PHYSICS 210A : EQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #2

(1) Consider a single classical anharmonic oscillator with the Hamiltonian

$$h = \frac{p^2}{2m} + V(q)$$

with $V(q) = \frac{1}{2}\kappa q^2 - \frac{1}{3} b q^3 + \frac{1}{4} u q^4$.

(a) Write down an expression for the partition function $\zeta(T)$. You may the p integral exactly, but you will have to leave the rest of the expression in the form of an integral over q.

(b) At low temperatures, the amplitude of the thermal oscillations is small. This means that you may perturb in the coefficients b and u. Find the free energy up to terms of order T^2.

(c) Write down an expression for the heat capacity $c(T)$ valid to first order in T.

(2) A nonlinear quantum oscillator has the Hamiltonian

$$h = \hbar\omega(n + \tfrac{1}{2}) + \lambda\hbar\omega(n + \tfrac{1}{2})^2 \quad .$$

Here λ is the dimensionless anharmonicity.

(a) Find the partition function $\zeta(u, \lambda)$ to first order in λ, where $u \equiv \hbar\omega / k_{\rm B} T$.

(b) Find the heat capacity to the same order in λ.

(3) Consider a one-dimensional system of free identical nonrelativistic particles of mass m, each of which is endowed with an internal degree of freedom S which takes on values $S \in \{-1, 0, 1\}$. The Hamiltonian is

$$H = \sum_{j=1}^{N} \left(\frac{p_j^2}{2m} - h S_j \right) \quad ,$$

where h is a magnetic field (with dimensions of energy). The linear dimension of the system is L.

(a) Find the Helmholtz free energy $F(T, L, N, h)$.

(b) Find the magnetic susceptibility $\chi_{MM} = \frac{1}{L} \frac{\partial M}{\partial h}$, where $\hat{M} = \sum_j S_j$ is the magnetization operator and $M = \langle \hat{M} \rangle$. Do not set $h = 0$ at the end of the calculation.

(c) Define the operator $\hat{Q} = \sum_{j=1}^{N} S_j \, p_j$. Let λ be the conjugate force. Find the susceptibility $\chi_{QQ}(T, h = 0, \lambda = 0)$.

(d) Working in the grand canonical ensemble, find an expression for $\Omega(T, L, \mu, h, \lambda)$.

(e) What is the lowest order term in h and λ which contributes to the cross-susceptibility $\chi_{QM} = -\frac{1}{L}\frac{\partial^2\Omega}{\partial h\,\partial\lambda}$?

(4) Atoms and ions with partially filled shells experience a magnetic field according to the effective Hamiltonian

$$H_{\text{eff}} = g_{\text{L}}\mu_{\text{B}}\,\boldsymbol{J}\cdot\boldsymbol{H}/\hbar \quad ,$$

where $\mu_{\text{B}} = e\hbar/2mc$ is the Bohr magneton (m is the electron mass), J is the total angular momentum, and

$$g_{\text{L}} = \tfrac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

is the Landé g-factor.

(a) For a gas or solution of N such objects in a volume V, find an expression for the magnetization density, $m = \frac{M}{V} = -\frac{N}{V}\frac{\partial F}{\partial H}$ at finite H and T. Show that $m = n\gamma J B_J(J\gamma H/k_{\text{B}}T)$, where $n = N/V$, $\gamma = g_{\text{L}}\mu_{\text{B}}$, and $B_J(x)$ is a function called the *Brillouin function*. Find and sketch $B_J(x)$ for a few different values of J.

(b) Taking the limit $H \to 0$, you should find $m = \chi H$, where $\chi(T)$ is the magnetic susceptibility. Find $\chi(T)$.