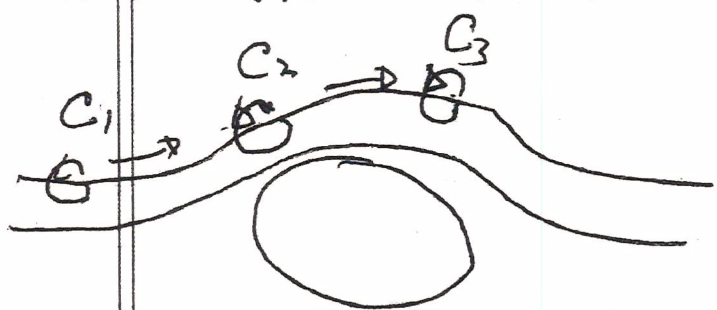
ii.) Potential Flow

- Consider fluid streamlines:



if $\underline{\omega} = 0$ at any point along streamline, then Kelvin thm $\Rightarrow$ $\underline{\omega} = 0$ everywhere on streamline.

Easily seen by considering circulation around infinitesimal loop "pulled" along streamline. Thus, if

$$\oint_{C_1} \underline{v} \cdot d\underline{\ell} = \int_{A_1} \underline{\omega} \cdot d\underline{s} = 0, \text{ then } \oint_{C_n} \underline{v} \cdot d\underline{\ell} = \int_{A_n} \underline{\omega} \cdot d\underline{s} = 0$$

for all C_n.

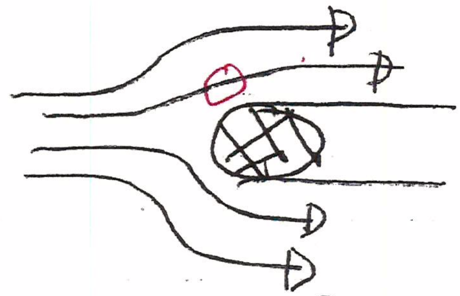
- flow with $\underline{\omega} = \underline{\nabla} \times \underline{v} = 0$ in all space is defined as:
  $\Rightarrow$ potential, irrotational flow

  $\Leftrightarrow$ $\underline{\omega} \neq 0$ rotational, vortical flow

- Important to note breakdown of Kelvin thm

Applicability, namely to flows with separation

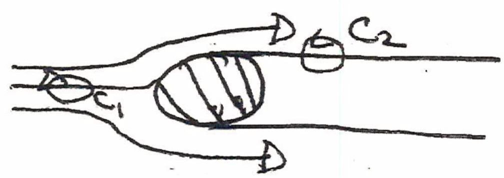
i.e. consider flow and sphere



i.e. - streamlines separate from body
- surface of tangential discontinuity appears (velocity component)

⇒ - Kelvin Thm not applicable

i.e.



- cannot infer $\oint_{C_2} \underline{V}\cdot d\underline{\ell}$ from $\oint_{C_1} \underline{V}\cdot d\underline{\ell}$ due to separation-induced tangential discontinuity.
- Also, viscosity important in (boundary layer) region of discontinuity. As viscous effects $\sim \nu k^2$, deviation from potential flow naturally most significant in small scale region of boundary layer!

Now, for isentropic fluids:

$W \equiv$ enthalpy

$$\partial \underline{V}/\partial t + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{\nabla} W$$

stream function

For potential flow, $\underline{V} = \underline{\nabla}\phi \Rightarrow \underline{\omega} = 0$

$$\underline{V}\cdot\underline{\nabla}\,\underline{V} = -\underline{V}\times\cancel{\underline{\omega}} + \underline{\nabla}(V^2/2)$$

$$= 0 + \underline{\nabla}(V^2/2), \quad \text{for potential flow}$$

$\therefore$

$$\frac{\partial \underline{V}}{\partial t} + \underline{\nabla}\left(V^2/2\right) = -\underline{\nabla}W$$

$$\underline{V} = \underline{\nabla}\phi$$

$$\Rightarrow \underline{\nabla}\left(\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} + W\right) = 0$$

$\therefore$ have equation for dynamics of potential flow:

Bernoulli — along streamline

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} + W = f(t)}$$

f(t) defined for each streamline

- for $\partial\phi/\partial t = 0$, recover Bernoulli's Law
- obvious that potential not uniquely defined, as $\underline{V} = \underline{\nabla}\phi$

what does incompressibility mean?

Now, consider incompressible fluid potential flow, i.e.,

- flows leaving density constant (no compression, expansion)

$\nabla \cdot \underline{V} = 0$

- $\nabla \cdot \underline{V} = 0 \iff \frac{d\rho}{dt} = 0$ (ρ constant)

$\Rightarrow$ if $\underline{V} = \nabla \phi \Rightarrow$
$$\boxed{\begin{gathered} \nabla^2 \phi = 0 \\ \frac{\partial \phi}{\partial t} + \frac{(\nabla\phi)^2}{2} + \frac{P}{\rho} = f(t) \end{gathered}}$$

$\therefore$ for static flow, with gravity, $\Rightarrow$ Bernoulli Eqn.:
$$\boxed{V^2/2 + P/\rho + gz = \text{Const.}}$$

Criterion for "incompressibility":

- "incompressibility" is valid description for certain classes of flows, dependent on time scales, speeds, etc.

- for stationary flows $\partial \underline{V}/\partial t = 0$,
$$\frac{P}{\rho} + \frac{V^2}{2} = \text{const.}$$

Now, for $\begin{Bmatrix}\text{adiabatic}\\ \text{isentropic}\end{Bmatrix}$ fluid (T const) $\Rightarrow$

$$\Delta P = \left(\frac{\partial P}{\partial \rho}\right)_s \Delta \rho$$

but $\quad \frac{P}{\rho} + \frac{V^2}{2} = \text{const.}$

$$\Rightarrow \quad \Delta\left(\frac{V^2}{2}\right) = -\left(\frac{\partial P}{\partial \rho}\right)_s \frac{\Delta \rho}{\rho}$$

"Incompressibility" $\Rightarrow$ $\Delta\rho/\rho \ll 1$

$(\partial P/\partial \rho)_s = c_s^2$ (sound speed in fluid)

$\therefore \quad V^2/c_s^2 \ll 1 \Rightarrow$ flow incompressible.

Note: $\begin{Bmatrix}\text{Supersonic flows always compressible} \Rightarrow \\ \text{fluid dynamics couples to acoustic waves.}\end{Bmatrix}$

– for dynamic flows (more generally);

need compare terms in continuity equation

$$\frac{d\rho}{dt} = -\rho\, \underline{\nabla} \cdot \underline{V}$$

Now $\tilde{\tau}$ → time scale for flow
ℓ → spatial scale for flow

Then, $\frac{d\rho}{dt} \sim \frac{\Delta\rho}{\tau}$

$\rho \nabla\cdot\underline{V} \sim \rho\frac{\tilde{V}}{\ell}$

$\nabla\cdot\underline{V} \sim 0$

$\frac{\rho\tilde{V}}{\ell} > \frac{\Delta\rho}{\tau}$

$> \frac{\partial P}{\tau}\cdot\frac{\psi}{c_s^2}$

To relate $\Delta\rho$ to $\tilde{V}$, consider Euler equation:

$$\frac{\partial\underline{V}}{\partial t} = -\frac{\nabla P}{\rho}$$

$\rho\frac{\tilde{V}}{\ell} > \frac{\not\rho}{\tau}\frac{\tilde{V}}{\tau}\frac{\rho\ell}{c_s^2}$

$c_s^2 > \ell^2/\tau^2$

$$\Rightarrow \frac{\tilde{V}}{\tau} \sim \frac{c_s^2\Delta\rho}{\rho\ell}$$

(tacitly, $\tilde{V} \lesssim \ell/\tau$)

$$\tilde{V} \sim \frac{\tau c_s^2\Delta\rho}{\rho\ell}$$

$\rho\nabla\cdot v > \frac{d\rho}{dt}$

$\rho\frac{\tilde{v}}{\ell} \sim \frac{\Delta\rho}{\tau}$

$$\therefore \frac{d\rho}{dt} \sim \left(\frac{c_s^2\tau}{\ell\rho}\right)^{-1}\frac{\tilde{V}}{\tau}$$

$\frac{\tilde{v}}{\tau} \sim \frac{-\partial P}{\partial}$

$\sim \frac{c_s^2\Delta\rho}{\rho\ell}$

$$\rho\nabla\cdot\underline{V} \sim \frac{\tilde{V}}{\ell}\rho$$

$\nabla\cdot\underline{V} \cong 0$ if $\frac{d\rho}{dt} \ll \rho\nabla\cdot\underline{V}$

$$\frac{\tilde{V}}{\ell}\rho \overset{?}{\gg} \frac{\ell\rho}{c_s^2\tau^2}\tilde{V} \Rightarrow \boxed{c_s^2 \gg \frac{\ell^2}{\tau^2}}$$

Thus, dynamics incompressible if $\begin{cases} c_s^2 \gg \ell^2/\tau^2 \\ \quad\gg \omega^2/k^2_{\text{(wave)}} \end{cases}$

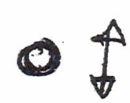
- note can synthesize static, dynamic conditions to obtain incompressibity criterion:

$$c_s^2 > \begin{cases} \tilde{V}^2 \\ \ell^2/\tau^2 \end{cases}$$

i.e. $\left\{\right.$ time slow compared to time to traverse 1 spatial scale at acoustic speeds.

Some furthur facts about potential flows (generally incompressible):

- for body (i.e. rigid sphere) immersed in fluid, if amplitude oscillation $\ll$ dimensions of body $\Rightarrow$ motion describable by potential flow



i.e. $a \equiv$ amplitude motion
$u \equiv$ body velocity
$f \equiv$ frequency of oscillation
$\ell \equiv$ size of body

Simply compare $\frac{\partial V}{\partial t}$ to $V\cdot\nabla V$, noting

OR for flow $\frac{\partial v}{\partial t}$ [illegible] $\underline{0 \cdot \nabla v}$

If $\partial \underline{v}/\partial t \gg \underline{v}\cdot\underline{\nabla}\underline{v} \Rightarrow \frac{\partial \underline{v}}{\partial t} \cong -\underline{\nabla} W$

so

$$\underline{\nabla} \times \underline{v} = 0 \Rightarrow \left\{ \begin{array}{l} \text{Potential} \\ \text{flow} \end{array} \right.$$

Now $\omega \sim u/a$

$\partial \underline{v}/\partial t = -i\omega\,\underline{v} \sim u^2/a$ ($\underline{v} \sim \underline{u}$ near body)

$\underline{v}\cdot\underline{\nabla}\,\underline{v} \sim u^2/\ell$ (ℓ sets smallest scale in problem)

$$\left|\frac{\partial \underline{v}}{\partial t}\right| \overset{?}{\gg} |\underline{v}\cdot\underline{\nabla}\,\underline{v}| \Rightarrow \frac{\underline{u}^2}{a} \overset{?}{\gg} \frac{\underline{u}^2}{\ell}$$

$$\Rightarrow \ell \gg a$$

Thus, fluid dynamics resulting from small oscillation of body describable by potential flow.

- In potential flow, streamlines must be open, not closed.

are

To see, consider circulation about closed contour

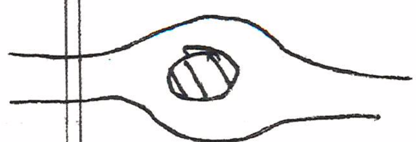
ϕ chnses

$$\oint_C \underline{V}\cdot d\underline{l} = \int_{S_{enc.}} d\underline{S}\cdot \underline{\omega} = 0$$

$\underline{\omega} = 0$ for potential flow

but, by definition, $\int_{\text{streamline}} \underline{V}\cdot d\underline{l} \neq 0$, $\Rightarrow$ streamlines must be open!

i.e.



sphere in $\underline{V} = V_0 \hat{z}$ flow is typical potential flow problem (describes flow at distance from sphere).

SY

- For incompressible flow, (not potential)

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V} - \cancel{\underline{\omega}\,\underline{\nabla}\cdot\underline{V}}$$

In 2D, $\underline{\omega}\cdot\underline{\nabla}\,\underline{V} = 0$ i.e. $\begin{cases} \underline{V} = (V_x(x,y),\ V_y(x,y)) \\ \underline{\omega} = \omega_z(x,y)\hat{z} \end{cases}$

Then, $\frac{d\omega}{dt} = 0$

Now, $\underline{\nabla}\cdot\underline{V} = 0 \Rightarrow \begin{matrix} V_x = \partial\psi/\partial y \\ V_y = -\partial\psi/\partial x \end{matrix}$

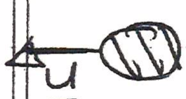
$$\underline{\omega} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{z} = \hat{z}\left(-\nabla^2\psi\right)$$

$$\frac{d\underline{\omega}}{dt} = 0 \quad \Rightarrow \quad \begin{cases} +\dfrac{\partial}{\partial t}\nabla^2\psi + \underline{\nabla}\psi \times z \cdot \nabla\nabla^2\psi = 0 \\ \text{2D incompressible fluid eqn.} \end{cases}$$

iv.) Problems in Potential Flow

a.) Incompressible Potential Flow Around Sphere

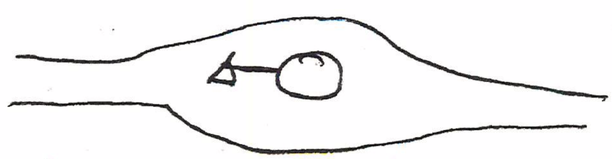
Consider rigid sphere in motion at $\underline{u}$ in infinite fluid



Flow Pattern ?

Now :

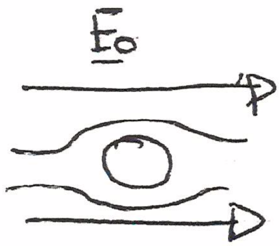
- intuitively, expect :



i.e. equivalent to $\begin{cases} \text{sphere at rest} \\ \underline{V}\big|_{\text{fluid},\ \infty} = -\underline{u} \end{cases}$

Electrostatic analogy: Conducting sphere in uniform electric field

i.e.



$\phi = -\underline{E}_0 \cdot \underline{r} + \phi_{sphere}$

ϕ_{sphere} is dipole field.

Dipole moment determined by b.c.
i.e. $\phi = \text{const} = 0$ on sphere surface

Now, for potential flow (incompressible):

$\nabla^2 \phi = 0 \qquad \underline{V} = \underline{\nabla} \phi$

$V_n = \underline{V} \cdot \hat{n} = \underline{u} \cdot \hat{n} \Big|_{surface}$

(i.e. normal velocity = sphere velocity on surface)

By analogy with electrostatics, can solve via:
- multipole expansion
- B.C.'s determine effective "charge" distribution

Recall e.s. $\Rightarrow$ $\nabla^2\phi = -4\pi\rho$

$$\phi = \int d^3x' \frac{\rho(x')}{|\underline{x}-\underline{x}'|}$$

For $\underline{x}$ outside region ρ :

$$\phi(\underline{x}) = \int d^3x' \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$= \int d^3x' \frac{\rho(x')}{|\underline{x}-\cancel{\underline{x}'}|} - \int d^3x' \, \underline{x}'\rho(x')\cdot\nabla\left(\frac{1}{|\underline{x}|}\right) + \cdots$$

$$= \frac{Q}{|\underline{x}|} - \underline{d}\cdot\nabla\left(1/|\underline{x}|\right) + \cdots$$

$\downarrow$ monopole $\qquad$ $\downarrow$ dipole $\qquad$ $\downarrow$ quadrupole

Thus, can write down general solution for potential flow streamlines around body as multipole expansion.

$\rightarrow$ $Q = 0$ (no sources, sinks)

$\therefore$ in general dipole dominates

$\therefore$ in 2D, same story with $\ln|x-x'| \Rightarrow 1/|x-x'|$

Here: $\underline{u} = u\,\hat{z}$ (flow velocity) (spherical symmetry) (body velocity)

$V_n\big|_R = V_r\big|_R = u\,\hat{z}\cdot\hat{n} = u\cos\theta$ → boundary condition

$u \to \hat{z}$.

Now, $\phi(\underline{x}) = \underline{A}\cdot\underline{\nabla}(1/|\underline{x}|)$

$\underline{A} = A\,\hat{z}$ (dipole moment in $\hat{z}$ direction)

$\therefore \quad \phi = -A\,\dfrac{\cos\theta}{r^2}$

$V_r = 2A\cos\theta/r^3$

$V_r = u\cos\theta$ on surf

$\Rightarrow \quad \dfrac{2A\cos\theta}{R^3} = u\cos\theta$

$\Rightarrow \quad A = \dfrac{R^3}{2}u$

$$\therefore \quad \boxed{\begin{aligned}\phi &= -uR^3\cos\theta/2r^2 \\ \underline{V} &= \nabla\phi\end{aligned}}$$

determines general flow field

Note:

regularity at ∞

- can recover from $\phi = \sum_{\ell} \left(a_\ell r^\ell + b_\ell / r^{\ell+1} \right) P_\ell(\cos\theta)$

expansion and b.c.'s.

- if sphere in uniform field:

$$\phi = U_0 r\cos\theta + \phi_{sphere}$$

ϕ_{sphere}: determine from $V_n = 0$

- to determine pressure distribution on sphere,

recall: $\rho \frac{\partial \phi}{\partial t} + \frac{\rho V^2}{2} + p = p_0$ } incompressible Bernoulli Eqn.

p_0: ambient pressure at ∞

Thus, can immediately write:

$$p(\underline{x}) = p_0 - \frac{\rho}{2} \nabla\phi \cdot \nabla\phi - \rho \frac{\partial \phi}{\partial t}$$

- $\phi(\underline{x}) \equiv$ determined a la' above via $\nabla^2\phi = 0$ and b.c.'s.

As sphere in motion (but uniform):

$$\frac{\partial \phi}{\partial t} = -\underline{u} \cdot \nabla \phi + \cancel{\frac{\partial \phi}{\partial \underline{y}} \cdot \dot{\underline{y}}} \quad \dot{u} = 0$$

so

$$\boxed{P(\underline{x}) = P_0 - \frac{\rho}{2} \nabla \phi \cdot \nabla \phi - \underline{u} \cdot \nabla \phi}$$

Generally, leads to concept of stagnation point

i.e. for Bernoulli Eqn. for incompressible fluid:

$$\frac{P}{\rho} + \frac{V^2}{2} = \text{const.} = P_0$$

Now, consider fixed body in fluid with $\begin{cases} V_{\infty} = u_0 \\ P_{\infty} = P_0 \end{cases}$

As $\underline{V} = 0$ on surface body:

$$P_{max} = P\Big|_{bdy} = P_0 + \frac{1}{2} \rho u^2$$

- stagnation point ($\underline{V} = 0$) on body is point of maximal pressure
- maximal pressure determined by $\begin{cases} P_0 \\ \text{speed} \end{cases}$

a.)  → fish skeleton strongest on front face, weakest elsewhere

→ front face is point of maximal pressure (head)

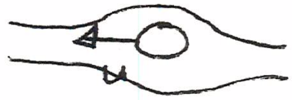
→ eye lens adjusts to allow for speed-induced pressure changes.

b.) Drag Force and Induced Mass

bubble

→ what force on ping pong ball $\underline{u'}$

→ Heuristics: Consider rigid body in water.



Slow __body__ motion ⇒ potential flow around sphere
⇒ energy in __fluid__ motion, too!

Thus, for $\underline{F}_{ext}$ to move body in fluid, need work against
- inertia of body (obvious)
- inertia of __fluid__, excited into potential flow

Thus, for body in water, need interpret Newton's 2nd Law as:

$$\underline{F}_{ext} = M_{eff} \frac{d\underline{u}}{dt}$$

$$M_{eff} = \underset{\substack{\downarrow \\ \text{mass of body}}}{M} + m_{induced}$$

→ induced mass of fluid in potential flow around body

(mass of fluid flow which 'dresses' the body)

[illegible]

[illegible]

To calculate induced mass:

- calculate energy in potential flow around rigid body in uniform motion in fluid

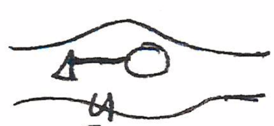
- use $\boxed{dE = d\underline{P} \cdot \underline{u}}$ to determine momentum in fluid

as $\underline{P} = \underline{P}(\underline{u}) \Rightarrow \boxed{P_i = m_{ik} u_k}$

$\therefore m_{ik}$ is induced mass tensor!

→ Calculation: Consider rigid body moving in fluid

i.e.



Now, for flow field outside body, multipole expansion solution to $\nabla^2 \phi = 0$ yields

$$\phi = \cancel{\frac{q}{r}} + \underline{A}\cdot\underline{\nabla}\left(1/r\right) + \cdots$$

monopole (vanishes → no sources)

dipole (dominant multipole at large radius)

→ dipole moment : $A = c\, R_s^3 \underline{u}$

($c = 1/2$, sphere)

$$\therefore\quad \phi = \underline{A}\cdot\underline{\nabla}\left(1/r\right)$$

$$= -\underline{A}\cdot\underline{r}/r^3 = -\underline{A}\cdot\hat{n}/r^2 .$$

$\Phi = -A\frac{\cos\theta}{r^2}$

$$\underline{V} = \underline{\nabla}\phi = \underline{A}\cdot\nabla\;\nabla\left(1/r\right)$$

$$= \left(\underline{A}\cdot\underline{\nabla}\right)\left(-\underline{r}/r^3\right)$$

$$\underline{V} = \left(3\left(\underline{A}\cdot\hat{n}\right)\hat{n} - \underline{A}\right)/r^3$$

$v_r = \frac{2A\cos\theta}{r^3}$

$u\cos\theta = \frac{2A\cos\theta}{R^3}$

$A = \frac{u}{2}R^3$

Now, for energy, seek calculate fluid energy in volume V enclosed within radius R around body. Take $R^3 \gg V_0 \equiv$ volume of body.

~~Thus~~:

$$E = \frac{1}{2}\rho\int dV\, |\underline{V}|^2$$

$$= \frac{1}{2}\rho\int d^3x\left(\underline{u}^2 + |\hat{V}|^2 - \underline{u}^2\right)$$

out $\quad |\nabla\phi|^2 - u^2 = (\underline{V} + \underline{u})\cdot(\underline{V} - \underline{u})$

$$= \nabla(\phi + \underline{u}\cdot\underline{r})\cdot(\underline{V} - \underline{u})$$

$$= \underline{\nabla}\cdot\left[(\phi + \underline{u}\cdot\underline{r})(\underline{V} - \underline{u})\right]$$

as $\quad \underline{V} = \underline{\nabla}\phi \qquad \underline{\nabla}\cdot\underline{V} = 0$

$\quad\;\;\; \underline{u} = \text{const.} \qquad \underline{\nabla}\cdot\underline{u} = 0$

$$\therefore \quad E = \frac{1}{2}\rho\int d^3x\left[u^2 + \nabla\cdot\left[(\phi + \underline{u}\cdot\underline{r})(\underline{V} - \underline{u})\right]\right]$$

$$= \frac{1}{2}\rho u^2(\underbrace{V}_{\text{volume space}} - \underbrace{V_0}_{\text{volume object/body}}) + \frac{1}{2}\rho\int d\underline{s}\cdot\left[(\phi + \underline{u}\cdot\underline{r})(\underline{V} - \underline{u})\right]$$

$V = \frac{4\pi}{3}R^3$ —

$\left\{ (\underline{V} - \underline{u})\cdot d\underline{s} = 0 \right.$ on R_0 surface

Now, $\quad d\underline{s} = \hat{n}\, R^2 d\Omega$, on outer surface

$$E = \frac{1}{2}\rho u^2(V - V_0)$$

$$+ \frac{1}{2}\rho\int R^2 d\Omega\left[(\hat{n}\cdot\underline{V} - \hat{n}\cdot\underline{u})(\phi + \underline{u}\cdot\underline{R})\right]$$

$$E = \frac{1}{2}\rho u^2 (V - V_0)$$

$$+ \frac{1}{2}\rho \int R^2 d\Omega \left[\left(2\frac{(\underline{A}\cdot\vec{n})}{R^3} - \underline{u}\cdot\vec{n} \right) \left(\frac{-\underline{A}\cdot\hat{n}}{R^2} + R\,\underline{u}\cdot\hat{n} \right) \right]$$

vanishes for large R

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int R^2 d\Omega \left[\cancel{-2\frac{(\underline{A}\cdot\vec{n})^2}{R^5}} \right.$$

$$\left. + \frac{(\underline{u}\cdot\vec{n})(\underline{A}\cdot\vec{n})}{R^2} + \frac{2(\underline{A}\cdot\vec{n})(\underline{u}\cdot\vec{n})}{R^2} - R(\underline{u}\cdot\vec{n})^2 \right]$$

$$= \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int \cancel{R}^3 d\Omega \left[\frac{3(\underline{A}\cdot\vec{n})(\underline{u}\cdot\vec{n})}{\cancel{R}^2} - R^3(\underline{u}\cdot\vec{n})^2 \right]$$

Thus finally,

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho \int d\Omega \left[3(\underline{A}\cdot\underline{\hat{n}})(\underline{u}\cdot\underline{\hat{n}}) - R^3(\underline{u}\cdot\vec{n})^2 \right]$$

$$d\Omega = d\theta \sin\theta \, d\phi$$

if $\int d\Omega \, (\;) = \langle (\;) \rangle$

$$\Rightarrow \langle (\underline{A}\cdot\vec{n})(\underline{B}\cdot\vec{n}) \rangle = \frac{1}{2}\delta_{ij} A_i B_j = \frac{1}{3}\underline{A}\cdot\underline{B}$$

$$E = \frac{1}{2}\rho u^2 (V - V_0) + \frac{1}{2}\rho\left[4\pi \underline{A}\cdot\underline{u} - \frac{4\pi}{3}R^3 u^2\right]$$

$$= \frac{1}{2}\rho\left[4\pi \underline{A}\cdot\underline{u} - u^2 V_0\right]$$

Thus finally, $\boxed{E = \frac{\rho}{2}\left[4\pi \underline{A}\cdot\underline{u} - u^2 V_0\right]}$ energy in potential flow and body

Now, $\underline{A} = \underline{A}(\underline{u}) \Rightarrow E = \frac{1}{2} m_{ik} u_i u_k$ defines induced mass tensor

$$dE = \underline{u}\cdot d\underline{P}$$

$\Rightarrow \boxed{\underline{P} = \rho_{H_2O}\left[4\pi \underline{A} - V_0 \underline{u}\right]}$ momentum in potential flow

Now, consider external force acting system, where system = body + fluid (in pot. flow)

i.e. $$\underline{f}_{ext} = \frac{d\underline{P}_{fluid}}{dt} + M_{body}\frac{d\underline{u}}{dt}$$

$$\Rightarrow \quad f_i = \left(M\delta_{ik} + m_{ik}\right)\frac{du_k}{dt}$$

∴ → effective mass of "system" is sum of
- body mass
- induced mass of fluid in potential flow around body

→ note induced mass is determined purely by body shape (i.e. via volume and dipole moment)

i.e. for sphere $$\underline{A} = \frac{R_0^3}{2}\underline{u}$$

$$\underline{P} = \rho\left[4\pi\frac{R_0^3}{2}\underline{u} - \frac{4\pi}{3}R_0^3\underline{u}\right]$$

$$= \rho\,\frac{2}{3}\pi R_0^3\,\underline{u}$$

$$m_{induced} = \rho\,\frac{2}{3}\pi R_0^3$$

In general $\quad M_{induced} \sim \# \rho R_0^3$

$$\sim \# \underbrace{\rho V_0}$$

↓ numerical factor, shape dependent

↳ displaced mass fluid

→ Example of "renormalization" in classical physics "dressing field" in continuum i.e. {renorm. polaron, debye shield, etc}

i.e. in quantum electrodynamics → electron polarizes vacuum

$$\rightarrow m_e = m_e^{bare} + m_e^{V.P.}$$

$$(E = mc^2)$$

in classical potential flow → moving a sphere in H_2O requires that some energy go into surrounding media (the water!)

(skip)

→ Enhanced inertia due induced mass may, alternatively, be viewed as drag force on body mom. transmitted to fluid (careful of phase!)

i.e. $\quad f_{ext} = \dfrac{dP_{fluid}}{dt} + M\dfrac{dv}{dt}$

drag!

$$\therefore \quad M \frac{d\underline{u}}{dt} = \underline{f}_{ext} - \frac{d\underline{P}_{fluid}}{dt}$$

$$= \underline{f}_{ext} + \underline{f}_{drag,\,lift} \qquad \underline{f}_{drag} \sim \dot{u}$$

$\underline{f}_{drag} = -\frac{d\underline{P}_{fluid}}{dt}$, along direction motion.

$\underline{f}_{lift} = -\frac{d\underline{P}_{fluid}}{dt}$, $\perp$ direction of motion.

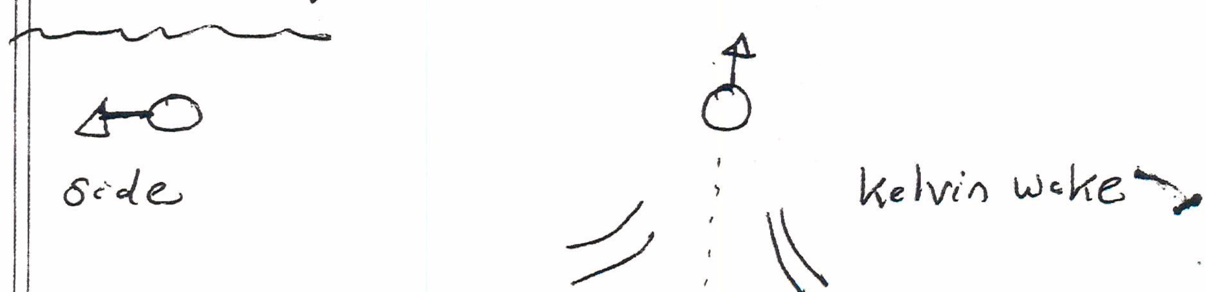
Note: → if body is uniform motion in ideal (fantasy) fluid $\underline{f}_{drag} = \underline{f}_{lift} = 0$ } D'Alembert's Paradox

→ need no external force to maintain uniform motion

as – no dissipation (ideal fluid)

– no loss of energy to ∞ ($v \sim 1/R^3$)

→ but if body near surface



body will radiate surface waves to ∞ (wake) ⇒ wave drag induced energy loss!

example: Obtain:

a) - eqn. of motion for sphere oscillating in fluid

b) - sphere in oscillating fluid

a) for sphere $\underline{A} = \frac{1}{2} R^3 \underline{U}$

for oscillating sphere

$$f_{ext} = m a_{sphere} + \underbrace{(m \dot{\underline{V}})_{induced}}_{\text{acceleration of dressing}}$$

$$m \dot{\underline{V}} = M_{ind} \dot{\underline{U}}$$

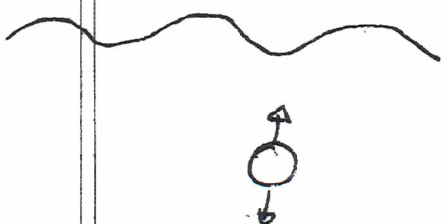
$$M_{ind} = \frac{2}{3} \pi R^3 \rho_{H_2O}$$

virtual mass

$$\therefore \quad f_{ext} = \frac{4\pi}{3} R^3 \left(\rho_{sph} + \frac{\rho_{H_2O}}{2} \right) \frac{dU}{dt}$$

→ Related Problem:

- consider body in fluid, which is set in motion by external agent.



Relate $\underline{u}$ body to $\underline{V}$ fluid !?

- Now $\underline{V} \equiv$ velocity of unperturbed flow

$$\frac{\|\nabla \underline{V}\| R_0}{\|\underline{V}\|} \ll 1 \Rightarrow \underline{V} \sim \text{const over scale of body}$$

(potential flow valid)

so if body fully carried along by fluid ($\underline{V} = \underline{u}$), then force on it would equal force on volume of displaced fluid

i.e. $$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{V}}{dt}$$

but body moves relative to fluid, so that fluid aquires momentum

→ drag due relative motion

i.e. $$d\underline{P}_{fluid}/dt = -\underline{\underline{m}} \cdot \frac{d}{dt}[\underline{u} - \underline{V}]$$

$\therefore$ so really,

$$\frac{d}{dt}(M\underline{u}) = \rho V_0 \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}(\underline{u}-\underline{v})$$

$$\frac{d}{dt}(Mu_i) = \rho V_0 \frac{dV_i}{dt} - m_{ik}\frac{d}{dt}(u_k - V_k)$$

$\Rightarrow$

$$Mu_i = \rho V_0 V_i - m_{ik}(u_k - V_k)$$

$\therefore$

$$(M\delta_{ik} + m_{ik})u_k = (\rho V_0 \delta_{ik} + m_{ik}) V_k$$

$$\boxed{u_k = \left(\frac{\rho V_0 \delta_{ik} + m_{ik}}{M\delta_{ik} + m_{ik}}\right) V_k}$$

Note: $\rho V_0 < M$ (body heavier than displaced fluid) $\rightarrow$ body lags

$\rho V_0 > M$ $\rightarrow$ body leads

$\rho V_0 = M$ $\quad u_k = V_k$.

Thus

$$M \frac{d\underline{u}}{dt} = \rho_f V \frac{d\underline{v}}{dt} - \underline{\underline{m}} \cdot \frac{d}{dt}\left[\underline{u} - \underline{v}\right]$$

$$(M \delta_{ij} + m_{ij}) \frac{du_j}{dt} = M_f \delta_{ij} + m_{ij} \frac{dV_j}{dt}$$

$$\therefore \quad u_j = \left[(M_f \delta_{ij} + m_{ij}) / (M \delta_{ij} + m_{ij}) \right] V_j$$

$$M_f = \rho_f V_0$$
$$M = \rho V_0$$

$\Rightarrow$ $\underline{u} = \underline{v}$ if $\rho_f = \rho$

$u < V$ if $\rho_f < \rho$ $\rightarrow$ heavy object lags

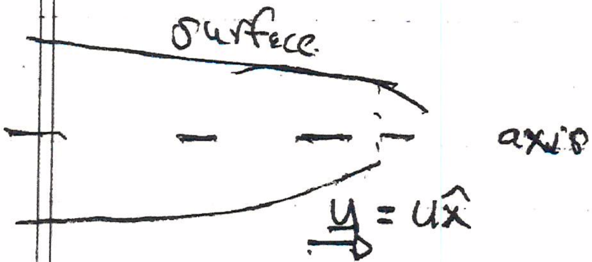
$\rho_f \equiv$ fluid density
$\rho \equiv$ body density

$u > V$ if $\rho_f > \rho$ $\rightarrow$ light object leads.

c.) Potential Flow – General Slender Body

- Till now, have considered simple body potential flows, i.e. sphere, cylinder,

Here consider general body from surface of revolution



i.e. – general, axially symmetric slender body

– slender $\Leftrightarrow$ $W/L \ll 1$

Now, observe analogy with electrostatics again,

i.e. e.s. $\Rightarrow$ $\phi(x) = \int d^3x' \, \rho(x') / |\underline{x} - \underline{x}'|$

potential flow $(A \sim uV)$

$$\phi(x) = \frac{1}{4\pi} \int d^3x' \, (\dot{\rho}(x')/\rho_0) / |\underline{x} - \underline{x}'|$$

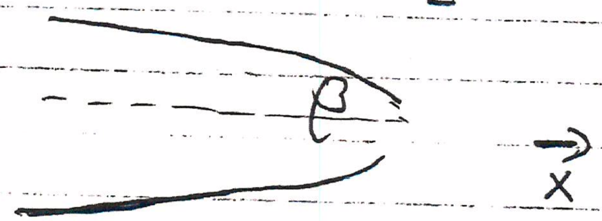
$\frac{\dot{\rho}(x')}{\rho_0} \equiv$ normalized density of fluid flowing accross cross-section of body

$\rightarrow$ yields $A \sim V_0 u$ etc.

$$\therefore \quad \phi(x) = \frac{1}{4\pi |x|^2} \int d^3x' \, \frac{\dot{\rho}(x')}{\rho_0} \, x' \quad + \text{ h.o.t.}$$

dipole term dominates

Now, – body slender $\rightarrow \frac{W}{L} < 1 \Rightarrow \beta \ll 1$



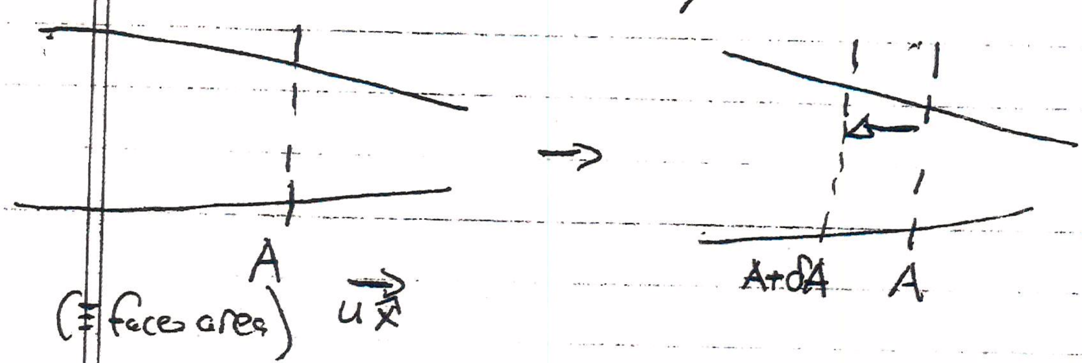
– $\nabla \cdot \underline{V} = 0$ and axial symmetry $\Rightarrow$

$$\frac{\partial V_x}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r V_r\right) = 0$$

$\therefore \frac{V_r}{V_x} \sim \frac{\Delta r}{\Delta x} \sim \beta \sim \frac{W}{L} \ll 1$

$\Rightarrow$ need only consider $\hat{x}$ fluid motion

$\therefore$ to compute dipole moment, need $\rho(x')/\rho_0$ for fluid flow accross body



Net $\frac{\dot{\rho}}{\rho_0} = u\left[A + \delta A - A\right] = u\frac{\partial A}{\partial x}dx$

$\Rightarrow \dot{\rho}(x')/\rho_0 = u \frac{\partial A}{\partial x}$

$$\therefore \quad \phi(x) = \frac{1}{4\pi |x|^2} \int dx' x' u \frac{\partial A(x')}{\partial x'}$$

$$= \frac{-u}{4\pi |x|^2} \int dx' A(x')$$

$$= \frac{-u}{4\pi |x|^2} V$$

$V \equiv$ volume of body $= \int dx' A(x')$

$\Rightarrow$ yields intuitive result:

$$\phi(\underline{x}) = \underbrace{-u V_{body}}_{\text{effective dipole moment for slender body.}} / 4\pi r^2$$