PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #6

(1) Consider the equation of state

$$p\sqrt{v^2 - b^2} = RT\,\exp\!\left(-\frac{a}{RTv^2}\right).$$

(a) Find the critical point (v_c, T_c, p_c).

(b) Defining $\bar{p} = p/p_c$, $\bar{v} = v/v_c$, and $\bar{T} = T/T_c$, write the equation of state in dimensionless form $\bar{p} = \bar{p}(\bar{v}, \bar{T})$.

(c) Expanding $\bar{p} = 1 + \pi$, $\bar{v} = 1 + \epsilon$, and $\bar{T} = 1 + t$, find $\epsilon_{liq}(t)$ and $\epsilon_{gas}(t)$ for $-1 \ll t < 0$.

(2) You are invited to contemplate the model

$$\hat{H} = -J \sum_{\langle ij \rangle} \hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j$$

on a regular lattice of coordination number z, where each local moment $\hat{\boldsymbol{n}}_i$ can take on one of $2n$ possible values: $\hat{\boldsymbol{n}}_i \in \{\pm\hat{\boldsymbol{e}}_1, \ldots, \pm\hat{\boldsymbol{e}}_n\}$, where $\hat{\boldsymbol{e}}_i \cdot \hat{\boldsymbol{e}}_j = \delta_{ij}$. You may assume $J > 0$.

(a) Making the mean field *Ansatz* $\boldsymbol{m} = \langle \hat{\boldsymbol{n}}_i \rangle$, find the dimensionless free energy density $f(m, \theta)$, where $\theta = k_B T/zJ$ and $f = F/NzJ$.

(b) Consider two possible orientations for the moment: $\boldsymbol{m}_A = m\,(1, 0, \ldots, 0)$, in which the moment lies along one of the $\hat{\boldsymbol{e}}_i$ directions, and $\boldsymbol{m}_B = m\left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$, in which the moment makes an angle $\cos^{-1}\left(\frac{1}{\sqrt{n}}\right)$ with each of the $\hat{\boldsymbol{e}}_i$. Which configuration will have the lower free energy?

(c) Analyze the mean field theory and show that for $n \le 3$ there is a second order transition. Find the critical temperature $\theta_c(n)$.

(d) Show that for $n > 3$ the transition is first order. Numerically obtain $\theta_c(n)$ for $n = 4, 5, 6$.

Hint: The case $n = 3$ is examined in example problem 7.16.

(3) A *ferrimagnet* is a magnetic structure in which there are different types of spins present. Consider a sodium chloride structure in which the A sublattice spins have magnitude S_A and the B sublattice spins have magnitude S_B with $S_B < S_A$ (*e.g.* $S = 1$ for the A sublattice but $S = \frac{1}{2}$ for the B sublattice). The Hamiltonian is

$$\hat{H} = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j + g_A \mu_0 H \sum_{i \in A} S_i^z + g_B \mu_0 H \sum_{j \in B} S_j^z$$

where $J > 0$, so the interactions are antiferromagnetic.

(a) Work out the mean field theory for this model. Assume that the spins on the A and B sublattices fluctuate about the mean values

$$\langle \boldsymbol{S}_{\rm A} \rangle = m_{\rm A}\, \hat{\boldsymbol{z}} \qquad , \qquad \langle \boldsymbol{S}_{\rm B} \rangle = m_{\rm B}\, \hat{\boldsymbol{z}}$$

and derive a set of coupled mean field equations of the form

$$\begin{aligned} m_{\rm A} &= F_{\rm A}(\beta g_{\rm A} \mu_0 H + \beta J z m_{\rm B}) \\ m_{\rm B} &= F_{\rm B}(\beta g_{\rm B} \mu_0 H + \beta J z m_{\rm A}) \end{aligned}$$

where z is the lattice coordination number ($z = 6$ for NaCl) and $F_{\rm A}(x)$ and $F_{\rm B}(x)$ are related to Brillouin functions.

(b) Show graphically that a solution exists, and find the criterion for broken symmetry solutions to exist when $H = 0$, *i.e.* find $T_{\rm c}$. Then linearize, expanding for small $m_{\rm A}$, $m_{\rm B}$, and H, and solve for $m_{\rm A}(T)$ and $m_{\rm B}(T)$ and the susceptibility

$$\chi(T) = -\frac{1}{2}\frac{\partial}{\partial H}(g_{\rm A}\mu_0 m_{\rm A} + g_{\rm B}\mu_0 m_{\rm B})$$

in the region $T > T_{\rm c}$. Does your $T_{\rm c}$ depend on the sign of J? Why or why not?