PHYSICS 210A : STATISTICAL PHYSICS
FINAL EXAM

(1) Consider the analog of the van der Waals equation of state for a gas if diatomic particles with *repulsive* long-ranged interactions,

$$p = \frac{RT}{v-b} + \frac{a}{v^2} \quad ,$$

where v is the molar volume.

(a) Does this system have a critical point? If not, give your reasons. If so, find (T_c, p_c, v_c).

(b) Find the molar energy $\varepsilon(T, v)$.

(c) Find the coefficient of volume expansion $\alpha_p = v^{-1}(\partial v/\partial T)_p$ as a function of v and T.

(d) Find the adiabatic equation of state in terms of v and T. If at temperature T_1 a volume $v_1 = 3b$ of particles undergoes reversible adiabatic expansion to a volume $v_2 = 5b$, what is the final temperature T_2?

(2) Consider a two-dimensional gas of ideal nonrelativistic fermions of spin-$\frac{1}{2}$ and mass m.

(a) Find the relationship between the number density n, the fugacity $z = \exp(\mu/k_B T)$, and the temperature T. You may choose to abbreviate $\lambda_T = \sqrt{2\pi\hbar^2/mk_B T}$. Assume the internal degeneracy (*e.g.*, due to spin) is g.

(b) A two-dimensional area A is initially populated with nonrelativistic fermions of mass m, spin-$\frac{1}{2}$, and average number density $n = N/A$ at temperature T. The fermions are noninteracting with the exception that opposite spin fermions can pair up to form spin-0 bosons of mass $2m$ and binding energy Δ. In other words, the fermion dispersion is $\varepsilon_f(\boldsymbol{k}) = \hbar^2\boldsymbol{k}^2/2m$ and the boson dispersion is $\varepsilon_B(\boldsymbol{k}) = -\Delta + \hbar^2\boldsymbol{k}^2/4m$. Assuming the reaction $f\!\uparrow + f\!\downarrow \rightleftharpoons B$ has achieved equilibrium, find the relationship between the initial number density n, fugacity z, and temperature T. *Hint: The total mass density of the system $\rho_{tot} = mn$ is conserved. Use this to first find the relation between the equilibrium densities n_f, n_B, and n.*

(c) Assuming the conditions in (b), in the limit $n\lambda_T^2 \gg 1$ at fixed T, what are the fermion and boson densities n_f and n_B, to leading order?

(d) Now suppose the initial particles are spin-0 bosons of mass m, which undergo the reaction $2b \rightleftharpoons B$, where B is a boson of mass $2m$. The initial density is again n. What is the relation between n, T, and z? What are n_b and n_B to leading order when $n\lambda_T^2 \gg 1$?

(3) On each site i of a (two-dimensional square) lattice exists a unit vector $\hat{\boldsymbol{n}}_i$ which can point in any of four directions: $\{\pm\hat{\boldsymbol{x}}, \pm\hat{\boldsymbol{y}}\}$. These vectors interact between neighboring sites. Of the $4^2 = 16$ configurations, two have energy $-J$ and the remaining 14 have energy zero. The nonzero energy configurations for horizontal and for vertical links are shown here:

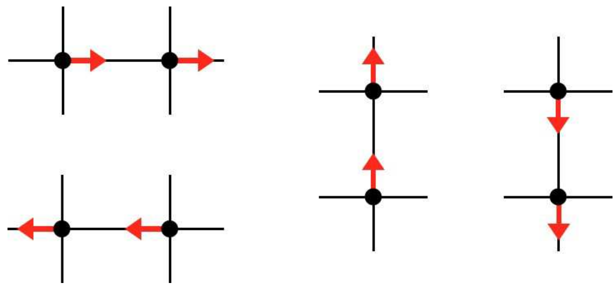

Figure 1: For both horizontal and vertical links, there are only two configurations with energy $E_{ij} = -J$, depicted here.

Consider a variational density matrix approach to this problem, based on the single site density matrix

$$\varrho_1(\hat{\boldsymbol{n}}) = \tfrac{1}{4}(1+3x)\,\delta_{\hat{\boldsymbol{n}}\,,\,\hat{\boldsymbol{x}}} + \tfrac{1}{4}(1-x)\,\delta_{\hat{\boldsymbol{n}}\,,\,-\hat{\boldsymbol{x}}} + \tfrac{1}{4}(1-x)\,\delta_{\hat{\boldsymbol{n}}\,,\,\hat{\boldsymbol{y}}} + \tfrac{1}{4}(1-x)\,\delta_{\hat{\boldsymbol{n}}\,,\,-\hat{\boldsymbol{y}}} \quad ,$$

where x is a variational parameter.

(a) What is the allowed range for x? Verify that the density matrix ϱ_1 is appropriately normalized.

(b) Taking $\varrho_{\rm var}\big(\{\hat{\boldsymbol{n}}_i\}\big) = \prod_i \varrho_1(\hat{\boldsymbol{n}}_i)$, find the average energy E. (Please denote the total number of lattice sites by N.)

(c) Find the entropy S.

(d) Find the dimensionless free energy per site $f \equiv F/NJ$ in terms of the variational parameter x and the dimensionless temperature $\theta \equiv k_{\rm B}T/J$.

(e) Find the Landau expansion of $f(x,\theta)$ to fourth order in x. *Hint:*

$$(1+\varepsilon)\ln(1+\varepsilon) = \varepsilon + \tfrac{1}{2}\varepsilon^2 - \tfrac{1}{6}\varepsilon^3 + \tfrac{1}{12}\varepsilon^4 - \tfrac{1}{20}\varepsilon^5 + \ldots \quad .$$

(f) *Based on the fourth order Landau expansion of the free energy*, sketch the equilibrium curve of x *versus* θ and identify the location(s) any and all phase transitions, as well as their order(s).

(4) Provide brief but accurate answers to each of the following:

(a) For a single-component system, the Gibbs free energy G is a function of what state variables? Write its differential and all the Maxwell equations resulting from consideration of the mixed second derivatives of G.

(b) A system of noninteracting spins is cooled in a uniform magnetic field H_1 to a temperature T_1. The external field is then adiabatically lowered to a value $H_2 < H_1$. What is the final value of the temperature, T_2?

(c) For a two-level system with energy eigenvalues $\varepsilon_1 < \varepsilon_2$, the heat capacity vanishes in both the $T \to 0$ and $T \to \infty$ limits. Explain physically why this is so. What will happen in the case of a three-level system?

(d) Sketch the phase diagram of the $d = 2$ Ising model in the (T, H) plane. Identify the critical point and the location of all first order transitions. Then make a corresponding sketch for the $d = 1$ Ising model.