potential to the large scales is supported by a numerical simulation of the two-dimensional MHD equations for homogeneous isotropic turbulence on a grid of 32^2 points (Fyfe *et al.* 1976), but such a calculation cannot, of course, give anything like an inertial range.

4. The two-dimensional anti-dynamo theorem

At high Reynolds number, the interactions between the velocity field and the magnetic field dominate the viscous and Joule dissipative terms. The stretching of magnetic field lines by velocity gradients allows a seed magnetic field to grow in time. However, the maintenance of a large-scale magnetic field (through nonlinear interaction with the velocity field) against dissipation is not always possible. In three-dimensional MHD turbulence, it is the small-scale helicity (the correlation between the velocity and the vorticity, i.e. the kinetic helicity, or the correlation between the magnetic field and the vector potential, i.e. the magnetic helicity) which gives rise to the α-effect (Steenbeck, Krause & Rädler 1966; Pouquet *et al.* 1976):

A. Pouquet
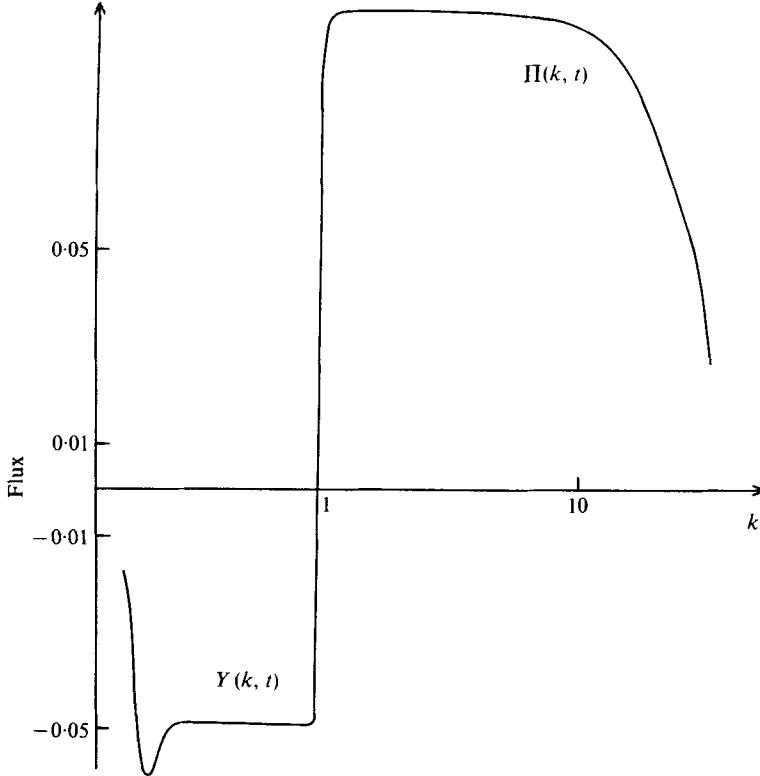
FIGURE 4. Energy flux $\Pi(k, t)$ and flux of squared magnetic
potential $Y(k, t)$. Same conditions as in figure 2.

when kinetic energy and kinetic helicity are injected, there is an inverse cascade of
magnetic helicity leading to the appearance of magnetic excitation in ever-increasing
scales (limited by the size of the system). In two-dimensional MHD, the kinetic and
magnetic helicities are equal to zero and one may ask what is the ultimate fate of a
seed field. It is known that, in the presence of Joule dissipation, it will eventually die
out (Lortz 1968; Vainshtein & Zeldovich 1972). However, one must realize that it
may take a very long time (compared with a typical eddy turnover time) to decay.
Indeed, because of the conservation of the magnetic potential by the nonlinear terms,
we have, starting from (1.2) and using homogeneity,

$$\tfrac{1}{2}d\langle a^2 \rangle / dt = \lambda \langle a \nabla^2 a \rangle = -\lambda \langle (\mathrm{curl}\, a e_3)^2 \rangle = -\lambda \langle b^2 \rangle. \qquad (4.1)$$

Integrating from 0 to T, we have

$$\int_0^T \langle b^2(t) \rangle \, dt \leqslant \frac{\langle a^2(0) \rangle}{2\lambda}. \qquad (4.2)$$

Therefore, if $\lim_{t \to \infty} \langle b^2(t) \rangle$ exists, it is necessarily zero for any positive magnetic diffusivity.
Notice that this result holds whether or not kinetic energy is injected (since it makes
use of only the equation for the magnetic field) but that it does not tell us anything
about the rate of decay of magnetic energy. An identical argument can be applied to the
EDQNM closure, which has the same conservation laws as the primitive equations.
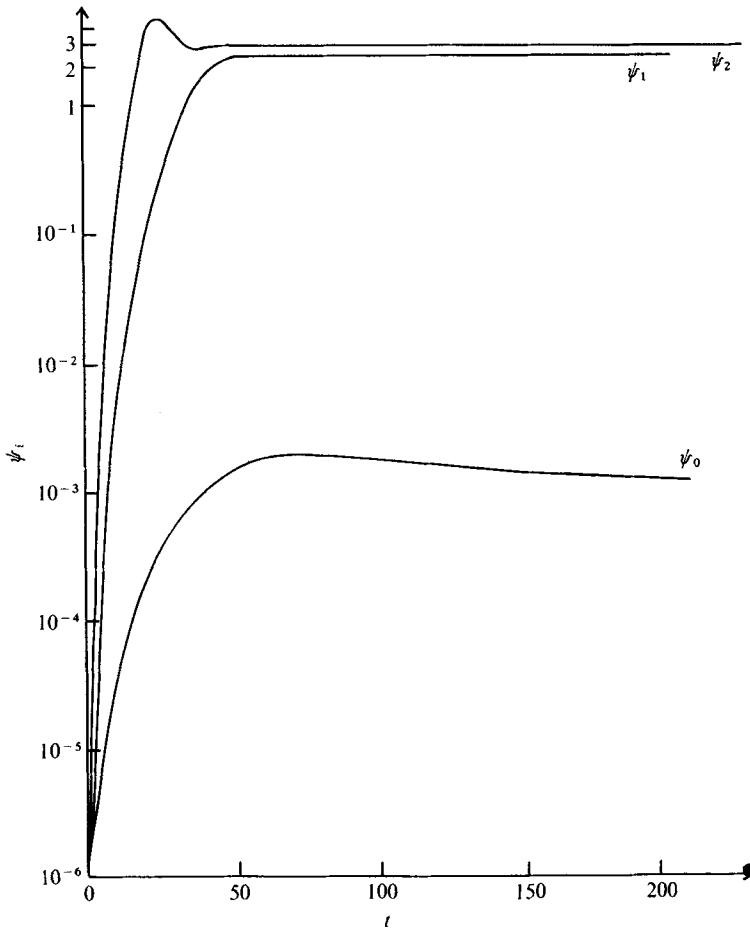
FIGURE 5. Growth of a seed field: ratios of magnetic and kinetic energies (ψ_0), enstrophies (ψ_1) and palinstrophies (ψ_2). Initially all ψ's are equal to 10^{-6}. Reynolds number $R = 8 \times 10^6$. Notice that ψ_1 and ψ_2 saturate at values of order one. Eventually the magnetic energy should decay (see figure 6).

A numerical calculation on the EDQNM closure indicates the following. A seed magnetic energy is first amplified by the velocity field as also observed by Weiss (1966) and Moss (1970). This can be seen in figure 5, which shows the time evolution of the ratios of the magnetic and kinetic energies (ψ_0), enstrophies (ψ_1) and palin-strophies (ψ_2), defined by

$$\psi_i(t) = \frac{\displaystyle\int_0^\infty k^{2i} E^M(k, t) \, dk}{\displaystyle\int_0^\infty k^{2i} E^V(k, t) \, dk}, \quad i = 0, 1, 2. \tag{4.3}$$

The initial kinetic energy spectrum is given by (2.11) and kinetic energy only is injected into a narrow band around $k = 1$ at a constant rate. The minimum and maximum wavenumbers are respectively 2^{-3} and 2^{13}. Initially, $\psi_i(t = 0) = 10^{-6}$ for $i = 0, 1$ and 2 and the Reynolds number is $R = 8 \times 10^6$. The magnetic energy grows rapidly (in a few large-eddy turnover times) by three orders of magnitude, ψ_0 saturating

at around 10^{-3}. But there is an excess of magnetic excitation in the small scales: ψ_1 and ψ_2 saturate at values slightly greater than unity. Figure 6 shows that the length of the plateau of magnetic energy increases (linearly?) with the Reynolds number. Only after this plateau does the magnetic energy decay and the total kinetic energy starts to grow, probably because the problem eventually becomes non-magnetic and an inverse energy cascade becomes possible. All this suggests that the two-dimensional anti-dynamo result is not uniform in λ, and we conjecture that in the limit of zero magnetic diffusivity a stationary state is obtained with non-vanishing magnetic energy.

5. Summary and comparison with direct numerical simulations

It has been shown that two-dimensional MHD turbulence differs basically from two-dimensional non-magnetic turbulence. Because of the relaxation of the vorticity constraint, the appearance of singularities at a finite time, as in three-dimensional turbulence, cannot be ruled out (at zero viscosity and zero magnetic diffusivity). Using a second-order closure which has all the required conservation laws and which is integrated numerically at very high kinetic and magnetic Reynolds numbers (3×10^7), a very sharp increase in the kinetic and magnetic enstrophies is indeed obtained at a finite time of the order of a few large-eddy turnover times; this is accompanied by a sudden onset of dissipation of energy (figure 1). Upon injection of kinetic and magnetic energy, a quasi-stationary state is obtained with a direct cascade of energy to small scales (as in three-dimensional turbulence) together with an inverse cascade of the variance of the magnetic potential (figure 3); the latter result supports a conjecture of Fyfe & Montgomery (1976). The inverse cascade can be linked to a negative magnetic diffusivity whereby the small-scale magnetic energy destabilizes the large-scale magnetic excitation. If only kinetic energy is injected, magnetic energy can be sustained against Joule dissipation for a time which increases with the magnetic Reynolds number (figure 6).

After a preliminary version of this paper had been written, Orszag & Tang (1978) carried out a direct numerical simulation of two-dimensional MHD using 256^2 modes. They found a rather sharp increase in the enstrophy at high Reynolds numbers, consistent with a singularity at a finite time when $\nu = \lambda = 0$. However, contrary to what is found with the present closure, the enstrophy dissipation rate did not seem to reach a finite non-zero value as ν, $\lambda \to 0$. Orszag & Tang therefore conclude that two-dimensional MHD turbulence is in a sense intermediate between two- and three-dimensional non-magnetic turbulence. In two-dimensional MHD, the enstrophy conservation law of non-magnetic turbulence is 'broken'. It may be however that it is only 'weakly broken': enough to allow singularities but not enough to allow an energy cascade to small scales (Fournier & Frisch 1978). If that is the case, an inverse energy cascade similar to the non-magnetic cascade should be possible. This could be tested numerically by feeding energy at intermediate wavenumbers through prescribed random forces.

If there are indeed singularities in two-dimensional MHD then it will be of interest to make numerical calculations for problems not accessible by traditional closures. An example is intermittency: in the light of several recent papers (Kraichnan 1974; Mandelbrot 1975; Nelkin 1975; Frisch et al. 1978), it appears that closure techniques
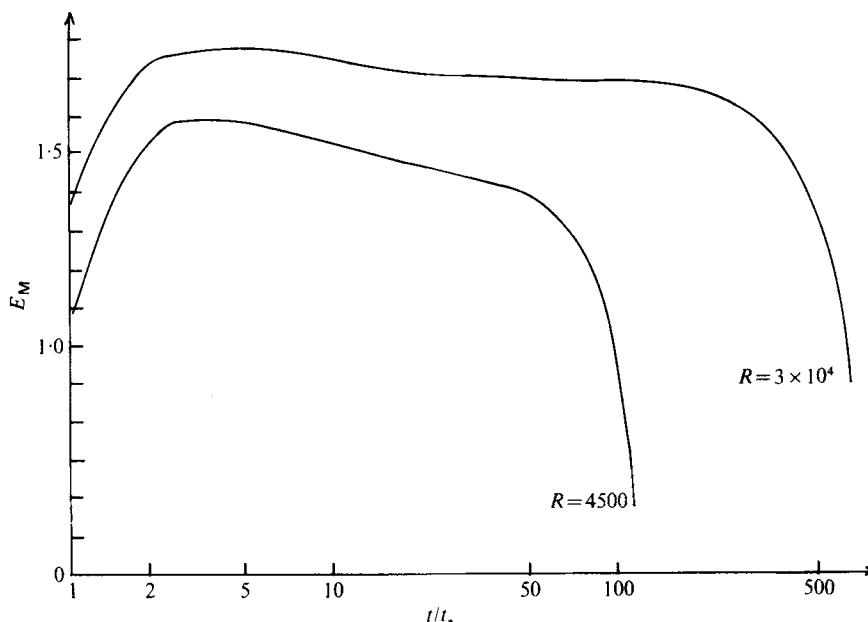
FIGURE 6. Maintenance of magnetic energy against Joule dissipation with only kinetic forcing near $k = 1$. The time evolution of magnetic energy is shown for $R = 4500$ and $R = 3 \times 10^4$. Notice that the magnetic energy plateau persists for times very long compared with the large-eddy turnover time and increases in length with increasing Reynolds number.

are probably very badly adapted to the small scales in which important deviations from the Kolmogorov 1941 theory are observed (Gibson, Stegen & McConnell 1970; Van Atta & Park 1972). Small-scale motions seem to fill a smaller and smaller fraction of the turbulent volume (Kuo & Corrsin 1971). There is also some observational evidence from the solar photosphere that MHD intermittency is much stronger than ordinary intermittency (Stenflo 1977; Stenflo & Lindegren 1977). The phenomenon of (internal) intermittency is at present one of the major problems in the statistical theory of turbulence. For non-magnetic three-dimensional turbulence, a numerical study of intermittency is well beyond reach at the moment because of the limitation on the Reynolds number (the largest simulation so far uses a grid with 128^3 points).

In *non-magnetic* two-dimensional turbulence, some intermittency in the small scales has been observed (Herring *et al.* 1974). According to Kraichnan (1975) however, this intermittency should be very different from three-dimensional intermittency and probably more akin to the intermittency of a passive scalar advected by prescribed large-scale velocity gradients.

For two-dimensional MHD turbulence, the numerical results of Orszag & Tang (1978) show considerable intermittency with strong localized vorticities and magnetic fields. We believe, therefore, that this problem offers a very good testing ground for theories of fully developed turbulence. It would be useful to extend the existing calculation to 512^2 and possibly 1024^2 modes to look for some sort of self-similar structure as suggested by phenomenological theories (Mandelbrot 1975; Frisch *et al.* 1978).

Appendix

Bounds on the inertial-range exponent of two- and three-dimensional MHD turbulence in the absence of boundaries are obtained, closely following a method of Sulem & Frisch (1975). In the case of finite energy turbulence, integrals can be taken instead of expectation values, so that the results are not of a statistical nature. The method consists essentially of the following. The space to which the solution belongs (for example the space of functions of finite total energy) is decomposed into orthogonal subspaces denoted by $\mathscr{S}_n$. More precisely, the unknown fields are written as a sum of functions whose Fourier transforms have their support in spherical shells S_n of exponentially increasing radii. The MHD equations are written in a form which is suitable in three as well as in two dimensions, namely

$$\partial \mathbf{v}/\partial t + (\mathbf{v}.\nabla)\,\mathbf{v} = -\nabla p + (\mathbf{b}.\nabla)\,\mathbf{b} + \nu\nabla^2\mathbf{v}, \tag{A 1}$$

$$\partial \mathbf{b}/\partial t + (\mathbf{v}.\nabla)\,\mathbf{b} = (\mathbf{b}.\nabla)\,\mathbf{v} + \lambda\nabla^2\mathbf{b}, \tag{A 2}$$

$$\nabla.\mathbf{v} = 0, \quad \nabla.\mathbf{b} = 0. \tag{A 3}$$

The negative rate of change of the total (kinetic and magnetic) energy in the first L shells S_n ($n = 1, ..., L$) is made up of two parts: (i) a viscous and Joule dissipation part which for finite L converges to zero as $\nu \to 0$ and $\lambda \to 0$ and (ii) a nonlinear contribution which is the energy flux through the wavenumber k_L, i.e.

$$\Pi(k_L) = \sum_{l=0}^{L} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{lmn}, \tag{A 4}$$

where

$$c_{lmn} = c_{lmn}^{vvv} - c_{lmn}^{vbb} + c_{lmn}^{bvb} - c_{lmn}^{bbv}, \tag{A 5}$$

$$c_{lmn}^{vvv} = (\mathbf{v}_l, (\mathbf{v}_m.\nabla)\,\mathbf{v}_n), \quad c_{lmn}^{vbb} = (\mathbf{v}_l, (\mathbf{b}_m.\nabla)\,\mathbf{b}_n), \tag{A 6}, (A 7)$$

$$c_{lmn}^{bvb} = (\mathbf{b}_l, (\mathbf{v}_m.\nabla)\,\mathbf{b}_n), \quad c_{lmn}^{bbv} = (\mathbf{b}_l, (\mathbf{b}_m.\nabla)\,\mathbf{v}_n), \tag{A 8}, (A 9)$$

(,) denotes a scalar product (in $L^2(\mathbb{R}^d)$) and u_l is the orthogonal projection of the field u on the functional space $\mathscr{S}_l$ (see Sulem & Frisch for a precise definition). If only c_{lmn}^{vvv} is non-zero, one recovers the non-magnetic case (in Sulem & Frisch, c_{lmn}^{vvv} is denoted by b_{lmn}). It is easily verified that c_{lmn} is skew symmetric ($c_{lmn} = -c_{nml}$). Furthermore we have in d dimensions

$$|c_{lmn}^{\alpha\beta\gamma}| \leqslant \begin{cases} C^{st}k_n\,k_{\mathrm{Inf}(m,n)}^{\frac{1}{2}d}\,(E_l^\alpha E_m^\beta E_n^\gamma)^{\frac{1}{2}} & \text{for} \quad l > n, \\ C^{st}k_l^{1+\frac{1}{2}d}(E_l^\alpha E_m^\beta E_n^\gamma)^{\frac{1}{2}} & \text{for} \quad l < n, \end{cases} \tag{A 10}$$

where α, β and γ are either v or b in order that the coefficients previously defined in (A 6)–(A 9) are recovered and where E_l^V (E_l^M) is the kinetic (magnetic) energy contained in shell S_l. If we now assume power-law bounds

$$E_l^V \leqslant Ck_l^{-\delta}, \quad E_l^M \leqslant Ck_l^{-\delta}, \tag{A 11}$$

we find as in Sulem & Frisch (1975) that

$$\lim_{L \to \infty} \Pi(k_L) = 0 \tag{A 12}$$

provided that $$\delta > \begin{cases} \frac{5}{3} & \text{in three dimensions,} \end{cases} \tag{A 13}$$
$$\begin{cases} \frac{4}{3} & \text{in two dimensions.}† \end{cases} \tag{A 14}$$

In other words, the energy spectra defined by ($\hat{v}$ and $\hat{b}$, Fourier transforms of v and b)

$$E^V(k) = k^{d-1}|\hat{v}(k,t)|^2, \quad E^M(k) = k^{d-1}|\hat{b}(k,t)|^2$$

cannot be steeper than $k^{-\frac{8}{3}}$ in three dimensions and $k^{-\frac{7}{3}}$ in two dimensions if the energy flux is to have a finite, non-zero limit.

REFERENCES

ALEMANY, A., MOREAU, R., SULEM, P. L. & FRISCH, U. 1978 Influence of an external magnetic field on homogeneous MHD turbulence. *J. Méc.* (in press).

BATCHELOR, G. K. 1969 *Phys. Fluids Suppl.* **12**, II 233.

BRISSAUD, A., FRISCH, U., LÉORAT, J., LESIEUR, M., MAZURE, A., POUQUET, A., SADOURNY, R. & SULEM, P. L. 1973 *Ann. Géophys.* **29**, 539.

DESBOIS, M. 1975 *J. Atmos. Sci.* **32**, 1838.

FOURNIER, J. D. & FRISCH, U. 1978 *Phys. Rev.* A **17**, 747.

FRISCH, U. & BARDOS, C. 1975 Global regularity of the two dimensional Euler equation for an ideal incompressible fluid. Unpublished manuscript, Observatoire de Nice.

FRISCH, U., LESIEUR, M. & BRISSAUD, A. 1974 *J. Fluid Mech.* **65**, 145.

FRISCH, U., SULEM, P. L. & NELKIN, M. 1978 *J. Fluid Mech.* **87**, 719.

FYFE, D. & MONTGOMERY, D. 1976 *J. Plasma Phys.* **16**, 181.

FYFE, D., MONTGOMERY, D. & JOYCE, G. 1977 *J. Plasma Phys.* **17**, 369.

GIBSON, C. H., STEGEN, G. R. & McCONNELL, S. 1970 *Phys. Fluids* **13**, 2448.

HERRING, J. R., ORSZAG, S. A., KRAICHNAN, R. H. & FOX, D. G. 1974 *J. Fluid Mech.* **66**, 417.

KOLESNIKOV, Y. B. & TSINOBER, A. B. 1972 *Magnitaya Gidrodinamica* **3**, 23.

KOLMOGOROV, A. N. 1962 *J. Fluid Mech.* **12**, 82.

KRAICHNAN, R. H. 1966 *Phys. Fluids* **9**, 1728.

KRAICHNAN, R. H. 1974 *J. Fluid Mech.* **64**, 737.

KRAICHNAN, R. H. 1975 *J. Fluid Mech.* **67**, 15.

KRAICHNAN, R. H. 1976 *J. Atmos. Sci.* **33**, 1521.

KRAUSE, F. & RÜDIGER, G. 1974 *Astron. Nachr.* **295**, 185.

KRAUSE, F. & RÜDIGER, G. 1975 *Solar Phys.* **42**, 107.

KUO, A.-Y. & CORRSIN, S. 1971 *J. Fluid Mech.* **50**, 285.

LEE, T. D. 1952 *Quart. Appl. Math.* **10**, 69.

LÉORAT, J. 1975 Thèse d'Etat, Université Paris VII.

LORTZ, D. 1968 *Phys. Fluids* **11**, 913.

MANDELBROT, B. B. 1975 *Les Objets Fractals: Forme, Hasard et Dimension.* Paris: Flammarion. (English edition: *Fractals: Form, Chance and Dimension.* San Francisco: W. H. Freeman & Comp. Publ., 1977.)

† There is a slight mistake in Sulem & Frisch concerning the bound on the enstrophy range of two-dimensional (non-magnetic) turbulence: equation (8) of their paper should read

$$|\Pi_{\text{enst}}(k_L)| \leqslant C^{st}\left\{ E_L^{\frac{1}{2}}\,\Omega_{L+1}^{\frac{1}{2}}\left(\sum_{n=0}^{L} k_n^2\,\Omega_n^{\frac{1}{2}}\right) + k_L\,\Omega_L^{\frac{1}{2}}\,\Omega_{L+1}^{\frac{1}{2}}\left(\sum_{m=0}^{L-1} k_m E_m^{\frac{1}{2}}\right) \right.$$
$$\left. + \left(\sum_{l=0}^{L} k_L^2\,\Omega_L^{\frac{1}{2}}\right)\left[E_{L+1}^{\frac{1}{2}}\,\Omega_{L+1}^{\frac{1}{2}} + E_{L+1}^{\frac{1}{2}}\,\Omega_{L+2}^{\frac{1}{2}} + \sum_{m=L+2}^{\infty} E_m^{\frac{1}{2}}(\Omega_{m-1}^{\frac{1}{2}} + \Omega_m^{\frac{1}{2}} + \Omega_{m+1}^{\frac{1}{2}})\right]\right\}. \tag{A 15}$$

A correct estimate yields the following result: in the enstrophy inertial range, if it exists, the energy spectrum cannot be steeper than $k^{-\frac{11}{3}}$. This $-\frac{11}{3}$ result can also be obtained by a dimensional argument similar to the argument for three dimensions (Sulem & Frisch 1975).

MOFFATT, H. K. 1967 *J. Fluid Mech.* **28**, 571.

MOSS, D. J. 1970 *Mon. Not. Roy. Astr. Soc.* **148**, 173.

NELKIN, M. 1975 *Phys. Rev.* **A11**, 1737.

ORSZAG, S. A. 1977 *Statistical Theories of Turbulence*, 1973 *Les Houches Summer School of Physics* (ed. R. Balian & J. L. Peube), p. 235. Gordon & Breach.

ORSZAG, S. A. & TANG, C. M. 1978 Small-scale structure of two-dimensional magnetohydrodynamic turbulence. Submitted to *J. Fluid Mech.*

POUQUET, A., FRISCH, J. & LÉORAT, J. 1976 *J. Fluid Mech.* **77**, 321.

POUQUET, A., LESIEUR, M., ANDRÉ, J. C. & BASDEVANT, C. 1975 *J. Fluid Mech.* **72**, 305.

ROSE, H. & SULEM, P. L. 1978 *J. Phys.* **39**, 441 (Paris).

STEENBECK, M., KRAUSE, F. & RÄDLER, K. H. 1966 *Z. Naturforsch.* **21a**, 369.

STENFLO, J. O. 1977 *IAU Coll.* no. 36 (ed. R. M. Bonnet & Ph. Delache), p. 143. Clermont-Ferrand: G. de Bussac.

STENFLO, J. O. & LINDEGREN, L. 1977 *Astron. Astrophys.* **59**, 367.

SULEM, C. 1977 *C.R. Acad. Sci. Paris* A **285**, 365.

SULEM, P. L. & FRISCH, U. 1975 *J. Fluid Mech.* **72**, 417.

TAPPERT, F. & HARDIN, R. 1971 Film: *Computer Simulated MHD Turbulence*. Bell Labs.

VAINSHTEIN, S. I. & ZELDOVICH, Y. B. 1972 *Geomagn. Aeron.* **13**, 123.

VAN ATTA, C. W. & PARK, J. 1972 In *Statistical Models of Turbulence* (ed. M. Rosenblatt & C. W. Van Atta), pp. 402–426. Springer.

WEISS, N. O. 1966 *Proc. Roy. Soc.* **A293**, 310.

WOLIBNER, W. 1933 *Math. Z.* **37**, 668.