The Vlasov Equation

→ For gas, (no mean field)

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f = C(f) \quad \rightarrow \text{Boltzmann Eqn.}$$

For plasma:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f + \frac{q}{m}\underline{E}\,\frac{\partial f}{\partial v} = C(f) \rightarrow$$

Boltzmann Eqn

where:

$$\nabla\cdot\underline{E} = 4\pi q\int dv\, f$$

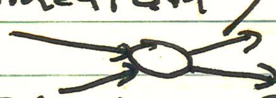
$$C(f) = C(\underset{\text{test}}{f},\ \underset{\hookrightarrow\ \text{field.}}{f})$$

$-\ C(f) \Rightarrow$ Boltzmann H-Thm

$$\frac{dS}{dt} \geq 0$$

$C(f) = 0 \rightarrow$ Local Maxwellian f.

H-Thm: $\left\{\begin{array}{l} \text{microscopic reversibility} \\ \text{momentum / energy conservation} \\ \\ f(1,2) = f(1)\,f(2) \end{array}\right.$



$$-C(f) = \begin{cases} \sim \sigma v_{rel} \text{ --- gas} \\ \sim \text{F-P operator, plasma} \\ -\frac{\partial}{\partial \underline{v}} \cdot \left[\underline{V}\langle f \rangle - \frac{\partial}{\partial \underline{v}} \cdot \underline{\underline{D}} \langle f \rangle \right] \\ \qquad \text{(glancing collns)} \\ \underline{\underline{D}}, \underline{V} \sim \langle f \rangle \end{cases}$$

Now, time scales:

$$\underbrace{\frac{\partial f}{\partial t}}_{\omega} + \underbrace{\underline{v} \cdot \frac{\partial f}{\partial \underline{x}}}_{v_{th}/L} + \underbrace{\frac{q}{m} \underline{\underline{E}} \cdot \frac{\partial f}{\partial \underline{v}}}_{\frac{q}{m} E / v_{th}} = \underbrace{C(f)}_{\nu}$$

so if $\omega, v_{th}/L \gg \nu$, dynamics described by:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{q}{m} \underline{E} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

Vlasov or Collisionless Boltzmann Eqn.

closed by Maxwell Eqns.

- in practice:

$$f = \langle f \rangle + \delta f$$

$\langle f \rangle$ → unperturbed; δf → perturbation

$\langle f \rangle$ set by collisional processes

$\langle f \rangle \to \langle f_0 \rangle$ — Maxwellian

$\delta f \to$ collisionless perturbation dynamics
$\to$ Vlasov

- n.b. $f(x, \underline{v}, t) \to$ all moments info.

- Vlasov Eqn. is statement of conservation of phase space density along particle orbits

i.e. $f \Leftrightarrow \rho$ density (phase space)

$$\frac{\partial f}{\partial t} + \underbrace{\underline{v}}_{d\underline{x}/dt} \cdot \frac{\partial f}{\partial \underline{x}} + \underbrace{\frac{q}{m} \underline{E}}_{d\underline{v}/dt} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$d\underline{x}/dt$ $d\underline{v}/dt$

characteristic Eqns — Hamiltonian EoM's.

$$\frac{\partial f}{\partial t} + \frac{d\underline{x}}{dt} \cdot \frac{\partial f}{\partial \underline{x}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

so

$$\nabla \cdot \underline{F} \Rightarrow \frac{df}{dt} = 0$$

- Collisions violate phase space density conservation
- obviously $dS/dt = 0$ for Vlasov Eqn.

→ Analogy:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = -\rho \cancel{\underline{\nabla} \cdot \underline{v}}$$

incompressible flow $= 0$

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla} \rho = \frac{d\rho}{dt} = 0$$

$$\begin{cases} \rho \text{ const. along trajectories} \\ \dfrac{d\underline{x}}{dt} = \underline{v}(\underline{x}, t) \end{cases}$$

- Connection:

Vlasov eqn. is $\underline{1}$ particle Liouville eqn. for Hamiltonian system

with: $d\underline{x}/dt = \underline{v}$

$d\underline{v}/dt = \underline{U}$

i.e. Liouville Thm $\underline{\nabla} \cdot \underline{V}_{\Gamma} = 0$.

— BUT: Plasma obeys Liouville Thm for $N \ggg 1$ particles

i.e. $N \sim 6.023 \times 10^{23}$.

How go from $f_N \to f$?

i.e.

$$\frac{dx_i}{dt} = v_i = p_i/m$$

$$\frac{d\underline{p}_i}{dt} = -\underline{\nabla}_i \left(\sum_{j \neq i} q/|x_i - x_j| \right)$$

Hamiltonian system

then Liouville Eqn:

$f_N = f_N(t, \underline{x}_1, v_1, \ldots \underline{x}_N, v_N) \to$ N body distribution

$$\partial_t f_N + \{H, f_N\} = 0$$

— How simplify: BBGKY Hierarchy

B - Bogoliubov
B - Born
G - Green
K - Kirkwood
Y - Yvon

$\uparrow t$

- strategy:
  - construct a hierarchy

$$\frac{\partial}{\partial t} f_{N-1} \ldots = \int dx \int dv \, \underline{L} \, f_N$$

- truncate: - RHS small
  - weak interaction

why weak:

Diluteness
- Gas: $d^3 n \lll 1$
- Plasma: $n \lambda_D^3 \gg 1$
  $T \gg e^2 n^{1/3}$

- Outline:

$$\partial_t f_{N-1} + \underbrace{L_{N-1} f_{N-1}}_{\text{evolution}} = -\underbrace{\int d\Gamma_N \, L_N f_N}_{\text{interaction of } N \text{ with all } N-1}$$

rpt

$$\frac{\partial}{\partial t} f_2 + L_2 f_2 = -\int d\Gamma_3 \, L_3 f_3$$

so for RHS negligible:

→ easy to see for gas:

$$\int dV\, \underline{\nabla} V_{i,N} \cdot \frac{\partial f_N}{\partial v} \sim \frac{\text{interaction volume}}{\text{total volume}}$$

↓

volume integral $\sim \dfrac{h^3 (N-1)}{V}$

$\sim n h^3 \ll 1$

kill RHS.

→ so

$$\partial_t f_2 + L_2 f_2 = 0$$

$f(1,2) \to f(1) f(2)$

→ diluteness

now

$$\underbrace{\partial_t f_1 + L_1 f_1}_{\text{LHS} \to \text{Vlasov operator}} = -\int d\Gamma_2\, L_2 f_2$$

$L_2 f_2 \to f(1,2)$

↓

Collision operator.

⇒ arrive at Boltzmann Eqn.

so

$$\frac{\partial f(1)}{\partial t} + L_1 f(1) = -\int d\Gamma_2 \, L_2 \, f(1,2)$$

but $\frac{d}{dt} f(1,2) = 0$

$$= -\int d\Gamma_2 \, L_2 \, f(1) \, f(2)$$

$$= \underbrace{C[f(1)]}_{\text{collision operator}}$$

$f(1,2)\Big|_{t=0}$ random (dilute), $\frac{d f(1,2)}{dt} = 0$

$\Rightarrow$ particles remain uncorrelated

so

$$\partial_t f(1) + L_1 f(1) = C(f(1))$$

— Boltzmann Eqn.

$C \to 0$

$$\partial_t f(1) + L_1 f(1) = 0$$

— Vlasov Eqn.

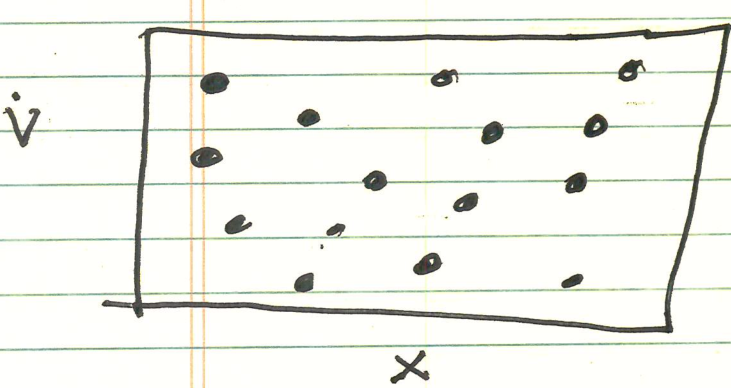
→ BBGKY hierarchy is
<u>weak</u> <u>correlation</u> expansion

weak: $\left\{ \begin{array}{l} n d^3 \ll 1 \\ 1/n\lambda_D^3 \Rightarrow \dfrac{e^2 n^{1/3}}{T} \ll 1 \end{array} \right.$

→ weak correlation allows:

- neglect h.o. correlation ($N > 2$)
- factorization of $f(1,2) \to f(1)\, f(2)$

→ Cartoon picture

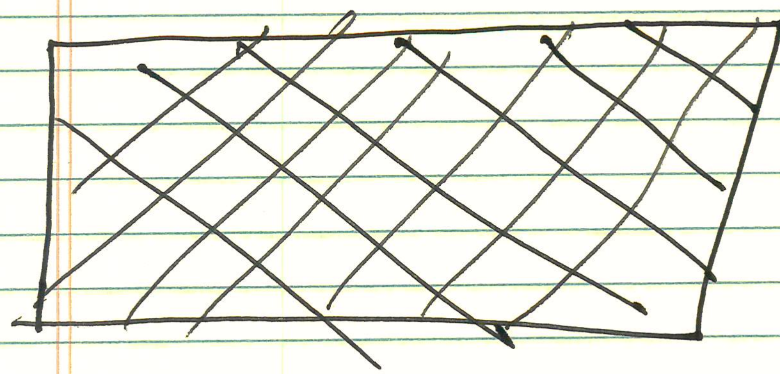


N particles
→ N 'peas'

→ f_N evolves
N peas in phase space.

crush peas ⇒
make pea soup

pea soup
(incompressible)



so, continuity equation for pea soup is:

$$\frac{d}{dt} \rho(\underline{x}, \underline{v}, t) = 0$$

$\Rightarrow$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0$$

Vlasov Eqn.

Obviously, integrations in BBGKY
$\Rightarrow$ "crushing the pea" (coarse graining)

- Boltzmann Eqn. retains husk/shell remnant.

N.B.

- Vlasov + Poisson System:

  Phase Space Fluid Dynamics

- Inviscid (pure) → ! ?

  N.B. In practice, any physical calculation introduces resolution scale → effective collisionality (albeit weak).

- Analogy:

  2D Fluid:

$\underline{\omega} = \underline{\nabla} \times \underline{V}$ → vorticity

$$\partial_t \underline{\omega} = \underline{\nabla} \times \underline{V} \times \underline{\omega} + \nu \nabla^2 \underline{\omega}$$

2D $\qquad \underline{\omega} = \omega(x,y)\,\hat{z}$

$$\partial_t \underline{\omega} + \underline{V} \cdot \nabla \underline{\omega} = \nu \nabla^2 \underline{\omega}$$

on

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z}\cdot\nabla\nabla^2\phi = \nu\nabla^2\nabla^2\phi$$

on

$$\partial_t \rho + \underline{v}\cdot\underline{\nabla}\rho = \nu\nabla^2\rho$$

$$\nabla^2\phi = \rho$$

- Vlasov Eqn. used heavily in stellar dynamics. See Binney and Tremaine

→ BBGKY questionable → 1 chaos
→ no ?

→ "Equilibrium" Collisionless ?!

- BGK solution - stationary Vlasov Soln.

$$\cancel{\frac{\partial f}{\partial t}} + \underline{v}\cdot\nabla f + \underline{\nabla\phi}\cdot\frac{\partial f}{\partial v} = 0$$

$$\nabla^2\phi = 4\pi G\int f\, dv$$

$f = f(IOM)$ → sets dist.

how constrain? - Violent Relaxation
Lynden-Bell.