PHYSICS 210A : EQUILIBRIUM STATISTICAL PHYSICS
HW ASSIGNMENT #1

(0) Read chapters 1 and 3 of the lecture notes. If you have time, look through some of the example problems and solutions.

(1) A six-sided die is loaded in such a way that it is twice as likely to yield an even number than an odd number when thrown.

(a) Find the distribution $\{p_n\}$ consistent with maximum entropy.

(b) Assuming the maximum entropy distribution, what is the probability that three consecutive rolls of this die will total up to seven?

(2) Show that the Poisson distribution $P_\nu(n) = \frac{1}{n!}\,\nu^n\,e^{-\nu}$ for the discrete variable $n \in \mathbb{Z}_{\geq 0}$ tends to a Gaussian in the limit $\nu \to \infty$.

(3) Frequentist and Bayesian statistics can sometimes lead to different conclusions. You have a coin of unknown origin. You assume that flipping the coin is a Bernoulli process, *i.e.* the flips are independent and each flip has a probability p to end up heads and probability $1-p$ to end up tails.

(a) You perform 14 flips of the coin and you observe the sequence $\{\textsf{HHTHTHHHTTHHHH}\}$. As a frequentist, what is your estimate of p?

(b) What is your frequentist estimate for the probability that the next two flips will each end up heads? If offered even odds, would you bet on this event?

(c) Now suppose you are a Bayesian. You view p as having its own distribution. The likelihood $f(\text{data}|p)$ is still given by the Bernoulli distribution with the parameter p. For the prior $\pi(p)$, assume a Beta distribution,

$$\pi(p|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\,p^{\alpha-1}\,(1-p)^{\beta-1} \quad .$$

where α and β are hyperparameters. Compute the posterior distribution $\pi(p|\text{data},\alpha,\beta)$.

(d) What is the posterior predictive probability $f(\textsf{HH}\,|\,\text{data},\alpha,\beta)$?

(e) Since *a priori* we don't know anything about the coin, it seems sensible to choose $\alpha=\beta=1$ initially, corresponding to a flat prior for p. What is the numerical value of the probability to get two heads in a row? Would you bet on it?

(4) Professor Jones begins his academic career full of hope that his postdoctoral work, on relativistic corrections to the band structure of crystalline astatine under high pressure, will eventually be recognized with a Nobel Prize in Physics. Being of Bayesian convictions, Jones initially assumes he will win the prize with probability θ, where θ is uniformly distributed on $[0,1]$ to reflect Jones' ignorance.

(a) After N years of failing to win the prize, compute Jones's chances to win in year $N+1$ by performing a Bayesian update on his prior distribution.

(b) Jones' graduate student points out that Jones' prior is not parameterization-independent. He suggests Jones redo his calculations, assuming initially the Jeffreys prior for the Bernoulli process. What then are Jones' chances after his N year drought?

(c) Professor Smith, of the Economics Department, joined the faculty the same year as Jones. His graduate research, which concluded that poor people have less purchasing power than rich people, was recognized with a Nobel Prize in Economics[1] in his fifth year. Like Jones, Smith is a Bayesian, whose initial prior distribution was taken to be uniform. What is the probability he will win a second Nobel Prize in year 11? If instead Smith were a frequentist, how would he assess his chances in year 11?

(5) Consider a system of N real degrees of freedom $x_j \in \mathbb{R}$ with energy $E = -JM^3/6N^2$, where $M = \sum_{j=1}^{N} x_j$. The vector $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ is constrained to lie on a sphere of radius $\sqrt{N}$ *i.e.* $\sum_{j=1}^{N} x_j^2 = N$.

(a) Evaluate the density of states like function,

$$D(\Lambda, N) = \int\limits_{-\infty}^{\infty} dx_1 \cdots \int\limits_{-\infty}^{\infty} dx_N \; \delta\Big(\Lambda - \sum_{j=1}^{N} x_j^2\Big)$$

by using the Laplace transform method outlined in chapter 4 of the lecture notes. You may assume that N is even, so there is no branch cut to consider when evaluating the inverse Laplace transform.

(b) Evaluate the second density of states like function,

$$D(\Lambda, M, N) = \int\limits_{-\infty}^{\infty} dx_1 \cdots \int\limits_{-\infty}^{\infty} dx_N \; \delta\Big(\Lambda - \sum_{j=1}^{N} x_j^2\Big)\, \delta\Big(M - \sum_{j=1}^{N} x_j\Big) \Big/ D(\Lambda, N) \quad . \tag{1}$$

Note that $\int\limits_{-\infty}^{\infty} dM \; D(\Lambda, M, N) = 1$.

(c) Let the Hamiltonian of our system be $H = -JM^3/6N^2$. The partition function is defined to be

$$Z(\beta, N) = \int\limits_{-\infty}^{\infty} dM \; \mathcal{D}(M, N)\, e^{-\beta H} \quad ,$$

[1] Strictly speaking, there is no such thing as a "Nobel Prize in Economics". Rather, there is a "Nobel Memorial Prize in Economic Sciences".

where $\mathcal{D}(M,N) = D(\Lambda = N, M, N)$, *i.e.* the vector $\boldsymbol{x} = \{x_1, \ldots, x_N\}$ is constrained to lie along an $(N-1)$-dimensional sphere of radius $\sqrt{N}$. Show that $Z(\beta, N)$ may be written as

$$Z(\beta, N) = \int_{-1}^{1} dm\, e^{-N f(m,\theta) + \mathcal{O}(\ln N)} \quad ,$$

where $\theta = k_{\rm B} T/J$ is the dimensionless temperature. In the limit $N \to \infty$, we are licensed to compute the integral by the steepest descents approximation, which entails finding the minimum of $f(m, \theta)$ as a function of m for fixed θ.[2]

(d) Sketch $f(m,\theta)$ *versus* m for $\theta = 0.25$, $\theta = 0.185$, $\theta = 0.1703$, and $\theta = 0.15$. Comment on how the minimum value of m evolves as a function of θ. *Hint*: You should have found that the minimum $m_{\rm min}(\theta)$ changes *discontinuously* at a critical temperature $\theta_{\rm c}$. Later on in the course, we will learn how this is the hallmark of a *first order phase transition*.

(e) What is the entropy per degree of freedom, $s(m)$, in the limit $N \to \infty$?

[2] The model you have just solved is called the *three spin spherical model*.