PHYSICS 220 : GROUP THEORY
FINAL EXAMINATION

This exam is due in my office, 5438 Mayer Hall, at 10 am, Friday, June 15. You are allowed to use the course lecture notes, the Lax text, and the character tables (link from lecture notes web page), but no other sources, and please do not discuss the exam with anyone other than me. If you have any urgent questions regarding the exam, send me email.

[1] Show that the Lie algebra structure constants are given by the expression

$$f_{ab}{}^{c} = \left(\left. \frac{\partial S_b{}^{c}}{\partial x^a} \right|_{\boldsymbol{x}_e} - \left. \frac{\partial S_a{}^{c}}{\partial x^b} \right|_{\boldsymbol{x}_e} \right) \quad ,$$

where

$$S_a{}^{b}(\boldsymbol{x}) = \left. \frac{\partial f^b(\boldsymbol{x}, \boldsymbol{u})}{\partial x^a} \right|_{\boldsymbol{u}=\boldsymbol{x}^{-1}} \quad ,$$

with $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ the group composition function. Thus, the structure constants depend on the parameterization of the associated Lie group G, but are representation-independent.

[2] Consider a chromium ion in a D_{3d} environment.

(a) First consider the case of Cr^{2+}, with electronic configuration $[\mathrm{Ar}]\,3d^4$. Hund's first two rules say that $S = 2$ and $L = 2$ (D). According to Hund's third rule, what is the atomic ground state term?

(b) The character table for the double group D'_{3d} is given in Tab. 1. With an even number of electrons, only the unbarred elements, which comprise D_{3d}, need be considered[1]. Ignoring spin-orbit, decompose D into IRREPs of D_{3d} (the decomposition will be the same as in D'_{3d}). Then decompose the $\Gamma_{S=2}$ spin representation into IRREPs of D_{3d}. Finally, decompose the product $\Gamma_2 \times \mathrm{D} = {}^5\mathrm{D}$ into IRREPs of D_{3d}. You may find the tables in the first appendix to chapter six to be useful.

(c) Starting on the dominant LS coupling end, decompose ^{5}D first into $\mathsf{O}(3)$ IRREPs via addition of angular momentum. Then decompose your result into D'_{3d} IRREPs and show your result agrees with that of part (b).

(d,e,f) Repeat parts (a), (b) and (c) for Cr^{3+}, with electronic configuration $[\mathrm{Ar}]\,3d^3$, where Hund's first two rules tell us $S = \frac{3}{2}$ and $L = 3$. Now you need to worry about the double group.

Hint : This problem is quite similar to problem 3 on the Spring 2016 exam, the solutions of which are available on the Spring 2018 Homework page. Studying that solution should help you approach this problem, but please note that the group considered in S16 was D_4, which is a proper point group, hence there was no need to evaluate the parity η. D_{3d} is not a proper point group.

[1] Remember D_{3d} has 12 elements and D'_{3d} has 24 elements since it is the double group of D_{3d}.

D'_{3d}	E	$3C'_2$	$2C_3$	I	$3\sigma_d$	$2S_6$	$\overline{E}$	$3\overline{C}'_2$	$2\overline{C}_3$	$\overline{I}$	$3\overline{\sigma}_d$	$2\overline{S}_6$
A_{1g}	1	1	1	1	1	1	1	1	1	1	1	1
A_{2g}	1	-1	1	1	-1	1	1	-1	1	1	-1	1
E_g	2	0	-1	2	0	-1	2	0	-1	2	0	-1
A_{1u}	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1
A_{2u}	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
E_u	2	0	-1	-2	0	1	2	0	-1	-2	0	1
Θ_g	1	i	-1	1	i	-1	-1	$-i$	1	-1	$-i$	1
Θ_g^*	1	$-i$	-1	1	$-i$	-1	-1	i	1	-1	i	1
Δ_g	2	0	1	2	0	1	-2	0	-1	-2	0	-1
Θ_u	1	i	-1	-1	$-i$	1	-1	$-i$	1	1	i	-1
Θ_u^*	1	$-i$	-1	-1	i	1	-1	i	1	1	$-i$	-1
Δ_u	2	0	1	-2	0	-1	-2	0	-1	2	0	1

Table 1: Character table for D'_{3d}

[3] The n-string braid group B_n has $(n-1)$ generators $\{\tau_1, \ldots, \tau_{n-1}\}$ obeying the relations

$$\begin{aligned} \tau_i\,\tau_j &= \tau_j\,\tau_i \qquad \text{if } |i-j|>1 \\ \tau_i\,\tau_{i+1}\,\tau_i &= \tau_{i+1}\,\tau_i\,\tau_{i+1} \qquad (1\le i\le n-2) \quad . \end{aligned}$$

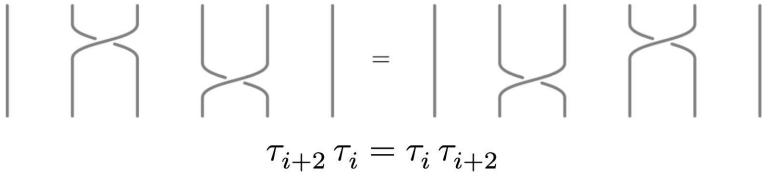
(a) Find all one-dimensional unitary representations of B_n, for all n.

(b) Find all two-dimensional unitary representations of B_3. Recall that a general element $g \in \mathsf{SU}(2)$ may be written as $g = a + ib\hat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}$, where $a^2+b^2=\hat{\boldsymbol{n}}^2=1$ and $\boldsymbol{\sigma}$ are the Pauli matrices. Recall also that $\sigma^a\,\sigma^b = \delta^{ab} + i\epsilon_{abc}\,\sigma^c$.

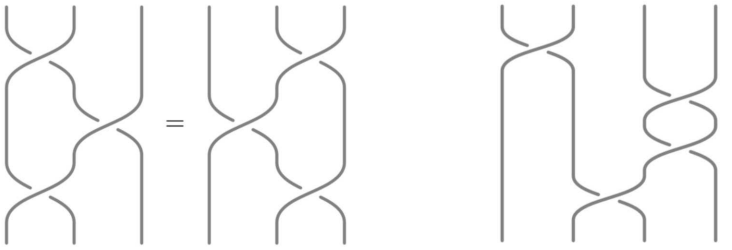
(c) Show that

$$\tau_i = \left[\begin{array}{c|ccc|c} \mathbb{I}_{i-1} & 0 & 0 & 0 & \\ \hline 0 & 1-t & t & 0 & \\ 0 & 1 & 0 & 0 & \\ \hline 0 & 0 & 0 & \mathbb{I}_{n-i-1} & \end{array}\right] \quad , \tag{1}$$

comprises an n-dimensional representation of B_n for any $t \in \mathbb{R}$. This is known as the *Burau representation*. The Burau representation is faithful for $n=2$ and $n=3$, but is known to be not faithful for $n \ge 5$. Whether it is faithful for $n=4$ is an open problem.



$$\tau_{i+2}\,\tau_i = \tau_i\,\tau_{i+2}$$



$$\tau_i\,\tau_{i+1}\,\tau_i = \tau_{i+1}\,\tau_i\,\tau_{i+1} \qquad \tau_1\,\tau_3\,\tau_3\,\tau_2$$

Figure 1: Braid group relations and construction of a braid group element.

[4] The point groups D_{4d} and D_{6d} are relevant to molecular chemistry, but are not among the 32 crystallographic point groups. Why not? `[50 quatloos extra credit]`