Life {after / beyond} the sphere....

Physics 216/116

Winter 2018

Lecture VI – Instabilities ~~III~~ I.

So far – basic ideas of instability and relaxation
– R-T } interfacial instability → in depth.
K-H }
– includes discussion of waves, surface tension, etc.

Acheron Lxl

{Chandrasekhar / Drazin + Reid}

Now
– More on K-H

then – $\boxed{\text{Convection}}$ (R-B) – distributed vorticity
– ideal physics
~ Schwarzchild criterion
– R-B physical picture
~ Rayleigh number Ra.
– R-B eqns
– R-B threshold ⇒ Ra crit
– discussion

{gas / liquid}

– Rotating Convection ⇒ {discussion to Lecture VII with rotation}
– freezing-in law,
– Taylor – Proudman Theorem, implications

- rotating convection (relate HW)

$$\omega^2 = \frac{k_H^2}{k^2}\, g\, \frac{\partial S}{\partial z} + \frac{k_V^2\, 4\Omega^2}{k^2}$$

→ Taylor columns, Proudman pillars.

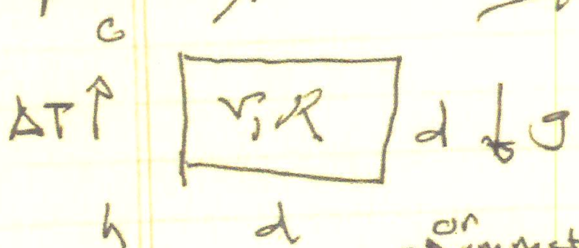
- physics of inertial waves *
- relation to magnetic field, magneto-convection

→ Convection (Rayleigh-Benard)

~ intensively studied → thousands of papers

~ transport of heat in astrophysical bodies, atmosphere, plasma confinement, etc.

~ prototype: Rayleigh-Benard problem

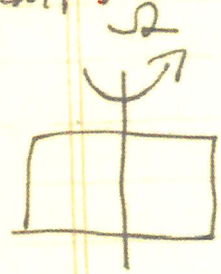


~ no slip b.c.

~ variation on B.C. or asymmetric

- critical ΔT, or Ra (~ΔT), for instability?
- pattern, structure?

variant:



box rotates, often fast.

What is effect of rotation on convection?

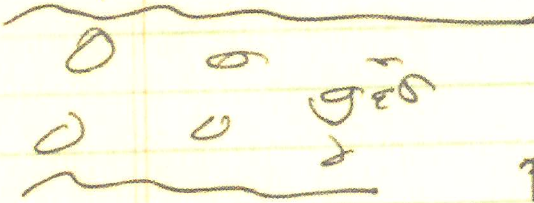
(connects to aim of Module II).

→ Ideal Fluid, ∞ Medium → { Schwarzschild Criterion

i.e. stellar atmosphere

⇓

{ onset convection in astrophysical systems



↑ z

$$\frac{d\rho}{dz} < 0$$

$$\frac{dP}{dz} < 0$$

R-T stable

$$\frac{dP}{dz} = -\rho g \quad (g>0)$$

$P\rho^{-\gamma} \simeq$ const — eqn. state

As basic idea of convection, consider the virtual displacement of a slug/blob of gas upward.

(identity maintenance)

ρ_1' ~ test slug

↑ background profile

$\rho_1' > \rho_2$ → blob sinks, system stable

$\rho_2' < \rho_2$ → blob buoyant, rises.

For infinitesimal displacement,

$\varepsilon \sim \Delta z$

$$\rho_2 = \rho_1 + \frac{\partial \rho}{\partial z}\Delta z$$

For ρ_1' → system obeys equation of state

$$P\rho^{-\gamma} = \text{const.}$$

→ displaced blob (i.e. ①') rapidly comes to pressure equilibration with s

$\delta p \sim 0$ →

blop

what "incompressible" means.

$$\boxed{\frac{\Delta z}{c_s} \ll \tau_{Rise}} \Longleftrightarrow v \ll c_s$$

~ nearly incompressible

∴

$$P_1' = P_2 = P_1 + \Delta z \frac{dP_1}{dz}$$

so

$$P_1 \rho_1^{-\gamma} = P_1' \rho_1'^{-\gamma}$$

→ solve

P_2

$$P_1 \rho_1^{-\gamma} = \left(P_1 + \Delta z \frac{dP_1}{dz}\right) \rho_1'^{-\gamma}$$

so

$$\left(\frac{\rho_1'}{\rho_1}\right)^{\gamma} = 1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}$$

$$\boxed{\begin{aligned}(\rho_1'/\rho_1) &= \left(1 + \frac{\Delta z}{P_1} \frac{dP_1}{dz}\right)^{1/\gamma} \\ &\simeq 1 + \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz}\end{aligned}}$$

$$\boxed{(\rho_2/\rho_1) = 1 + \frac{1}{\rho_1} \Delta z \frac{d\rho_1}{dz}}$$

so blob buoyant if — unstable)

$$\frac{\rho_1'}{\rho_1} < \frac{\rho_2}{\rho_1} \Rightarrow \frac{\Delta z}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{\Delta z}{\rho_1} \frac{d\rho_1}{dz}$$

$$\Rightarrow \boxed{\frac{1}{\gamma} \frac{1}{P_1} \frac{dP_1}{dz} < \frac{1}{\rho_1} \frac{d\rho_1}{dz}}$$

or, as both gradients negative:

$$\frac{1}{\gamma}\left|\frac{1}{P_1}\frac{dP_1}{dz}\right| > \frac{1}{\rho_1}\left|\frac{d\rho}{dz}\right|$$

and as $S = \alpha \ln(P\rho^{-\gamma})$

$$\frac{dS}{dz} = \frac{\alpha}{P}\frac{dP}{dz} - \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

really free energy availability

$\Rightarrow$ buoyant blob - instability - if:

$$\boxed{\frac{dS}{dz} < 0}$$ $\rightarrow$ "superadiabatically stratified"

(blob sinks)

stable - if:

$\frac{dS}{dz} > 0$ $\rightarrow$ "subadiabatically stratified"

$\frac{dS}{dz} = 0$ $\rightarrow$ adiabatically stratified (marginal)

Schwarzschild criterion for convection instability:

$$\boxed{\frac{dS}{dz} < 0}$$

or

$$\boxed{\frac{1}{P}\frac{dP}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}}$$

free energy criterion

or $P = k_B \rho T$

$$\frac{1}{\rho}\frac{d\rho}{dz} + \frac{1}{T}\frac{dT}{dz} < \frac{\gamma}{\rho}\frac{d\rho}{dz}$$

$$\Rightarrow \boxed{\frac{1}{T}\frac{dT}{dz} < \frac{(\gamma-1)}{\rho}\frac{d\rho}{dz}}$$

→ i.e. 'sufficiently steep' temperature gradient, rel. to density

'sufficient' ⇒ $\gamma - 1$

$\gamma \equiv$ captures essential thermal properties

↓

E.O.S.

Note: Convenient to work with S in full analysis, as (in ideal fluid)

$$\frac{\partial S}{\partial t} + \mathbf{v} \cdot \nabla S = 0$$

(isentropic dynamics)

$$s = \langle s \rangle + \tilde{s}$$

$\langle s \rangle$ ↳ mean profile; $\tilde{s}$ → fluctuation

and,

$$\frac{\partial \tilde{s}}{\partial t} + \tilde{v}\frac{\partial \langle s \rangle}{\partial z} = 0$$

Now
$$s \sim \ln(P\rho^{-\gamma})$$
$$\sim \ln\left(T\rho^{-(\gamma-1)}\right)$$

$$\tilde{s} = \frac{1}{T\rho^{-(\gamma-1)}}\left[\tilde{T}\rho^{-(\gamma-1)} + T(-(\gamma-1))\rho^{-\gamma}\tilde{\rho}\right]$$

$$\tilde{s} = \left[\frac{\tilde{T}}{T} - (\gamma-1)\frac{\tilde{\rho}}{\rho}\right]$$

Anticipate: $\underline{\nabla}\cdot\underline{V} = 0$ so no sound wave in convection dynamics

slow rise

$$\delta p = 0 \Rightarrow \delta\rho\, T = -\delta T\, \rho$$

$$\boxed{\tilde{\rho}/\rho = -\frac{\tilde{T}}{T}}$$

→ [relates buoyancy to temperature.

aside on incompressibility, Boussinesq:

Check: $\frac{\partial \underline{v}}{\partial t} = -\frac{\underline{\nabla} p}{\rho} + \underline{g}$

$\underline{\nabla} \cdot \underline{\tilde{v}} \Rightarrow$

$\partial_t \, \underline{\nabla} \cdot \underline{\tilde{v}} = -\frac{\nabla^2 \tilde{p}}{\rho} + \cancel{\underline{\nabla} \underline{g}}$

$\underline{\nabla} \cdot \underline{\tilde{v}} = 0 \Rightarrow \nabla^2 \tilde{p} = 0$

$k^2 \tilde{p}_{\underline{k}} = 0$

so

$\frac{\tilde{\rho}_k}{\rho} = -\frac{\tilde{T}_k}{T}$

N.B.: Essence of "Boussinesq, incompressible" convection is:

① – dynamics slow, relative to sound wave

② – vertical wave vector k s/t $k L_{s,z} \gg 1$

↑ scale height

so,

$$\underset{①}{\frac{\partial \tilde{\rho}}{\partial t}} + \underset{②}{\tilde{v}_z \frac{d\rho_0}{dz}} + \underset{③}{\rho_0 \nabla \cdot \tilde{\underline{V}}} = 0$$

$$\frac{1}{\tau}\frac{\tilde{\rho}}{\rho_0} \qquad \frac{\tilde{v}_z}{L_\rho} \qquad \rho_0 k_z \tilde{v}$$

Take $L_\rho \sim L_s$

$\rightarrow$ $\tau \gg (k c_s)^{-1}$ $\Rightarrow$ drop ①

$\rightarrow$ $k_z L_s \gg 1$ $\Rightarrow$ drop ②

$\boxed{\underline{\nabla} \cdot \tilde{\underline{V}} = 0 \text{ emerges as effective condition.}}$

$\rightarrow$ simplest subsonic extension is:

$\nabla \cdot (\rho \underline{V}) = 0$ $\rightarrow$ incompressible <u>mass</u> flow.

(so called "anelastic eqn.")

$$\underline{\nabla} \cdot \tilde{\underline{V}} + \frac{\tilde{v}_z}{\rho}\frac{d\rho_0}{dz} = 0$$

$\rightarrow$ decouples sound wave

and modify freezing-in law. $\rightarrow$ retains finite scale height.

(show).

⊗

$$\tilde{s} = \gamma \frac{\tilde{T}}{T} \Longleftrightarrow$$ entropy fluctuation tied directly to temperature fluctuation.

Now, at level of estimation: For buoyancy time scale

$$\frac{\partial \tilde{v}_z}{\partial t} = -\frac{\partial_z \tilde{p}}{\rho_0} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$$= g\frac{\tilde{T}}{T_0}\hat{z}$$

$$\frac{\partial \tilde{T}/T_0}{\partial t} = -\tilde{v}_z \frac{\partial S_0}{\partial z}$$

$$= \frac{\tilde{\tau}_b}{\gamma} g \frac{\partial S_0}{\partial z}\tilde{v}_z\hat{z}$$

$\tilde{\tau}_b \rightarrow$ buoyancy time scale

$$\frac{\tilde{v}_z}{\tilde{\tau}_b^2} \sim \frac{1}{\gamma} g \frac{\partial S_0}{\partial z}\tilde{v}_z$$

gives:

$$\boxed{\frac{1}{\tilde{\tau}_b^2} \sim \frac{g}{\gamma}\frac{\partial S_0}{\partial z}}$$

buoyancy time scale.

If entertain now dissipation:

→ viscosity, i.e. momentum diffusion:

$$\partial_t \underline{\tilde{v}} \rightarrow \partial_t \underline{\tilde{v}} - \nu \nabla^2 \underline{\tilde{v}}$$

[smears out rise motion

and $1/\tau_\nu \sim \dfrac{\nu}{l^2}$ → specifies viscous time on scale l

→ thermal diffusivity

heat lost from blob

$$\partial_t \tilde{T} \rightarrow \partial_t \tilde{T} - \underbrace{\kappa}_{\text{thermal conduction/diffusion}} \nabla^2 \tilde{T}$$

and

$$1/\tau_\kappa \sim \kappa/l^2$$ → specifies thermal diffusion time on scale l

And can then note:

Diffusion effects will limit buoyancy
→ i.e. smear out heat parcel (moving) if

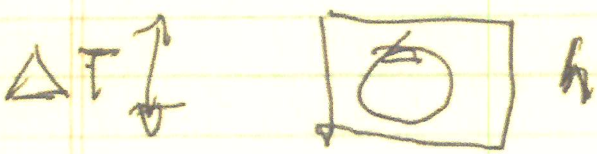
$$1/\tau_b^2 \sim 1/\tau_\nu \tau_\kappa$$

point → instability needs free energy sufficient to overcome dissipation so, taking $\partial s/\partial z$

$$\boxed{\frac{\tau_\nu \tau_\kappa}{\tau_b^2} \sim g \frac{\partial \langle s \rangle}{\partial z} l^4/\nu\kappa \equiv Ra}$$

Rayleigh number.
→ 2nd most popular dimensionless # in fluid mechanics

For convective fluid in a box:



$$\delta\rho = -\alpha\, \delta T$$

coefficient of thermal expansion

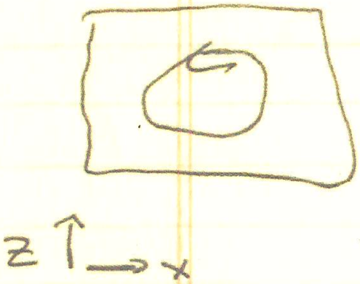
$$\frac{\partial T}{\partial z} \sim \frac{\Delta T}{h}$$

$$\boxed{Ra \equiv g\, \Delta T \alpha h^3/\nu\kappa}$$

clearly need $Ra > \# \sim 1$ for convective instability to occur.

Point of analysis is to determine $(Ra)_{crit}$.

Some calculation



- Consider vorticity $\perp$ to x, z

$\Rightarrow \omega_y$.

- $\underline{v} = \underline{\nabla}\phi \times \hat{y}$

then

$$\underline{\omega}_y = -\partial_x^2\phi - \partial_z^2\phi$$

Linearized ideal eqns:

$$\frac{\partial \tilde{\underline{v}}}{\partial t} = -\frac{\underline{\nabla}\tilde{p}}{\rho} - g\tilde{\rho}\hat{z}$$

$\tilde{\rho}$ fluctuation drives convection

convective eqbm.

i.e.

$$\cancel{\frac{\partial \underline{v}_0}{\partial t}} + \frac{\partial \tilde{\underline{v}}}{\partial t} = \cancel{-\frac{\underline{\nabla} p_0}{\rho_0}} - \frac{\underline{\nabla}\tilde{p}}{\rho_0} - \cancel{g\frac{\rho_0}{\rho_0}\hat{z}} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

hydrostatic eqbm.

Also, "Boussinesq Approximation" (consistent with $\underline{\nabla}\cdot\underline{v} = 0$)

$\Rightarrow$

Boussinesq.

- treat ρ as constant, except where deviation $\tilde{\rho}$ is compared to zero.

- $\tilde{\rho}$ only in buoyancy force

$\Rightarrow$

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla \tilde{p}}{\rho} - g\frac{\tilde{\rho}}{\rho_0}\hat{z}$$

$$= -\frac{\nabla \tilde{p}}{\rho_0} + g\frac{\tilde{T}}{T_0}\hat{z}$$

$\rightarrow \hat{y}\cdot\nabla\times \Rightarrow$

$\rightarrow$ $$\frac{\partial}{\partial t}\left(-\nabla^2\phi\right) = g\frac{\partial}{\partial x}\left(\frac{\tilde{T}}{T_0}\right) - \nu\nabla^2(-\nabla^2\phi)$$

$\nabla^2 = \partial x^2 + \partial z^2$

$\rightarrow$ $$\frac{\partial}{\partial t}\left(\frac{\tilde{T}}{T_0}\right) = -v_z\frac{d\langle s\rangle}{dz} + \chi\nabla^2 T$$

$\rightarrow$ convection roll equations, with B.C. and dissipation effects

$\rightarrow$ ideal: $$\omega^2 = g\frac{(\partial s/\partial z)}{\gamma}\frac{k_x^2}{k^2}$$

c.f. → Chandra. → Chapt. 2

Manneville → Chapt. 3, 4

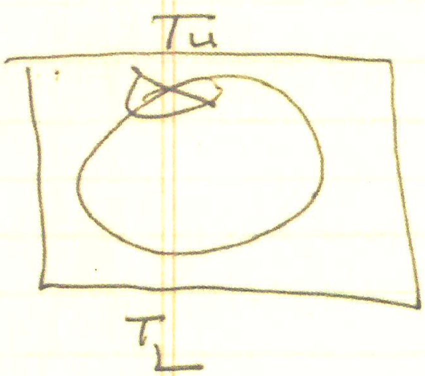
[dissipative structures and weak turbulence]

Now, universal notation for problem given by Chandrasekhar so switch:

$W \to \hat{v}_z$

$\Theta \to \hat{T}/T$ and $\begin{cases} d\rho = -\alpha\, dT \\ \beta \equiv -\dfrac{dT}{dz} = \dfrac{\Delta T}{h} \end{cases}$

$\omega \to \omega_z$



i.e. $\nabla^2 T = 0$ for eqbrm. → linear vertically with b.c. fixed

$\hat{z}\cdot(\nabla \times \nabla \times \text{NSE})$

$$\boxed{\begin{aligned} \frac{\partial}{\partial t}\nabla^2 W &= g\alpha \underbrace{\left(\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2}\right)}_{\nabla_h^2} + \nu\nabla^4 W \\ \frac{\partial \Theta}{\partial t} &= \beta W + \kappa \nabla^2\Theta \end{aligned}}$$

(before)

same, in slightly different notation.

n.b. $v_z = \partial_x \phi \to i k_x \phi$

Now, $T_0 = T_b - \beta z$

$\beta = \Delta T / h$.

Can de-dimensionalize:

length $\rightarrow h$

time $\rightarrow h^2/\kappa$

$h/\tau_\kappa \rightarrow \underline{V}$

$\kappa\nu/\alpha g h^3 \rightarrow T$

$\kappa^2/h^2 \rightarrow p/\rho$

So

$$\boxed{\begin{aligned} \partial_t \nabla^2 w &= P \nabla^4 w + \nabla_h^2 \Theta \\ \partial_t \Theta &= \nabla^2 \Theta + R_a w \end{aligned}}$$

← std.

4 w

2 Θ

↓

B.C.'s

need 6

2 parameters specify system:

$R_a = g \alpha \beta h^3 / \nu \kappa$

$P = \nu/\kappa \rightarrow$ Prandtl #

↳ relative strength of dissipations.

↓ another key dimless #.

Game now becomes:

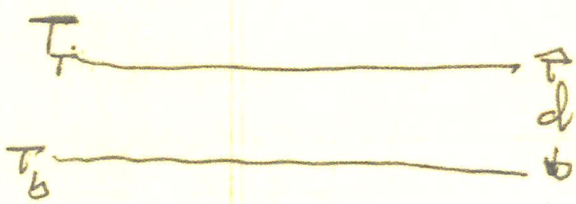
- compute Ra_{crit} for onset.

for

- P value given (~ 1 here)
- given $k_y \rightarrow k_x \Rightarrow$ scanned.
- boundary conditions. $\Rightarrow$ the lesson.

$\Rightarrow$ Task is laborious.

Lesson: Boundary conditions set effect/behavior of dissipation. ✓



assume wide tank, don't worry about lateral B.C.

$\tilde{\Theta} = 0$ at $z = 0, h$. [T fixed] $\rightarrow 2$

$w = v_z = 0$ at $z = 0, h$. [no flow thru wall] $\rightarrow 2$

need 2 more

For other B.C., can envision two scenarios (of several).

3

① → no-slip
② → stress free (Rayleigh – 2 free
(1916) boundaries)

① For no slip: (rigid)

$$-\tilde{v}_z\Big|_{0,h} = \tilde{w}\Big|_{0,h} = 0$$

$$\boxed{\tilde{v}_h\Big|_{0,h} = 0}$$

but worked with w: $\equiv$?? how express?

$$\nabla_h \tilde{v}_h + \partial_z \tilde{v}_z = 0$$

as all ∇_h of $\tilde{v}$ vanish → i.e. all horizontal derivatives of V vanish as V vanishes

$$\nabla_h \tilde{v}_h = 0 \quad \text{so} \quad \partial_z \tilde{v}_z = 0$$

$V_h = 0$
all $\nabla_h V_h = 0$

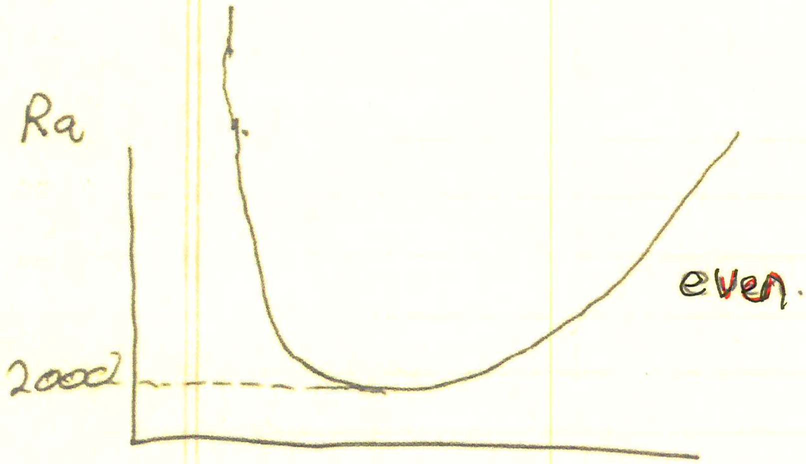
⇒ → other B.C.

$$\boxed{\partial_z W = 0} \quad \text{at } z = 0, H$$

②

so $\partial_z W = 0$
$W = 0$ at $0, h$.
$\Theta = 0$

5a



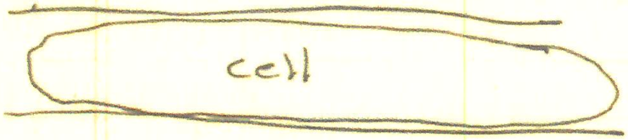
$\alpha = k_h h$ (norm. k_h)

Chandra. 39

$Ra_{crit} \sim 2000$.

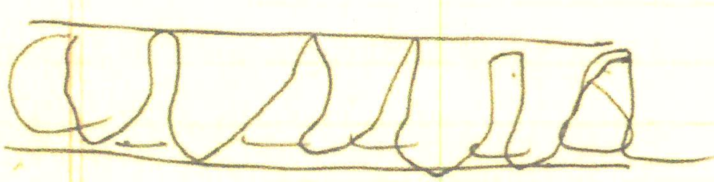
→ high k growth Ra_{crit} due rise $\sim k^2$, etc → $\sim (k_v^2 + k_h^2)$

→ low k branch due:

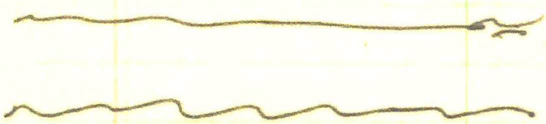


dissipation due visc. effects in upper, lower layer

i.e. above, vs.



② For stress free:



open top (bottom):

$W\big|_{0,h} = 0$, $\Theta\big|_{0,h} = 0$

but free surface: no stress

$\underline{\underline{\tau}} = -\eta\, \partial_z V_h$

shear stress delivered to surface → vanishes for free surface!

but have: $\partial_z V_h = 0$

and $\partial_h V_h = -\partial_z V_z$

so $\partial_z \partial_h V_h = -\partial_z^2 V_z$

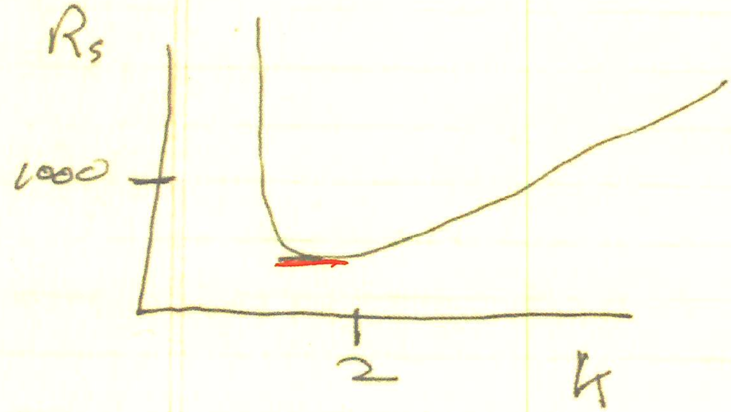
$$\partial_h \partial_z V_h = \boxed{0 = -\partial_z^2 V_z}$$

BC

So b.c. $\partial_z^2 w = 0$ | top, bttm

(replaces: $\partial_z w = 0$ for no slip)

and have:



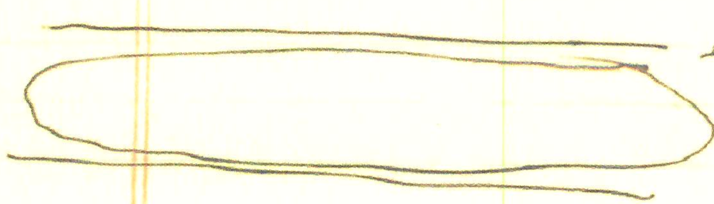
$Ra_{crit} \sim \frac{27\pi^4}{4}$ for $k_c = \pi/\sqrt{2}$

$\frac{27\pi^4}{4}$ ~~vs~~

lower Ra crit!

→ ⇒ { same material, but substantially low Ra crit due stress free B.C.

again, low k rise in Ra crit due:



→ layer dissipn.