PHYSICS 210A : EQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #5

(1) Consider a spin-1 Ising chain with Hamiltonian

$$\hat{H} = -J \sum_n S_n \, S_{n+1} \ ,$$

where each S_n takes possible values $\{-1, 0, 1\}$.

(a) Find the transfer matrix for the this model.

(b) Find an expression for the free energy $F(T, J, N)$ for an N-site chain and for an N-site ring.

(c) Suppose a magnetic field term $\hat{H}' = -\mu_0 H \sum_n S_n$ is included. Find the transfer matrix.

(2) Consider an N-site Ising ring, with N even. Let $K = J/k_{\rm B}T$ be the dimensionless ferromagnetic coupling ($K > 0$), and $\mathcal{H}(K, N) = H/k_{\rm B}T = -K \sum_{n=1}^{N} \sigma_n \, \sigma_{n+1}$ the dimensionless Hamiltonian. The partition function is $Z(K, N) = \text{Tr}\, e^{-\mathcal{H}(K,N)}$. By 'tracing out' over the even sites, show that

$$Z(K, N) = e^{-N'c} \, Z(K', N') \ ,$$

where $N' = N/2$, $c = c(K)$ and $K' = K'(K)$. Thus, the partition function of an N site ring with dimensionless coupling K is related to the partition function *for the same model* on an $N' = N/2$ site ring, at some *renormalized* coupling K', up to a constant factor.

(3) For each of the cluster diagrams in Fig. 1, find the symmetry factor s_γ and write an expression for the cluster integral b_γ.

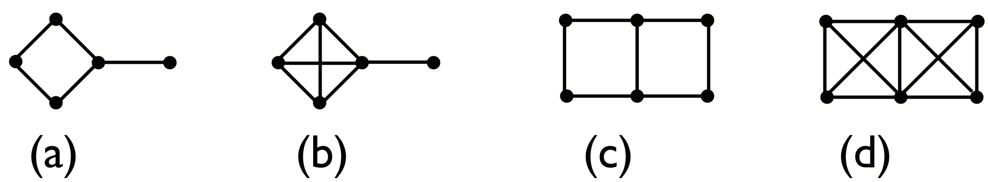

Figure 1: Cluster diagrams for problem 1.

(4) The grand potential for an interacting system in a finite volume V is given by

$$\Xi(z) = (1 + z)^M \prod_{j=1}^{j} \frac{1 - (z/\sigma_j)^{L_j + 1}}{1 - (z/\sigma_j)} \quad .$$

(a) Find all the zeros of $\Xi(z)$ in the complex plane, along with their orders.

(b) Define the normalized density of states like function,

$$g(\sigma) = \frac{1}{L} \sum_{j=1}^{J} L_j \, \delta(\sigma - \sigma_j) \quad ,$$

with $L = \sum_{j=1}^{j} L_j$. In the thermodynamic limit, take $V \to \infty$, $M \to \infty$, $L_j \to \infty$ with $v_0 \equiv V/M$ and $\alpha \equiv L/M$ constant. Then define the dimensionless density $\nu = N v_0 / V$ and dimensionless pressure $\pi \equiv p v_0 / k_B T$. Derive expressions for $\nu(z)$ and $\pi(z)$ in terms of z, α, and the function $g(\sigma)$. *Hint: you may find it helpful to consult Example Problem 6.12.*

(c) Suppose $g(\sigma) = A\,(b-\sigma)^t\,\Theta(b-\sigma)$ with $A = (t+1)\big/ b^{t+1}$ and $t > -1$. Show that there is a phase transition at all values of $b > 0$, and find expressions for $\nu_c(b)$ and $\pi_c(b)$.

(d) Find the leading singularity in $\pi(\nu)$ as a function of $(\nu - \nu_c)$ on either side of the critical point (*i.e.* for $\nu < \nu_c$ and $\nu > \nu_c$).