Lecture V : Kinetics – A Crash Course, I

Here, concerned with:

1) Thermal equilibrium fluctuations and Fluctuation - Dissipation Theorem
2) Pdf evolution, transport, diffusion.

1.) Thermal Equilibrium Fluctuations

– simplest possible dynamics question

⇒ what is spectrum of thermal equilibrium fluctuations in plasma?

– result:

(Spectrum) ~ (Dissp.) T

⇒ Fluctuation - Dissipation Theorem!

– follows from emission - absorption balance.

consider Brownian Motion (simplest)

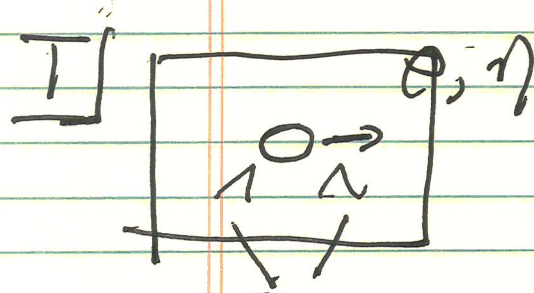
$$m \frac{d\underline{v}}{dt} = -\gamma \underline{v} + \underline{f}$$

Langevin Eqn.

here: $\tilde{f}$ → rndm forcing due thermal flctns

$$\gamma = 6\pi\eta l$$

→ stokes Drag



$\tilde{f}$ → thermal flucts

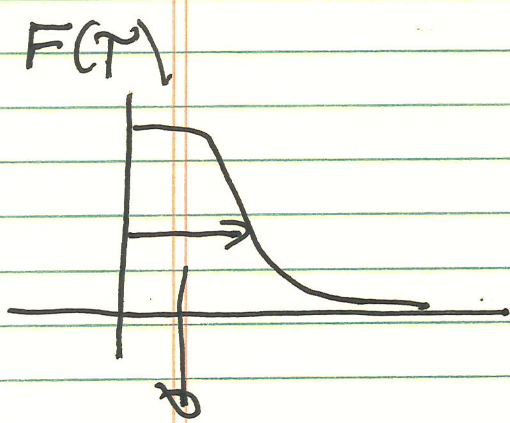
as $\tilde{f}$ random →

$$\langle \tilde{f}(t_1)\,\tilde{f}(t_2)\rangle = f_0^2\, F(t_2 - t_1)$$

f_0^2 → strength

$F(t_2-t_1)$ → stationary series

$t_2 - t_1$ → τ



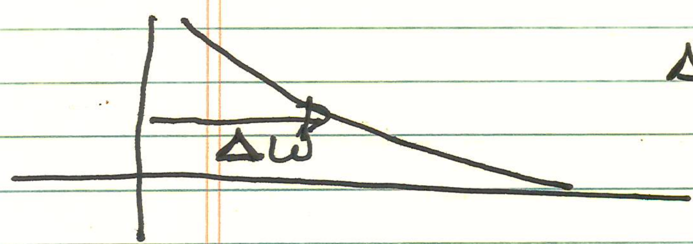
$\tau_{c} \equiv$ self-correlation / coherence time

↓

spectral autocorrelation time

$$\int e^{+i\omega\tau} F(\tau) = F(\omega)$$

forcing spectrum (in frequency)



$\Delta\omega \equiv$ bandwidth

$$\Delta\omega\, \tau_{ac} \sim 1 \Rightarrow \tau_{ac} \sim 1/\Delta\omega$$

so, for white noise: $\Delta\omega \to \infty$, $\tau_{ac} \to$ small

$$\langle \tilde{f}(t_1)\tilde{f}(t_2)\rangle = 2|\tilde{f}|^2 \tau_{ac}\, \delta(t_2 - t_1)$$

time order. dims delta - correlated

Now, to solve for motion:

$$\frac{d\tilde{v}}{dt} + \frac{\gamma}{m}\tilde{v} = \frac{\tilde{f}}{m}$$

$$\tilde{v}(t) = e^{-\frac{\gamma}{m}t}\,\tilde{v}(0) + \int_0^t dt'\, e^{-\frac{\gamma}{m}(t-t')}\,\frac{\tilde{f}(t')}{m}$$

so

$$|\tilde{v}|^2 = e^{-2\frac{\gamma}{m}t}|\tilde{v}(0)|^2 + \text{cross terms}$$

T

$$+ \int^t dt' \, e^{-\frac{\gamma}{m}(t-t')} \frac{\tilde{f}(t')}{m} \int^t dt'' e^{-\frac{\gamma}{m}(t-t'')} \frac{\tilde{f}(t'')}{m}$$

so

$$\langle|\tilde{v}|^2\rangle = e^{-\cancel{2}\frac{\gamma}{m}t}|\tilde{v}(0)|^2 + \int_0^t dt' \int_0^t dt'' \, e^{-\frac{\gamma}{m}(t-t'')} e^{-\frac{\gamma}{m}(t-t')} * \langle \tilde{f}(t') \tilde{f}(t'') \rangle / m^2$$

ensemble, statistical average

but $\langle \tilde{f}(t_1) \tilde{f}(t_2) \rangle = 2|\tilde{f}_0|^2 \tau_{ac} \, \delta(t_2 - t_1)$

$$\langle|\tilde{v}|^2\rangle = e^{-2\frac{\gamma}{m}t}\langle|\tilde{v}(0)|^2\rangle + \int_0^t dt' \int_0^t dt'' \left[e^{-\frac{\gamma}{m}(t-t'')} e^{-\gamma(t-t')} * \frac{2|\tilde{f}_0|^2 \tau_{ac}}{m^2} \delta(t'-t'') \right]$$

integrating:

$$= e^{-2\frac{\gamma}{m}t}\langle|\tilde{v}(0)|^2\rangle + e^{-2\frac{\gamma}{m}t} \frac{2|\tilde{f}_0|^2 \tau_{ac}}{m^2} \frac{1}{2\frac{\gamma}{m}} \left(e^{2\frac{\gamma}{m}t} - 1\right)$$

$$\langle|\tilde{v}|^2\rangle = e^{-\frac{2\gamma}{m}t}\langle|v(0)|^2\rangle + \frac{\cancel{2}|\tilde{f}_0|^2\tilde{\tau}_{ac}}{\cancel{2}\gamma m}\left(1-e^{-\frac{2\gamma}{m}t}\right)$$

so, for long time: $\gamma t >> 1$

$$\langle|\tilde{v}|^2\rangle \cong \frac{|\tilde{f}_0|^2\tilde{\tau}_{ac}}{\gamma m}$$

but, as particle in thermal bath (fluid) at T:

$$\frac{m\langle|\tilde{v}|^2\rangle}{2} = T$$

$$T \cong \frac{|\tilde{f}_0|^2\tilde{\tau}_{ac}}{2\gamma}$$

$$\Rightarrow \boxed{\gamma T = \frac{|\tilde{f}_0|^2}{2}\tilde{\tau}_{ac}}$$

simple version: Fluctuation-Dissipation Theorem.

i.e.

(Noise) ~ (Damping) T

– clearly,

$$\text{Noise} \sim |\tilde{f}_0|^2 \tilde{\tau}_{ac}$$

$$\text{Damping} \sim \gamma$$

– given 2 of 3, deduce the third.

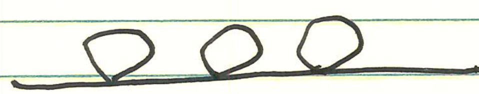
– equilibrium: → emission by noise

→ absorption by damping

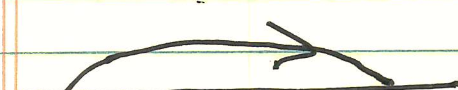
∴ balance matches T.

N.B.: Additional assumption: Linear system

→ emission and absorption at same scale

i.e. 

not



Alternatively:

① $\partial_t \tilde{v} + \frac{\gamma}{m}\tilde{v} = \frac{\tilde{f}}{m}$

$$\partial_t \cancel{\frac{\langle \tilde{v}^2 \rangle}{2}} + \frac{\gamma}{m}\langle \tilde{v}^2 \rangle = \frac{\langle \tilde{f}\tilde{v} \rangle}{m}$$

stationary:

$$\gamma \langle \tilde{v}^2 \rangle = \langle \tilde{f}\tilde{v} \rangle$$

but $\tilde{v}(t) = e^{-\gamma/m\, t}\,\tilde{v}(0) + \int_0^t e^{-\frac{\gamma}{m}(t-t')}\frac{\tilde{f}(t')}{m}dt'$

$\Rightarrow$

$$\langle \tilde{v}^2 \rangle = \frac{2T}{m} = \frac{\cancel{m}}{\gamma}\left\langle \tilde{f}\int_0^t dt'\, e^{-\frac{\gamma}{m}(t-t')}\frac{\tilde{f}(t')}{m}\right\rangle$$

$$\langle \tilde{f}(t)\tilde{f}(t') \rangle = 2|\tilde{f}_0|^2 \tau_{ac}\, \delta(t-t')$$

$$\langle \tilde{v}^2 \rangle = \frac{\cancel{2}T}{\cancel{m}} = \left(\frac{\cancel{2}}{\gamma}\right)\frac{|\tilde{f}_0|^2 \tau_{ac}}{\cancel{m}}$$

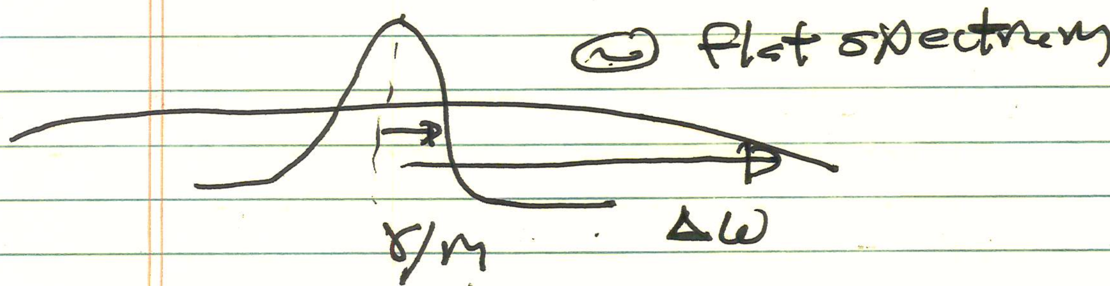
$$\gamma T = 2|\tilde{f}_0|^2 T_{a\omega} \qquad \text{(ignoring \#)}$$

② Transform in time:

$$(-i\omega + \gamma/m)\,\tilde{v}_\omega = \tilde{f}_\omega/m$$

$$|\tilde{v}_\omega|^2 = |\tilde{f}_\omega|^2 / m^2\left(\omega^2 + (\gamma/m)^2\right)$$

for white noise:



$$\Delta\omega \gg \gamma/m$$

$$\int d\omega\, |\tilde{v}_\omega|^2 = \int \frac{d\omega\, |\tilde{f}_\omega|^2}{m^2\left(\omega^2 + (\gamma/m)^2\right)}$$

$\cong \dfrac{|\tilde{f}_\omega|^2}{m^2}$ [redacted] $\displaystyle\int d\omega \Big/ \left(\omega^2 + \left(\frac{\gamma}{m}\right)^2\right)$

$$\cong \frac{|\tilde{f}_\omega|^2}{m^2}\,\frac{m}{\gamma} \quad (\#)$$

Now, integrating:

$$\int d\omega \, |\tilde{V}_\omega|^2 = |\tilde{V}|^2 \cong T/M$$

$$\int d\omega \, |\tilde{f}_\omega|^2 d\omega \sim \Delta\omega \, |\tilde{f}_\omega|^2$$

$$\sim |\tilde{f}_0|^2$$

$\Delta\omega \, \tau_{ac} \sim 1$

$$|\tilde{f}_\omega|^2 \sim \tau_{ac} |\tilde{f}_0|^2$$

$\int d\tau \langle \tilde{F}^2(\tau) \rangle$

$\Downarrow$

$\int d\tau \langle F(0) F(\tau) \rangle$

$= \int d\omega \langle \tilde{F}^2(\omega) \rangle$

$\langle \tilde{F}^2 \rangle_\omega = \int e^{-\omega\tau} \langle \tilde{F}^2(\tau) \rangle$

so

$$\frac{T}{\cancel{M}} \sim \frac{|f_0|^2 \tau_{ac}}{\cancel{M^2}} \frac{M}{\gamma}$$

$$\boxed{|\tilde{f}_0|^2 \tau_{ac} \sim \gamma T}$$

Generally: $$|\tilde{V}_\omega|^2 = \frac{|\tilde{f}_\omega|^2/M^2}{|r(\omega)|^2}$$

$\Downarrow$

response function

(damping $\Longleftrightarrow$ width)

$$\frac{I}{m} = \int d\omega \, \frac{\overbrace{|\tilde{a}(\omega)|^2}^{\text{acceleration}}}{\underbrace{|r(\omega)|^2}_{\text{response}}}$$

if now harmonically bound particle:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \tilde{f}/m$$

$$|\tilde{x}_\omega|^2 = \frac{|\tilde{f}_\omega|^2/m^2}{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}$$

$$|\tilde{x}_\omega|^2 = \frac{|\tilde{f}_\omega|^2/m^2}{|r_{real}(\omega)|^2 + |r_{Im}(\omega)|^2}$$

and

$$\int |\tilde{x}_\omega|^2 = \langle \tilde{x}^2 \rangle$$

$$2\left(\frac{1}{2} k \langle \tilde{x}^2 \rangle\right) = 2\left(\frac{m\omega_0^2}{2} \langle \tilde{x}^2 \rangle\right) = T$$

so

$$T = m\omega_0^2 \langle \tilde{x}^2 \rangle = m\omega_0^2 \int d\omega \frac{|\tilde{a}_\omega|^2}{|r_r(\omega)|^2 + |r_{IM}(\omega)|^2}$$

$$\frac{1}{|r|^2} = \frac{1}{(\omega-\omega_0)^2 \left(\frac{\partial r}{\partial \omega}\right)^2 + |r_{IM}(\omega_0)|^2}$$

→ "pole approx"

→ expansion abt resonance $\left\{ \begin{array}{l} \text{resonance} \\ \text{damping} \end{array} \right.$

for flat $|\tilde{a}_\omega|^2$

$$T = m\omega_0^2 |\hat{a}_{\omega_0}|^2 \int \frac{d\omega}{(\omega-\omega_0)^2 |\dot{r}|^2 + |r_{IM}(\omega_0)|^2}$$

$$= m\omega_0^2 \frac{|\tilde{f}_\omega|^2}{m^2} \pi \Big/ |r_{IM}(\omega_0)| \left|\frac{\partial r_r}{\partial \omega}\right|_{\omega_0}$$

so key to FDT relation:

→ collective modes

→ mode damping

So, for Plasma:

Collective Modes, Damping → Vlasov Eqn.

Warm Plasma Wave ✓

Landau damping

Langevin Eqn for Plasma case?

$$m \frac{dv}{dt} = qE$$

$$\frac{dv}{dt} = \frac{q}{m} \tilde{E}(x,t) \quad , \quad \frac{dx}{dt} = v$$

stochastic (field from "other" particles)

Damping? → Collective

→ Particles emit (damped) waves

⇒ Maintains steady state.