Physics 4B — Quiz 5 Solutions — Spring 2018

Problem 1

$$y(x,t) = A\cos(kx - \omega t) = 0.45\cos(2.4x - 18t)$$

$\omega = 18$ rad/s, $k = 2.4\,\mathrm{m}^{-1}$

(a) Wavelength $\lambda = \frac{2\pi}{k} = \frac{2\pi}{2.4}\,\mathrm{m} = \boxed{2.62\,\mathrm{m} = \lambda}$

(b) Frequency $\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi} = \frac{18}{2\pi}\,\mathrm{s}^{-1} = \boxed{2.86\,\mathrm{Hz} = f}$

(c) $v = \frac{\omega}{k} = \frac{18}{2.4}\,\mathrm{m/s} = \boxed{7.5\,\mathrm{m/s} = v}$

(d) Speed of particles in the cord is magnitude of

$$\dot{y}(x,t) = +\omega A\sin(kx - \omega t)$$

Maximum speed: $v_{max} = \omega A = 18 \times 0.45\,\frac{\mathrm{m}}{\mathrm{s}} = \boxed{8.1\,\mathrm{m/s} = v_{max}}$

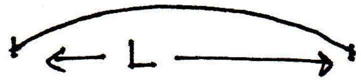
Minimum speed: $\boxed{v_{min} = 0}$

(e) Acceleration of particles in the cord is

$$\ddot{y}(x,t) = -\omega^2 A\cos(kx - \omega t)$$

Maximum acceleration: $a_{max} = \omega^2 A = 18^2 \times 0.45\,\frac{\mathrm{m}}{\mathrm{s}^2} = \boxed{146\,\frac{\mathrm{m}}{\mathrm{s}^2} = a_{max}}$

Problem 2



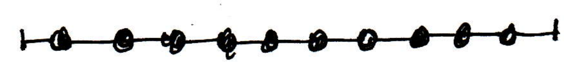
$v = \lambda f = \sqrt{\frac{T}{\mu}}$; $T = 40\,N$, $\mu = \frac{100 g}{1 m} = 100\,g/m \Rightarrow$

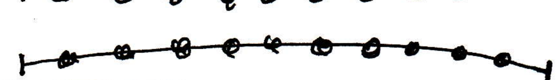
$$\boxed{v = \sqrt{\frac{40\ N \cdot m}{0.1\ kg}} = 20\ m/s}$$

The lowest frequency is for the largest λ. Largest λ is $\lambda_1 = 2L$

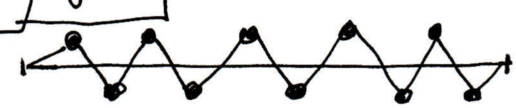
$\lambda_1 = 2L = 2m \Rightarrow f_1 = \frac{v}{\lambda_1} = \frac{20\,m/s}{2\,m} = \boxed{10\,Hz = f_1}$ (a)

(b) $\lambda = \frac{v}{f} = \frac{20\,m/s}{1000\,s^{-1}} = \boxed{2\,cm = \lambda(f = 1000\,Hz)}$

(c) 10 masses instead of continuous string : 

approximately: lowest frequency : 

almost same as continuum $\Rightarrow$ $\boxed{f_1 \simeq 10\,Hz} = f_{min}$

highest frequency : 

so 5 wavelengths in 1 m $\Rightarrow \lambda = 20\,cm \Rightarrow$

$\Rightarrow f \approx \frac{v}{\lambda} = \frac{20\,m/s}{20\,cm} = \boxed{100\,Hz \simeq f_{max}}$

Exactly: $\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right) = 2\pi f_n$

$\omega_0 = \sqrt{\frac{T}{m\ell}}$, $m = 10\,g = 0.01\,kg$, $\ell = 10\,cm = 0.1\,m \Rightarrow \omega_0 = \sqrt{\frac{40}{0.01 \times 0.1}}$ rad/s $\Rightarrow$

$\Rightarrow \boxed{\omega_0 = 200\,rad/s}$, $N = 10 \Rightarrow$ $f_n = \frac{1}{\pi} \cdot \omega_0 \cdot \sin\left(\frac{n\pi}{2 \cdot 11}\right) \Rightarrow$

lowest frequency: $f_1 = \frac{1}{\pi} \cdot 200 \cdot \sin\left(\frac{\pi}{22}\right) Hz = \boxed{9.1\,Hz = f_1}$ min

highest frequency: $f_{10} = \frac{1}{\pi} \cdot 200 \cdot \sin\left(\frac{10\pi}{22}\right) Hz = \boxed{63\,Hz = f_{10}}$ max

Problem 3

$y(x,t) = A\sin(kx)\cos(\omega t)$

$A = 2\,cm,\ k = 0.2\pi\,cm^{-1},\ \omega = 80\pi\,s^{-1}.\ L = 20\,cm,\ m = 40\,g$

(a) The velocity of the string at position x, time t is

$\dot{y}(x,t) = -\omega A\sin(kx)\sin(\omega t)$

For a small element of mass, dm, at position x and time t, kinetic energy is

$dK = \frac{1}{2}dm\,\dot{y}^2 = \frac{1}{2}\cdot dm\cdot \omega^2A^2\sin^2(kx)\sin^2(\omega t)$

Minimum kinetic energy is e.g. for $t=0$, then $\boxed{K=0}$

Maximum kinetic energy is when $\sin(\omega t) = 1$. Integrating over the string,

$K = \frac{1}{2}m\omega^2 A^2\cdot\frac{1}{2} = \frac{1}{4}m\omega^2A^2$ using that $\langle\sin^2(kx)\rangle = \frac{1}{2}$

$\Rightarrow K = \frac{1}{\cancel{4}}\cdot 40g\cdot 80^2\pi^2\cdot\cancel{2}^2\frac{cm^2}{s^2} = \boxed{2.5\times10^6\,erg = K_{max}}$

(b) The wavelength is $\lambda = \frac{2\pi}{k} = \frac{2\pi}{0.2\pi}\,cm = 10\,cm = L/2$

The possible wavelengths are $\lambda_n = 2L/n \Rightarrow n = 4$

$\omega_n = n\omega_1 = 80\pi s^{-1} \Rightarrow \omega_1 = 20\pi s^{-1} = 2\pi f \Rightarrow \boxed{f_1 = 10\,Hz}$

So the three lowest frequencies are $\boxed{10\,Hz,\ 20\,Hz,\ 30\,Hz}$

(c) $\boxed{A\sin(kx)\cos(\omega t) = \frac{A}{2}\sin(kx-\omega t) + \frac{A}{2}\sin(kx+\omega t)}$

speed of wave: $v = \omega/k = 80/0.2\ cm/s = \boxed{400\,cm/s}$

(d) Using the formula: $\bar{P} = 2\pi^2\mu v f^2 A^2$ but A should be $\frac{A}{2}$

$\mu = \frac{40g}{20cm},\ f = 40\,Hz,\ A = 1\,cm \Rightarrow$

$\bar{P} = 2\pi^2\times 2\frac{g}{cm}\times 400\frac{cm}{s}\cdot\frac{40^2}{s^2}\cdot 1\,cm^2 = \boxed{2.5\times10^7\,\frac{ergs}{s}} = \bar{P}$

Derivation of the formula for power $\bar{P}$ for traveling wave:

$$y = A \sin(kx - \omega t)$$

$$\dot{y} = A\omega \cos(kx - \omega t)$$

Consider 1 wavelength: kinetic energy is

$$K = \frac{1}{2} m \dot{y}^2 \quad , \quad m = \mu \lambda \Rightarrow \quad (\mu = \text{mass / unit length})$$

$$K = \frac{1}{2} \mu \lambda A^2 \omega^2 (\cos(kx - \omega t))^2 .$$ On average, $\langle \cos^2 \rangle = 1/2$.

There is also potential energy which is = on average; the average energy in one wavelength is then $\bar{E} = \frac{1}{2} \mu \lambda A^2 \omega^2$, it flows in 1 period $T = \frac{1}{f} \Rightarrow$

$$\Rightarrow \quad \bar{P} = \frac{\bar{E}}{T} = \frac{1}{2} \mu \lambda f A^2 \omega^2 = \boxed{2\pi^2 \mu v f^2 A^2}$$

using that $\omega = 2\pi f$ and $v = \lambda f$

(e) Relation between result in (a) and (d):

In (a), the maximum kinetic energy = average kinetic + potential energy. There are 2 wavelengths in the string, and the standing wave is superposition of 2 traveling waves, so energy per wavelength of traveling wave is on average $\frac{K_{max}}{4}$ in one wavelength. The period is

$T = \frac{1}{f_4}$, $f_4 = f_1 \cdot 4 \Rightarrow f_4 = 40$ Hz, so energy per unit time is

$$\boxed{\bar{P} = \frac{K_{max}}{4} \cdot f_4 = K_{max} \cdot \frac{10}{s}}$$