On steady laminar flow with closed streamlines at large Reynolds number

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SUMMARY

Frictionless flows with finite vorticity are usually made determinate by the imposition of boundary conditions specifying the distribution of vorticity 'at infinity'. No such boundary conditions are available in the case of flows with closed streamlines, and the velocity distributions in regions where viscous forces are small (the Reynolds number of the flow being assumed large) cannot be made determinate by considerations of the fluid as inviscid. It is shown that if the motion is to be exactly steady there is an integral condition, arising from the existence of viscous forces, which must be satisfied by the vorticity distribution no matter how small the viscosity may be. This condition states that the contribution from viscous forces to the rate of change of circulation round any streamline must be identically zero. (In cases in which the vortex lines are also closed, there is a similar condition concerning the circulation round vortex lines.)

The inviscid flow equations are then combined with this integral condition in cases for which typical streamlines lie entirely in the region of small viscous forces. In two-dimensional closed flows, the vorticity is found to be uniform in a connected region of small viscous forces, with a value which remains to be determined—as is done explicitly in one simple case—by the condition that the viscous boundary layer surrounding this region must also be in steady motion. Analogous results are obtained for rotationally symmetric flows without azimuthal swirl, and for a certain class of flows with swirl having no interior boundary to the streamlines in an axial plane, the latter case requiring use of the fact that the vortex lines are also closed. In all these cases, the results are such that the Bernoulli constant, or 'total head', varies linearly with the appropriate stream function, and the effect of viscosity on the rate of change of vorticity at any point vanishes identically.

1. GENERAL REMARKS

The work described herein concerns the steady laminar motion of fluids of small viscosity, and is based on the generally accepted premise that, when the Reynolds number of a flow field is very large, viscous forces acting on the fluid are small everywhere, except perhaps in the neighbourhood of certain surfaces in the fluid. 'Small viscous forces' here means forces that are small compared with unity when made non-dimensional using a length and a velocity typical of the flow as a whole (those used in the definition of Reynolds number would be suitable). It will usually happen that pressure forces on the fluid are of order unity when made non-dimensional in this same way, and the above premise is then equivalent to the statement that viscous forces are small compared with pressure forces nearly everywhere.

It is well known, partly as a matter of observation and partly from mathematical analysis, that, for certain steady flow fields amenable to study, the above premise is undoubtedly true. It seems that, if the Reynolds number of the flow is allowed to approach infinity, without any other change in the conditions of these flow fields, the region of the fluid in which viscous forces are not small becomes smaller and smaller, and ultimately reduces, at most, to a number of thin layers, usually in the form of boundary layers and wakes. In the limit of infinite Reynolds number, the region in which viscous forces are not small either disappears altogether or becomes a number of stream-surfaces (i.e. surfaces whose tangent planes everywhere contain the local velocity vector), which usually coincide with, or are connected with, rigid boundaries in the fluid. Across such singular stream-surfaces there may be a discontinuity in velocity, as for instance at a rigid boundary where the presence of a singular stream-surface ensures that the no-slip condition is satisfied even in the limit of zero viscosity. In what follows, the above-mentioned premise will simply be accepted as valid generally.

We shall consider those steady flows that take place in a confined region, the motion of the fluid being generated by steady tangential motion of the surrounding boundaries (which need not all be rigid). It will be assumed that the Reynolds number is large enough for the thicknesses of the associated viscous layers to be small compared with the linear dimensions of the region of fluid under consideration. It will also be assumed that the motion of the fluid is laminar, despite the high Reynolds number (which clearly will correspond with reality only if the velocity distribution happens to have strong inherent stability). Most of the flows of this type that are capable of practical reproduction involve plane rigid boundaries moving in their own planes, or rigid surfaces of revolution rotating about their axes. Cases in which part of the boundary is rigid and stationary, the remaining part of the boundary of the region of closed streamlines being a 'free boundary layer' —for example, the motion in a cavity opening off a plane wall over which fluid is streaming—are of interest in a wide range of practical problems in aerodynamics and hydraulics. Examples of steady flow in a closed region are not common, but they perhaps occur often enough in practice to warrant notice of their peculiar features.

The equations governing the steady laminar motion of a uniform incompressible fluid are

$$\nabla . \mathbf{u} = 0, \tag{1.1}$$

$$\mathbf{u} \times \boldsymbol{\omega} - \nabla(p/\rho + \tfrac{1}{2}q^2) + \nu\nabla^2\mathbf{u} = \partial\mathbf{u}/\partial t = 0, \tag{1.2}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, $q = |\mathbf{u}|$, and the other symbols have their usual meanings. When the Reynolds number of the motion is large, viscous forces, according to our premise, are small everywhere except in the neighbourhood of certain singular surfaces, and equation (1.2) reduces approximately to

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla H, \tag{1.3}$$

where $H = p/\rho + \frac{1}{2}q^2$ is the local 'total head' in the fluid. H is here constant over stream-vortex surfaces, or 'Bernoulli surfaces', each such surface containing the local vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ in its tangent plane everywhere and being swept out by a material vortex line. When $\nu \neq 0$, stream-vortex surfaces will not exist, in general, because $\mathbf{u} \times \boldsymbol{\omega}$ is then (see (1.2)) not necessarily parallel everywhere to the gradient of some scalar function.

It is well known that equation (1.3), which is approximately valid everywhere except in the neighbourhood of the singular surfaces, is not sufficient to allow the velocity distribution to be determined from the condition of zero normal velocity at specified boundaries in the field. In the case of two-dimensional steady motion, the indeterminacy takes a form such that the vorticity—which is constant along a streamline—may vary arbitrarily from one streamline to another. It often happens that the inviscid flow equations can be made sufficient to determine $\mathbf{u}$ with the help of additional information, for instance about the variation of $\boldsymbol{\omega}$ from one streamline to another far upstream (most often in the form of a statement that the velocity is uniform at infinity).

However, in the case of flow in a confined region, for which all the streamlines are necessarily closed, the possibility of making equation (1.3) sufficient to determine $\mathbf{u}$ in the region of small viscous forces, by introducing boundary conditions which specify $\boldsymbol{\omega}$ everywhere in a region 'far upstream', no longer exists. Other means of making the velocity distribution determinate must be found, and it is apparent that there is no further information to be found from considerations of the fluid as inviscid; this is the feature that makes the study of flow with closed streamlines novel and interesting. It will be shown that the action of viscosity imposes certainly one, and, when the vortex lines also are closed, two, integral conditions on the distribution of $\boldsymbol{\omega}$, and that in the cases of two-dimensional flow and of rotationally symmetric flow (with suitable restrictions) these integral conditions render the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ determinate in the region of small viscous forces. The question of what further conditions are needed to make the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ determinate in general three-dimensional flow with closed streamlines is left unresolved. We begin with a derivation of the integral conditions in the general three-dimensional case.

2. Integral conditions arising from the effect of viscosity

To obtain a condition which arises from the effect of viscosity and which is valid no matter how small the value of ν may be, we operate on the complete equation of motion (1.2) in such a way that the contributions from all terms other than the term containing ν vanish identically. Such an

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operation is to take the line integral around a closed contour (in space) of certain shape, with line element $d\mathbf{l}$, giving

$$\oint (\mathbf{u}\times\boldsymbol{\omega}).d\mathbf{l} - \oint d\mathbf{l}.\nabla H - \nu\oint (\nabla\times\boldsymbol{\omega}).d\mathbf{l} = \frac{\partial}{\partial t}\oint \mathbf{u}.d\mathbf{l}. \tag{2.1}$$

The term on the right-hand side vanishes in view of the steadiness of the motion. (It is important to notice that the motion is regarded as exactly steady, even though the value of ν will later be taken as very small and even though steady motion would take a long time to develop from arbitrary initial conditions. In other words, we are considering flows subject to the double limiting operation $t \to \infty$, $\nu \to 0$, *in that order*, corresponding in effect, to the procedure that would be used in a real observation of the forms of the steady flows set up at several different large values of the Reynolds number.) The second term on the left-hand side vanishes since H is a single-valued function of position. We now make the first term on the left-hand side zero by choosing the closed contour to coincide with a streamline, of which a line element will be donated by $d\mathbf{s}$. Then, since ν is non-zero (although it will later be assumed to be small), we have the exact integral condition

$$\oint (\nabla\times\boldsymbol{\omega}).d\mathbf{s} = 0, \tag{2.2}$$

to be satisfied for every closed streamline.

It is natural to enquire if there are any other choices of the closed contour for which the integral $\oint (\mathbf{u}\times\boldsymbol{\omega}).d\mathbf{l}$ vanishes identically. If there existed a family of surfaces such that their normals were everywhere parallel to $\mathbf{u}\times\boldsymbol{\omega}$, we could make the integral zero by choosing the contour as any closed curve on one of these surfaces. However, as already remarked, such a family of surfaces does not exist in general; the surface formed by all the streamlines passing through a given vortex line will in general be intersected by other vortex lines (except when $\nu = 0$, which is irrelevant, since we are seeking an integral condition which is exact for small but non-zero values of ν). For reasons related to the symmetry, surfaces which everywhere contain the local vectors $\mathbf{u}$ and $\boldsymbol{\omega}$ exist in cases of two-dimensional motion and of rotationally symmetric motion without azimuthal swirl when $\nu \neq 0$, but we shall see that the condition (2.2) alone is then sufficient to make the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ determinate and no other choice of closed contour is needed.

There exists the possibility that vortex lines are closed and that the integral $\oint (\mathbf{u}\times\boldsymbol{\omega}).d\mathbf{l}$ may be made zero by choosing the contour to coincide with a closed vortex line. Vortex lines may end at a rigid boundary, but there is no reason why at least some of the vortex lines in a confined flow should not be closed; for such lines we have the additional exact integral condition

$$\oint (\nabla\times\boldsymbol{\omega}).d\mathbf{v} = 0, \tag{2.3}$$

where $d\mathbf{v}$ is an element of a vortex line. It appears to be difficult to decide whether vortex lines are, or are not, closed in any given flow field, but in one case, described in § 4, this is possible and the condition (2.3) is utilized.

Other contours which are closed, and which are such that each line element is orthogonal to $\mathbf{u} \times \boldsymbol{\omega}$, may be imagined (one natural choice is a contour which consists of four segments, of which the first and third coincide with streamlines, and the second and fourth with vortex lines); however, their existence depends on the particular properties of the flow, and in any case it is not clear how the corresponding integral condition could be made use of.

The integral condition (2.2) (and likewise (2.3), where it is applicable) is valid independently of the Reynolds number of the flow. If now the Reynolds number is assumed to be large, the equation of inviscid flow, (1.3), is approximately valid over nearly all the flow field, and there exists the possibility of making use of *both* equation (1.3) and the condition (2.2). Provided the streamline, around which the integral in (2.2) is taken, lies entirely in the region of small viscous forces, the integrand in (2.2) may be evaluated with the aid of (1.3). It will be shown that, in this way, the form of the velocity distribution in the region of small viscous forces may be determined in certain cases.

The proviso of the penultimate sentence is equivalent to the requirement that the shortest distance from the streamline to any singular surface does not tend to zero as $\nu \to 0$. Now the velocity in the region of small viscous forces will in general be of the same order of magnitude as the tangential velocity of the boundaries (as can be verified experimentally in certain simple cases, and can in any case be examined *a posteriori*). Consequently, in cases of two-dimensional flow and of rotationally symmetric flow without azimuthal swirl, a typical streamline passing through the region of small viscous forces, on which the value of the appropriate stream function ψ differs from that for the outer enclosing boundary or singular surface by a finite amount, cannot come close to the boundary of this region without the velocity there being infinite. The above proviso is therefore satisfied for typical streamlines in these two cases. However, in more general types of flow, it is not certain that streamlines lying entirely in the region of small viscous forces exist; indeed, there are some closed flows for which *all* the closed streamlines pass through a boundary layer region.

3. Steady two-dimensional flow with closed streamlines

When the flow is two-dimensional, we can introduce the stream function ψ, and use (ψ, ξ) as orthogonal curvilinear coordinates, the lines ξ = const. being everywhere normal to the streamlines. The displacements corresponding to increments in ψ and ξ are $d\psi/q$ and $hd\xi$, where h is an unknown function of ψ and ξ. The inviscid flow equation (1.3) can then be written as

$$\omega(\psi) = \frac{dH(\psi)}{d\psi}. \tag{3.1}$$

This equation will be approximately valid, when the Reynolds number is large, everywhere except in the neighbourhood of certain singular curves (in the plane of motion) which are themselves members of the family of streamlines. Even when the shapes of outer and inner streamlines (perhaps given by the shape of enclosing rigid boundaries) bounding a region in which (3.1) holds is given, there will in general be a solution of this equation for any choice of function $\omega(\psi)$, and the flow in the region of small viscous forces can be made determinate only with the aid of further conditions. The considerations given in § 2 supply the integral condition (2.2), which we proceed to apply.

When evaluating the integral in (2.2) for streamlines lying wholly in the region of small viscous forces, we can make use of the approximation (3.1), whence $\nabla \times \boldsymbol{\omega}$ becomes a vector parallel to the local streamline, and (2.2) takes the form

$$\frac{d\omega}{d\psi}\oint q\,ds = 0. \tag{3.2}$$

Hence
$$\omega(\psi) = \frac{dH(\psi)}{d\psi} = \omega_0(\text{const.}) \tag{3.3}$$

everywhere in a connected region of small viscous forces. (The possibility of there being an exception to (3.3) on a streamline for which $\oint q\,ds = 0$ —which is possible only if the velocity is zero at all points on the streamline —with a possible discontinuity in ω across such a streamline, is irrelevant since the viscous forces would then be large at such a streamline). The distribution of velocity in the region of small viscous forces can be determined from (3.3) without difficulty when the shape of the streamline bounding this region is known*.

The argument leading to the simple result (3.3) may be summarized as follows. In view of the fact that ν is small, convection of vorticity will dominate viscous diffusion of vorticity when both processes occur, so that ω is approximately constant *along* streamlines. But, in exactly steady motion, the net viscous diffusion of vorticity across a closed streamline must be exactly zero (even when the streamline encloses a rigid boundary), and this is then possible only if ω is also approximately constant *across* streamlines. It seems that two-dimensional flows with closed streamlines cannot be exactly steady until the slow but persistent effect of viscous diffusion of vorticity across streamlines has evened out any variation of vorticity that may have been present initially; the time required for this asymptotic steady state to be set up will of course increase as ν decreases.

* The notion of a two-dimensional ' inviscid ' core with uniform vorticity has already been inferred from arguments rather less general or rigorous than those given above, and has been applied to some problems of two-dimensional free convection in closed regions. Pillow (in a dissertation submitted for the degree of Ph.D. at the University of Cambridge, 1952) and Batchelor (1954) have used it to calculate the heat transfer across rectangular cavities, and Carrier (1953) has done so for a circular cavity. The temperature in the core was shown to be uniform in all these cases from an argument based on symmetry of the streamlines, but in fact this result is true generally in a simply-connected ' non-conducting ' closed region in two dimensions, as may be seen from a proof like that used above.

The value of the constant ω_0 is undetermined as yet, and since the distribution (3.3) is such that $\nu\nabla^2\mathbf{u}$ is identically zero there is no further information to be gained from considerations of the region in which viscous forces are small. Surrounding the region in which (3.3) is valid there is a singular streamline, in the neighbourhood of which viscous forces are not small, and the value of ω_0 will presumably be determined by the need for steady motion to be possible near this singular streamline. The situation can be illustrated by reference to the simple case of flow inside a circular cylinder of radius a which rotates steadily with angular velocity $\frac{1}{2}\omega_1$, an inner sleeve of length $2\pi a\sigma$ being held stationary (figure 1). It is evident here that, when the Reynolds number is large, viscous forces will be small everywhere except near the outer circular boundary, so that the region of 'inviscid' flow is circular and in it the fluid rotates as a rigid body with angular velocity $\frac{1}{2}\omega_0$. The motion in the 'inviscid' core can be regarded as a standing eddy, which is driven by the motion of part of the outer boundary, the exact speed of rotation of the eddy being determined by the need for steadiness in the viscous boundary layer surrounding the eddy.

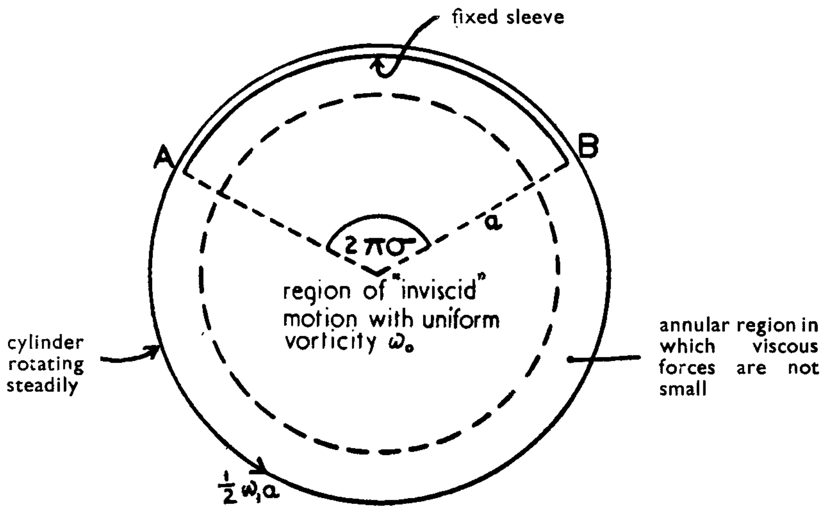

Figure 1. A case of two-dimensional flow in a closed region.

The present paper is concerned with general results rather than solutions for particular configurations of the boundaries, but it may be found illuminating to present some consideration of the boundary layer problem associated with the simple system shown in figure 1. Being unable to solve the boundary layer equation exactly, I at first solved the problem by linearizing it in the manner of Oseen* (which is equivalent to converting it to a time-dependent problem as Rayleigh did for a flat plate). However, it was

* Professor H. B. Squire has also conceived this simple boundary layer problem and has solved it in this same way, in a paper to be published in the *Journal of the Royal Aeronautical Society*.

later pointed out to me by Mr W. W. Wood that so far as the relation between ω_0 and ω_1 is concerned the problem may be solved exactly by using the von Mises form of the boundary layer equation. This form (see *Modern Developments in Fluid Dynamics*, Oxford University Press, 1938, Vol. 1, § 50) becomes

$$\frac{1}{h}\frac{\partial q}{\partial \xi} = \tfrac{1}{2}\nu\frac{\partial^2(q^2)}{\partial \psi^2} \tag{3.4}$$

in cases in which the velocity outside the boundary layer is uniform, where $h d\xi$ represents a displacement along the streamline ψ = const., as before, and q is the velocity in this same direction. Then, since the streamlines here are closed and q is single-valued, we have

$$0 = \oint \frac{\partial^2(q^2)}{\partial \psi^2}\, qh\, d\xi \doteqdot \frac{\partial^2}{\partial \psi^2}\oint q^2 h\, d\xi, \tag{3.5}$$

since the mesh parameter h is approximately constant across the boundary layer. The total variation of ψ across the boundary layer tends to zero as $\nu \to 0$, so that the solution of (3.5) is effectively

$$\oint q^2 h\, d\xi = \text{const.} \tag{3.6}$$

throughout the boundary layer. On evaluating the integral for the streamline at the wall and for one just outside the boundary layer, we find the required relation to be

$$\omega_0/\omega_1 = (1-\sigma)^{1/2}. \tag{3.7}$$

It is still necessary to rely on some approximate procedure like the Oseen linearization for details of the velocity distribution in the boundary layer at different values of ξ, but (3.7) represents the crucial piece of information. It should be noted that the basic assumption that the velocity in the region of small viscous forces does not tend to zero as $\nu \to 0$ is confirmed in this case.

Another problem whose examination involves a consideration of the exact shape of the boundaries, and which will not be attempted here, concerns the position of the singular streamlines. The location of these viscous layers will sometimes be evident from the nature of the conditions at the outer boundary, as in the very simple case represented in figure 1. Cases in which their location is not evident will clearly present great difficulties in any detailed analysis, akin to those in problems in which a boundary layer separates from a rigid wall. The natural assumption that the singular surfaces coincide everywhere with rigid boundaries (except where the contrary is evident) needs particularly careful scrutiny. For instance, in a case of flow in a region bounded externally by a rigid wall which has a 90° corner, a viscous boundary layer would not flow along the wall right up to the corner, in general, because there would then be a stagnation point of the inviscid flow at the corner; it is probable that a singular surface exists to divide the main body of the fluid from a secondary 'standing eddy' in the corner, and indeed there may be even a whole sequence of such singular surfaces and 'standing eddies' of diminishing size as the corner is approached.

Finally, it is worth noting that steady two-dimensional motion of a fluid relative to given boundaries is unaffected by steady rotation of the whole system about an axis normal to the plane of motion (Taylor 1921). Consequently, all the above remarks and results apply not only to two-dimensional flows enclosed by outer boundaries whose positions are fixed and whose velocities are steady, but also to flows enclosed by boundaries whose positions and velocities are steady relative to suitably chosen uniformly rotating axes.

4. Steady rotationally symmetric flow in a closed region

It is convenient here to introduce orthogonal curvilinear coordinates (ξ, η, ψ), where the ξ-lines (on which η and ψ are constant) are everywhere parallel to the component of $\mathbf{u}$ lying in an axial plane, the η-lines are azimuthal

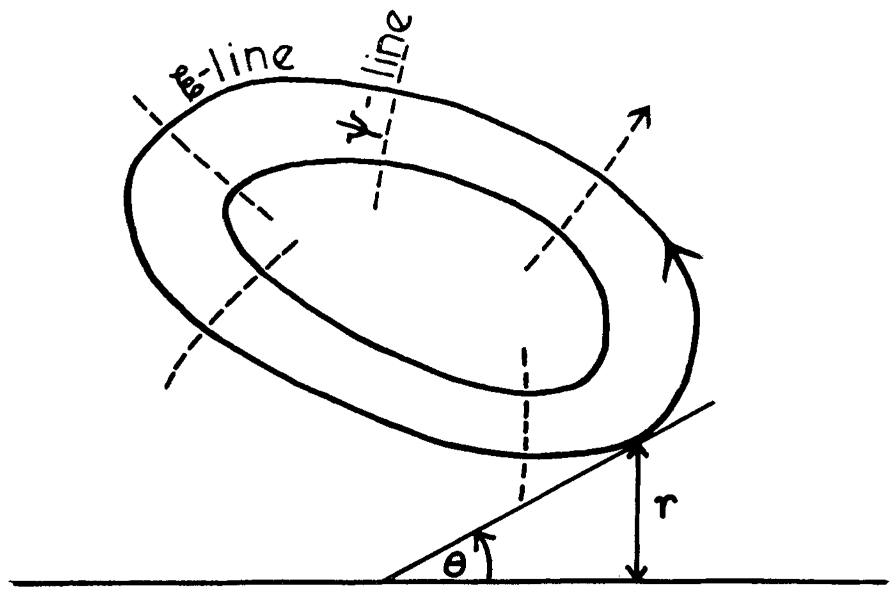

Figure 2. Coordinate system for rotationally symmetric flow.

circles, and the ψ-lines (ψ being the Stokes stream function of the component of the motion in an axial plane) lie in axial planes and are orthogonal to the ξ-lines (figure 2). The displacements corresponding to increments in ξ, η, ψ are

$$h_1\, d\xi, \qquad r\, d\eta, \qquad h_3\, d\psi,$$

where $h_1(\xi, \eta)$ is unknown, r is the distance from the axis of symmetry, and the definition of ψ supplies the relation

$$h_3 = (ru_1)^{-1}, \tag{4.1}$$

where $(u_1, u_2, 0)$ are the velocity components.

The ξ-lines, or streamlines of the components of velocity in an axial plane, are bounded externally, and perhaps internally also, by a singular curve which is itself a member of the family of ξ-lines. Provided the velocity component u_1 in the region of small viscous forces remains of order unity as $\nu \to 0$ —which is evident in the case of flow without swirl, since the 'inviscid' core would not otherwise be responding to the motion of the boundaries, but is not necessarily true in the case of flow with swirl since the motion generated in the 'inviscid' core may here be primarily an azimuthal swirl—a typical streamline will lie in the region of small viscous forces over the whole of its length. It will thus be possible to combine the exact integral condition (2.2) with the approximate inviscid flow equation (1.3). However, only in the case in which the azimuthal component of velocity is zero are these two equations sufficient to determine the form of the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ in the region of small viscous forces.

(a) *Flow without azimuthal swirl*

The velocity here has components $(u_1, 0, 0)$, and the components of the vorticity are $(0, \omega_2, 0)$ where

$$\omega_2 = \frac{1}{h_1 h_3}\frac{\partial(h_1 u_1)}{\partial\psi}.$$

Everywhere in the region of small viscous forces, (1.3) is satisfied approximately, whence

$$u_1\omega_2 = \frac{1}{h_3}\frac{dH(\psi)}{d\psi},$$

that is,

$$\frac{\omega_2}{r} = \frac{dH(\psi)}{d\psi}. \qquad (4.2)$$

This last relation describes the known proportionality, in inviscid flow, between the vorticity and the length of a material element of a vortex line. This is as far as one can go, making use only of the equations for inviscid motion, and (4.3) is the counterpart of (3.1) in two-dimensional flow. For information about the function $H'(\psi)$, which describes the unknown variation of vorticity across the streamlines, we must take some account of the action of viscosity, and this will be done by applying the integral condition (2.2) to closed streamlines lying entirely within the region of small viscous forces.

The components of $\nabla\times\boldsymbol{\omega}$ are given by

$$\nabla\times\boldsymbol{\omega} = \left(-\frac{1}{rh_3}\frac{\partial(r\omega_2)}{\partial\psi},\ 0,\ \frac{1}{rh_1}\frac{\partial(r\omega_2)}{\partial\xi}\right), \qquad (4.3)$$

and the condition (2.2) becomes

$$\oint \frac{\partial(r\omega_2)}{\partial\psi}\frac{h_1}{rh_3}\,d\xi = 0,$$

or, in view of (4.1) and (4.2),

$$\begin{aligned} H''(\psi)\oint r^2 u_1 h_1\,d\xi &= -H'(\psi)\oint \frac{\partial r}{\partial\psi}\frac{2h_1}{h_3}\,d\xi \\ &= 2H'(\psi)\oint \cos\theta\,ds \\ &= 0, \end{aligned} \qquad (4.4)$$

where θ is the angle between the tangent to the streamline (ξ and s increasing) and the axis of symmetry (with regard for the sense). Hence, we require

$$H''(\psi) = 0, \qquad H(\psi) = \alpha\psi, \tag{4.5}$$

where α is a constant (the other constant of integration being absorbed in the definition of ψ), and

$$\omega_2/r = H'(\psi) = \alpha \tag{4.6}$$

everywhere in a connected region of small viscous forces (the possibility of a different result applying on certain exceptional streamlines again being irrelevant, since viscous forces would not be small at such streamlines).

The result (4.6) is an obvious counterpart of the result that the vorticity is uniform in the region of small viscous forces in two-dimensional flow, and it is also true of (4.6) that the net effect of viscosity on the rate of change of the vorticity at any point vanishes identically. (But note that in axisymmetric motion the vorticity does not satisfy a heat-conduction type of equation, and it does not seem possible here to arrive at the result (4.6) by an argument in terms of diffusion of vorticity across streamlines.) There is also the common linear dependence of H on ψ, although the meanings of ψ in the two cases are not the same. The result established for the region of small viscous forces in two-dimensional flow is such that the net viscous force on any element of fluid vanishes identically, but this is not true of (4.6). The local viscous force per unit mass of fluid is $-\nu\nabla\times\boldsymbol{\omega}$, and it is readily seen from (4.3) and (4.6) that this is a uniform vector, of magnitude -2α and directed along the axis of symmetry. Thus the local viscous force has a simple character, and does not require the velocity distribution to be different (by even a small amount) from that for an inviscid fluid, since the viscous force can be balanced exactly by a uniform pressure gradient.

When the shape of the singular surfaces bounding the region of small viscous forces is known, the velocity distribution can be found from (4.6). The constant α, like ω_0, then remains to be determined from the condition that the surrounding viscous boundary layer is steady.

(b) *The general case* (*flow with swirl*)

The velocity components are now $(u_1, u_2, 0)$, and the components of the vorticity are $(\omega_1, \omega_2, \omega_3)$, where

$$\omega_1 = -\frac{1}{rh_3}\frac{\partial(ru_2)}{\partial\psi}, \qquad \omega_2 = \frac{1}{h_1h_3}\frac{\partial(h_1u_1)}{\partial\psi}, \qquad \omega_3 = \frac{1}{rh_1}\frac{\partial(ru_2)}{\partial\xi}. \tag{4.7}$$

The approximate equation (1.3) then yields the three scalar relations

$$\frac{1}{h_1}\frac{\partial H}{\partial\xi} = \frac{u_2}{rh_1}\frac{\partial(ru_2)}{\partial\xi}, \tag{4.8}$$

$$\frac{1}{r}\frac{\partial H}{\partial\eta} = 0 = -\frac{u_1}{rh_1}\frac{\partial(ru_2)}{\partial\xi}, \tag{4.9}$$

$$\frac{1}{h_3}\frac{\partial H}{\partial\psi} = \frac{u_1}{h_1h_3}\frac{\partial(h_1u_1)}{\partial\psi} + \frac{u_2}{rh_3}\frac{\partial(ru_2)}{\partial\psi}. \tag{4.10}$$

From (4.9), we have

$$ru_2 = C(\psi) \quad (\text{and } \omega_3 = 0), \tag{4.11}$$

which describes the constancy of circulation around a material curve in the form of a circle about the axis of symmetry, and then (4.8) reduces to

$$H \equiv H(\psi). \tag{4.12}$$

The last scalar equation, (4.10), can be written as

$$\frac{\omega_2}{r} = \frac{dH}{d\psi} - \frac{1}{2r^2}\frac{dC^2}{d\psi}, \tag{4.13}$$

which is the generalization of (4.2).

These relations appropriate to purely inviscid flow are now combined with the exact relation (2.2) which arises from the existence of viscous forces. This can be done only if there exist streamlines which lie entirely in the region of small viscous forces. As before, it follows that such streamlines will exist provided u_1 does not tend to zero, as $\nu \to 0$, everywhere in the region of small viscous forces. However, whereas the possibility of u_1 tending to zero could be rejected in cases of two-dimensional motion and of rotationally symmetric flow without swirl, it cannot immediately be rejected in cases in which u_1 is not the only component of velocity that may be finite. The assumption that u_1 does not tend to zero as $\nu \to 0$ is here a genuine restriction, which places some cases of rotationally symmetric flow with swirl outside the scope of the theory.

We first find that the components of $\nabla \times \boldsymbol{\omega}$ are given by expressions like those on the right-hand sides of (4.7), with ω_1 and ω_2 replacing u_1 and u_2. All these components are independent of η, and the integral (2.2) round a streamline reduces to an integral round a closed ξ-line, giving

$$\oint [(\nabla \times \boldsymbol{\omega})_1 u_1 + (\nabla \times \boldsymbol{\omega})_2 u_2] \frac{h_1}{u_1} d\xi = 0. \tag{4.14}$$

When the inviscid relations (4.11) and (4.13) are employed, this condition reduces to

$$\oint \left[r^2 \frac{d^2H}{d\psi^2} - \left(\frac{dC}{d\psi}\right)^2 + C\frac{dC}{d\psi}\frac{1}{h_1u_1}\frac{\partial(h_1u_1)}{\partial\psi}\right] u_1h_1\, d\xi = 0, \tag{4.15}$$

which is a more general version of (4.4).

It is not possible to determine the unknown functions $H(\psi)$ and $C(\psi)$ from (4.15), and the inviscid flow equations together with the integral condition (2.2) are not sufficient here to determine the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ in the region of small viscous forces. However, there is another condition that is applicable in this case of rotationally symmetric flow, at any rate provided the ξ-lines are not bounded internally by a solid boundary or a singular surface. Since $\omega_3 \to 0$ as $\nu \to 0$, the angle between the local components of $\mathbf{u}$ and $\boldsymbol{\omega}$ in the axial plane tends to zero as $\nu \to 0$, and the ξ-lines and the vector lines formed from the components of $\boldsymbol{\omega}$ in an axial plane (the latter being referred to as 'vortex lines in an axial plane') nearly coincide. This approximate coincidence of the two families of curves may take either of two forms. The ξ-line and 'vortex line in an axial plane' that pass through any given point may intersect once more at some other point along their lengths (or, more generally, may intersect at an odd number of additional points) or they may not intersect again (even number of additional points.) In the former case the 'vortex lines in an axial plane' are necessarily closed when ν is small, whereas in the latter case they have one end at a point exterior to all the ξ-lines and another end at a point interior to all the ξ-lines—which is possible only if the area intercepted on an axial plane by the region of small viscous forces is bounded internally. Thus, provided the ξ-lines have no inner boundary, we conclude that the vortex lines are closed when ν is small, and hence that, as explained in § 2,

$$\oint (\nabla \times \boldsymbol{\omega}).d\mathbf{v} = 0. \tag{4.16}$$

For small values of ν, when $\omega_3 \doteqdot 0$, this condition becomes

$$\oint \left[(\nabla \times \boldsymbol{\omega})_1 \, \omega_1 + (\nabla \times \boldsymbol{\omega})_2 \, \omega_2\right] \frac{h_1}{\omega_1} \, d\xi = 0. \tag{4.17}$$

Combining (4.17) with (4.14), we have

$$\oint \left(\frac{u_2}{u_1} - \frac{\omega_2}{\omega_1}\right) \frac{1}{h_3} \frac{\partial(h_1 \omega_1)}{\partial \psi} \, d\xi = 0,$$

which, after some reduction using (4.11) and (4.13), becomes

$$\oint r^2 \frac{\partial}{\partial \psi}\left(u_1 h_1 \frac{dC}{d\psi}\right) d\xi = 0,$$

that is

$$\frac{d}{d\psi}\left[\frac{dC}{d\psi} \oint r^2 u_1 h_1 \, d\xi\right] = 0. \tag{4.18}$$

It has already been assumed that the ξ-lines are not bounded internally, so that there will exist an inner limiting ξ-line which is merely a point. On this degenerate ξ-line, $h_1 = 0$ and $u_1 \, dC/d\psi$ is finite, in general, so that the integral in (4.18) is zero; hence

$$\frac{dC}{d\psi} \oint r^2 u_1 h_1 \, d\xi = 0. \tag{4.19}$$

The integral cannot be zero, except perhaps on an isolated ξ-line (since we have already supposed that u_1 is not zero everywhere, even when $\nu \to 0$), and (4.19) becomes

$$C = r u_2 = \text{const.} \tag{4.20}$$

Equation (4.15) now reduces to the form appropriate to flow without swirl, and the solution is

$$\frac{\omega_2}{r} = \frac{dH}{d\psi} = \text{const.} \tag{4.21}$$

as before.

The solution represented by (4.20) and (4.21) is such that the vortex lines are circles about the axis of symmetry, as for flow without azimuthal swirl. Moreover, the remarks of the preceding sub-section about the effect of viscosity on the rate of change of vorticity at any point, and on the acceleration of any fluid element, apply here also. When the azimuthal component of velocity is allowed to be non-zero, it seems that in truly steady motion this component is necessarily an irrotational velocity field corresponding to a circulation about the axis (provided, as above, that (*a*) u_1 does not tend to zero as $\nu \rightarrow 0$, and (*b*) there is a degenerate inner ξ-line). Note, however, that the result (4.20) does not apply in the neighbourhood of the axis of symmetry, since, in cases in which the region of flow does include part of the axis of symmetry, the axis is part of the streamline that bounds the whole flow (in the axial plane) and this streamline necessarily passes through a region in which viscous forces are appreciable.

In cases in which the area intercepted on an axial plane by the region of small viscous forces does have an inner boundary (as, for example, when the fluid lies between two anchor rings with a common axis of symmetry, one ring enclosing the other), it does not seem possible to deduce the distributions of $\boldsymbol{\omega}$ and $\mathbf{u}$ in the region of small viscous forces unless the vortex lines can first be shown to be closed.

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