Lecture II – Introduction to Collective Dynamics (Fluid)

A.) Basic Collective Modes → {Basic Collective Response

1.) EM
2.) Warm Plasma → The Players (Animals)
3.) Ion Acoustic

1.) EM

– well known ↔ Qual Problem

– $k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)$

$\epsilon(\omega) = 1 - \omega_p^2/\omega^2$

→ dispn. relation $\omega^2 = \omega_{p0}^2 + c^2k^2$

→ cut-off at $\omega = \omega_{p0}$ → $n_{crit.}$

Work out by linearizing:

$$m_e n \frac{d\underline{v}}{dt} = qn\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right) - \cancel{\nabla P_0}$$

$$\partial_t n + \nabla\cdot(n\underline{v}) = 0$$

+ Maxwell Eqns.

2) Warm Plasma Wave $\left\{ \begin{array}{l} \text{Bohm-Gross} \\ \text{Langmuir} \end{array} \right.$

$\underline{k} \cdot \underline{E} = kE \neq 0 \rightarrow$ electrostatic

- ions stationary
- w/o propagation $\rightarrow$ plasma oscillation $\omega = \omega_{pe}$

$\Downarrow$

pressure

$$n m_e \frac{d\underline{v}}{dt} = qn\underline{E} - \underline{\nabla} p$$

$$\sim \quad m_e \partial_t \underline{\tilde{v}} = q\underline{\tilde{E}} - \frac{\underline{\nabla}\tilde{p}}{n_0}$$

$$\partial_t n + \underline{\nabla} \cdot (n\underline{v}) = 0 \qquad\qquad q = -|e|$$

$$\sim \quad \partial_t \tilde{n} = -n_0 \underline{\nabla} \cdot \underline{\tilde{v}}$$

$$\sim \quad \underline{\nabla} \cdot \underline{\tilde{E}} = 4\pi q \tilde{n}$$

For p: equation of state

$$p = p_0 (n/n_0)^\gamma \quad \text{- adiabatic} \qquad (\text{fast})$$

$$p = \tilde{n} T \quad \text{- isothermal} \qquad (\text{slow})$$

$\Rightarrow$

$$\partial_t^2 \tilde{n} = -n_0 \, \nabla \cdot \partial_t \tilde{v}$$

$$\partial_t^2 \tilde{n} = -n_0 \left(\frac{q \nabla \cdot \tilde{E}}{m_e} - \frac{\nabla^2 \tilde{p}}{n_0 m} \right)$$

as $\nabla \cdot E = 4\pi q \tilde{n}$

$$\partial_t^2 \tilde{n} = -\omega_p^2 \tilde{n} + \frac{1}{m_e} \nabla^2 \tilde{p}$$

$$= -\omega_p^2 \tilde{n} + \frac{\gamma T_e}{m_e} \nabla^2 \tilde{n}$$

$$\boxed{\partial_t^2 \tilde{n} = -\omega_p^2 \tilde{n} + \frac{\gamma T_e}{m_e} \nabla^2 \tilde{n}}$$

$$\omega^2 = \omega_p^2 + \gamma k^2 v_{the}^2$$

$$\boxed{\omega^2 = \omega_p^2 \left(1 + \gamma k^2 \lambda_D^2 \right)}$$

Debye length

- $k \lambda_D \ll 1 \rightarrow$ plasma oscillation
- $k \lambda_D \gg 1 \rightarrow$ electron pressure wave

$\gamma \to 1$ simplicity

$$\nabla^2 \tilde{n} - \frac{1}{\lambda_D^2}\tilde{n} = \frac{1}{v_t^2}\frac{\partial^2 \tilde{n}}{\partial t^2}$$

$\Downarrow$

$\omega \to 0$ recovers
screened charge field.

N.B.: - Warm plasma wave propagates carrier wave momentum

- combines $\left\{ \begin{array}{l} \text{electron acoustic} \\ \text{plasma oscillation} \end{array} \right.$

aside: (Gas Dynamic) Sound

$$\partial_t \rho + \underline{\nabla} \cdot (\rho \underline{V}) = 0$$

$$\rho \frac{d\underline{V}}{dt} = -\underline{\nabla} P \qquad P = P_0 (\rho/\rho_0)^\gamma$$

$$\rho = \rho_0 + \tilde{\rho}$$

$\Rightarrow$

$$\rho_0 \partial_t \tilde{\underline{V}} = -\underline{\nabla}\tilde{P} = -\frac{P_0}{\rho_0}\gamma \underline{\nabla}\tilde{\rho}$$

$$\partial_t \tilde{\rho} = -\rho_0 \underline{\nabla}\cdot\tilde{\underline{V}}$$

$$\rho_0 \partial_t \underline{\nabla}\cdot\tilde{\underline{V}} = -\partial_t^2 \tilde{\rho} = -c_s^2 \nabla^2 \tilde{\rho}$$

$c_s^2 = \gamma P_0 / \rho_0$ → speed of sound.

$$\partial_t^2 \tilde{\rho} = c_s^2 \nabla^2 \tilde{\rho}$$

$$\omega^2 = c_s^2 k^2$$

B) Ion Acoustic Wave

- so far, 'single species' dynamics
  now 2 species – "2 fluid wave"
- consider: (contrast gas, MHD)

$$v_{th_i} < \frac{\omega}{k} < v_{Te}$$

↓ phase velocity

→ electrons screen ion plasma oscillation!

→ "acoustic" wave $\begin{cases} \text{electron pressure} \\ \text{ion inertia} \end{cases}$

Warm electrons: $\tilde{p} = \tilde{n}\, T_0$

(isothermal → slow electron temp. equalizes)

$$\partial_t \tilde{n} = -n_0 \underline{\nabla} \cdot \underline{\tilde{v}}$$

$$m_e \partial_t \underline{\tilde{v}} = |e| \underline{\nabla} \tilde{\phi} - T_e \frac{\underline{\nabla} \tilde{n}}{n_0}$$

$\Rightarrow$

$$\underbrace{\partial_t^2 \tilde{n}}_{O(\omega^2)} - \underbrace{v_{The}^2 \nabla^2 \tilde{n}}_{O(k^2 v_{The}^2)} = -n_0 \frac{|e|}{m_e} \nabla^2 \tilde{\phi}$$

(inertia) (compression)

so, for $k^2 v_{The}^2 \gg \omega^2$

$$\boxed{\frac{\tilde{n}_{k,\omega}}{n_0} = \frac{|e| \tilde{\phi}_{k,\omega}}{T_e}}$$

Boltzmann Response!

- limit electron inertia negligible
- obtain from kinetics.
- corrections;

$$\left(-\omega^2 + k^2 v_{The}^2\right) \tilde{n} = n_0 \frac{|e|}{m_e} k^2 \tilde{\phi}$$

$$\tilde{n}/n = \frac{|e| \tilde{\phi}}{T_e} \left[1 \Big/ \left(1 - \omega^2 / k^2 v_{The}^2\right) \right]$$

For ions: (take cold)

$$\partial_t \tilde{n} = -n_0 \underline{\nabla} \cdot \tilde{\underline{v}}$$

$$\partial_t \tilde{v}_i = \frac{|e|}{m_i} \tilde{\underline{E}}$$

$$\Rightarrow \frac{\partial^2 \tilde{n}}{\partial t^2} = \frac{n_0 |e|}{M_i} \nabla^2 \tilde{\phi}$$

$$\boxed{\frac{\tilde{n}_{i\,\underline{k},\omega}}{n_0} = \frac{|e|}{M_i} \frac{k^2}{\omega^2} \tilde{\phi}_{\underline{k},\omega}}$$

$$\nabla^2 \phi = -4\pi \rho$$

$$k^2 \tilde{\phi}_{\underline{k}} = 4\pi n_0 |e| \left(\frac{|e|}{M_i} \frac{k^2}{\omega^2} \tilde{\phi}_{\underline{k}} - \frac{|e| \tilde{\phi}_{\underline{k}}}{T_e} \right)$$

$$k^2 = \underbrace{\frac{\omega_{pi}^2}{\omega^2} k^2}_{\text{ion plasma}} - \underbrace{\frac{\omega_{pe}^2}{v_{Te}^2}}_{\to 1/\lambda_D^2}$$

screening!

$$\left(k^2 + 1/\lambda_D^2 \right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

$$\boxed{\omega^2 = k^2 c_s^2 / 1 + k^2 \lambda_{De}^2}$$

$$\boxed{c_s^2 = T_e / M_i}$$

ion-acoustic wave

N.B.:

→ Compare/contrast to gas-dynamic acoustic wave

	Hydro	2-Component Plasma
"spring"	p	T_e
"inertia"	ρ_0	M_i

$\left\{ \text{electrons short out ions} \right.$

→ ion acoustic wave is two component, hybrid oscillation

$$\left(k^2 + 1/\lambda_D^2\right) = \frac{\omega_{pi}^2}{\omega^2} k^2$$

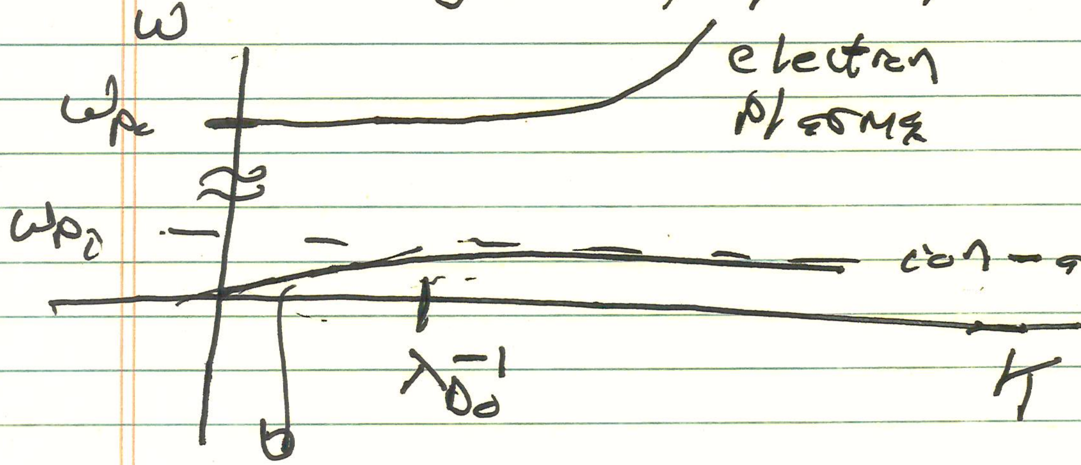
$$\underbrace{\left(1 + 1/k^2\lambda_D^2\right)}_{\text{Debye screening } (T_e)} = \underbrace{\frac{\omega_{pi}^2}{\omega^2}}_{\text{ion plasma oscillation}}$$

$k\lambda_D \ll 1 \rightarrow$ quasi-neutral $\quad \omega^2 = k^2 C_s^2$

so see ion acoustic wave as Debye shielded ion plasma oscillation

$k^2 \lambda_D^2 \gtrsim 1 \qquad \omega \to \omega_{pi}^2$

→ So: Basic electrostatic modes of unmagnetized plasma



$\omega \sim c_s k$

Scales:
- ω_{pe}, ω_{pi}
- λ_{De}
- V_{Th}, c_s

N.B: Many waves/instabilities of practical interest → Boltzmann electrons + fluid ions

B.) Energetics [and Quasi-Particle Picture]

- Wave Energy and Momentum necessary for understanding collective dynamics
- step toward quasi-particle picture

Seek general 'Poynting' theorem for plasma waves: i.e. of form:

$$\partial_t W + \nabla \cdot \underline{S} + Q = 0$$

energy density W — energy density flux $\underline{S}$ — dissipation Q.

- each 2nd order in amplitude
- "wave" $\Rightarrow$ includes $\left\{ \begin{array}{l} \text{medium} \\ \text{field} \end{array} \right.$ effects

in EM:

$$\partial_t \left(\frac{E^2}{8\pi} + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[\frac{c}{4\pi} \underline{E} \times \underline{H} \right] + \overline{\underline{E} \cdot \underline{J}} = 0$$

how generalize?

(a) consider build-up of dielectric energy in time. Allow carrier + slow envelope

(b) Principle Least Action; A, ϕ
$\Rightarrow$ Action density evolution.

Now, for ⓐ : (see LL, Cont. Media Kubo/ref)

$$\int W = \int d^3x \ \underline{E}^* \cdot \underline{D}$$

→ energy (density) of dielectric medium

$$\frac{dW}{dt} = \frac{1}{8\pi} \, \mathrm{re} \left(\underline{E}^* \cdot \frac{d\underline{P}}{dt} \right)$$

rate change energy density

energy builds up via media response

Consider:

$$\underline{E} = E_0(t, \underline{x}) \, e^{i(\underline{k}_0 \cdot \underline{x} - \omega_0 t)}$$

envelope (slow)

carrier fast

t → build-up of energy

$\underline{x}$ → spread of local perturbation

slow t → frequency α

$\alpha \ll \omega_0$

slow $\underline{x}$ → wave vector $\underline{q}$

$$|\underline{q}| \ll |\underline{k}_0|$$

$\underline{D} = \epsilon \underline{E}$, but

$\epsilon = \epsilon(\omega, k) \Rightarrow$

$\underline{D} \Leftrightarrow \underline{E}$ relation

non-local in space-time

so

$$D(\underline{k}, \omega) = \epsilon(\underline{k}, \omega)\, E(\underline{k}, \omega)$$

so if $\quad f(\underline{k}, \omega) = -i\omega\, \epsilon(\underline{k}, \omega)$

$$\frac{d\underline{P}}{dt} = \sum_{\underbrace{\alpha, \underline{q}}_{\text{envelope}}} f(\omega_0 + \alpha, \underline{k}_0 + \underline{q})\, e^{i(\underline{q}\cdot x - \alpha t)} * \underset{\alpha, q}{\underline{E}_0}\, e^{i(\underline{k}_0 \cdot \underline{x} - \omega_0 t)}$$

expand:

$$\frac{d\underline{P}}{dt} = \sum_{\alpha, \underline{q}} \left[-i\omega\, \epsilon(\underline{k}, \omega) + \alpha \frac{\partial}{\partial \omega}(-i\omega\epsilon) \right| + \underline{q}\cdot\frac{\partial}{\partial \underline{k}}(-i\omega\epsilon) \Big| \Big]_{\underline{k}_0, \omega_0} e^{i(\underline{q}\cdot x - \alpha t)} \underset{*}{\underline{k}_0\, \omega_0}$$

$$\underline{E}_0 \Big|_{\omega_0, \underline{k}_0} e^{i(\underline{k}_0 \cdot x - \omega_0 t)}$$

$\alpha, \underline{q} \rightarrow i\,\partial/\partial t, \quad -i\,\partial/\partial \underline{x}$

acting on envelope

So, re-summing:

$$\frac{d\underline{D}}{dt} = \Big[-i\omega\epsilon\,\underline{E}_0(t,x) + \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{k_0,\omega_0} \frac{\partial}{\partial t}E_0(t,x) - \frac{\partial}{\partial \underline{k}}(\omega\epsilon)\cdot\underline{\nabla}E_0(t,\underline{x})\Big]\, e^{i[\underline{k}_0\cdot\underline{x} - \omega_0 t]}$$

as $\dfrac{dW}{dt} = \dfrac{1}{8\pi}\,\mathrm{re}\left(E^{*}\cdot\dfrac{d\underline{D}}{dt}\right)$

$$\frac{dW}{dt} = \omega\epsilon_{IM}(k,\omega)\frac{|\underline{E}_0|^2}{8\pi} + \frac{\partial}{\partial t}\left[\frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\frac{|E_0|^2}{8\pi}\right] - \underline{\nabla}\cdot\left[\frac{\partial}{\partial\underline{k}}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\frac{|E_0|^2}{8\pi}\right]$$

thus have:

$$W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\left(\frac{|E_0|^2}{8\pi}\right) \rightarrow \text{total wave energy density}$$

$$\underline{S} = -\frac{\partial}{\partial \underline{k}}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\left(\frac{|E_0|^2}{8\pi}\right) \rightarrow \text{total wave energy density flux}$$

$$Q = \omega\,\epsilon_{IM}\left(\frac{|E_0|^2}{8\pi}\right) \rightarrow \text{energy dissipation rate}$$

Barring external source:

$$\partial_t W + \underline{\nabla}\cdot\underline{S} + Q = 0.$$

Note:

- For EM wave:

$$W \rightarrow \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\underline{k}_0,\omega_0}\left(|E_0|^2/8\pi\right) + \underbrace{\frac{\partial}{\partial\omega}(\omega\mu)\Big|_{\underline{k}_0,\omega_0}}_{\text{perm. factor}}\left(\frac{|H_0|^2}{8\pi}\right)$$

- $$\underbrace{\underline{S}}_{\text{momentum in matter}} \rightarrow \underline{S} + \frac{c}{4\pi}\underbrace{\underline{E}\times\underline{H}}_{\text{EM momentum}}$$

- at wave resonance, $\epsilon(k_0, \omega_0) = 0$

$$W = \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\underline{k}_0, \omega_0} \left(|E_0|^2 / 8\pi \right)$$

$$\underline{S} = -\omega_k \frac{\partial \epsilon}{\partial \underline{k}}\Big|_{\underline{k}_0, \omega_0} \frac{|E_0|^2}{8\pi}$$

$$= -\frac{\partial \epsilon / \partial k}{\partial \epsilon / \partial \omega}\Big|_{k, \omega_0} \omega_k \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E_0|^2}{8\pi}$$

but, along wave trajectory:

$$d\epsilon = 0 = \frac{\partial \epsilon}{\partial \omega} d\omega + \frac{\partial \epsilon}{\partial \underline{k}} d\underline{k}$$

$$\underline{v}_{gr} = d\omega / d\underline{k}$$

$$\Rightarrow \underline{v}_{gr} = -(\partial \epsilon / \partial \underline{k}) / (\partial \epsilon / \partial \omega)\Big|_{\underline{k}_0, \omega_0}$$

$$\underline{S} = \underline{v}_{gr} W$$

$$Q = \omega_r \epsilon_{I,M} \Big/ \frac{|E_0|^2}{8\pi}$$

On the physics:

$$- \quad W = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{k_0, \omega_0} \frac{|E_0|^2}{8\pi}$$

then FYI / FYA:

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

$$\Rightarrow W = \left(1 + \frac{\omega_p^2}{\omega^2}\right) \frac{|E_0|^2}{8\pi} \underset{\omega = \omega_p}{=} 2 \times \frac{|E_0|^2}{8\pi}$$

$$= W_{\text{Field}} + W_{\text{Quiver}}$$

field energy

kinetic energy of sloshing in wave

i.e.

$$\frac{1}{2} n_0 m |\tilde{v}|^2 = \frac{n_0}{2} \frac{e^2}{m^2} m \frac{|E_0|^2}{\omega^2}$$

$$= \frac{1}{8\pi} \underbrace{\frac{\omega_p^2}{\omega^2}}_{1} |E_0|^2 \quad \checkmark.$$

$-\underline{S} = -\omega \left.\frac{\partial \epsilon}{\partial \underline{k}}\right|_{k_0, \omega_0} |E_0|^2/8\pi$

for beam:

$$\epsilon = 1 - \omega_{pb}^2/(\omega - k V_b)^2$$

$\Longrightarrow$

beam plasma

$$\underline{S} = \omega \omega_{pb}^2 \, 2 k V_b / (\omega - k V_b)^2$$

$S \sim \underline{k}$.

- if collisions (with neutrals → friction)

$$\partial_t \underline{V} + \nu_{en} \underline{V} = \frac{q}{M} \underline{E}$$

$$\epsilon = 1 - \omega_p^2 / \omega(\omega + i\nu)$$

$$\epsilon_{IM} = \omega_p^2 \nu / \omega(\omega^2 + \nu^2)$$

so $Q = \dfrac{\omega_p^2 \nu}{\omega^2 + \nu^2} \dfrac{|E|^2}{8\pi}$

→ Yet more physics:

Semi-classical analogy!

$$\mathcal{E}_{wave} = \underbrace{\omega_k \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_0|^2}{8\pi}}_{?}$$

Semi-classically:

$$\mathcal{E} \sim N \hbar \omega$$

N: # quanta, "waves"

$$P \sim (\hbar k) N$$

Dimensionally:

$$\mathcal{E} = N \omega$$

$$N = \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k} \frac{|E_0|^2}{8\pi} \sim \mathcal{E}/\omega$$

→ Action density

– Action density $N(\underline{x}, \underline{k}, t)$ satisfies wave kinetic Eqn

(See posted notes on Adiabatic Theory for waves)

$$\partial_t N + \underline{v}_{gr}\cdot\underline{\nabla} N - \partial_{\underline{x}}\omega\cdot\underline{\nabla}_k N = C(N)$$

$$\frac{dN}{dt} = C(N)$$

along $\dfrac{d\underline{x}}{dt} = \underline{v}_{gr}$, $\dfrac{d\underline{k}}{dt} = -\partial_{\underline{x}}\omega$

For $N(\underline{x}, t)$

$$\partial_t N + \nabla\cdot(\underline{v}_{gr} N) = \cancel{\int d\underline{k}\, C(N)}$$

cons.

$N(\underline{x}, t) \Leftrightarrow$ packet

Can derive N, N. eqn. from

$S = S(A, \phi)$

$\delta S/\delta A = 0 \Rightarrow \epsilon = 0$

$\delta S/\delta \phi = 0 \Rightarrow$ N eqn. $\Leftrightarrow$ adiabatic invariant statement.

→ Positive, Negative Energy Waves.

Contrast:

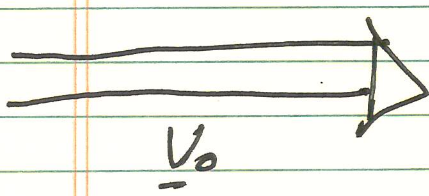
– i.e. usual, ⊕

$$\epsilon = 1 - \omega_p^2/\omega^2$$

$$W = \frac{|E_0|^2}{4\pi} > 0$$

~ positive energy wave. Need input energy into oscillator to excite medium

– now, try beam(-plasma)



Key: → active medium (energy)
→ frame.

$$\epsilon = 1 - \omega_p^2/(\omega - kV_0)^2$$

i.e. $\partial_t \tilde{v} + v_0 \partial_x \tilde{v} = \frac{q}{m} \tilde{E}$

$$\partial_t \tilde{n} + v_0 \partial_x \tilde{n} = -n_0 \underline{\nabla} \cdot \underline{\tilde{v}}$$

$$\underline{\nabla} \cdot \underline{\tilde{E}} = 4\pi n_0 q \left(\frac{\tilde{n}}{n_0}\right)$$

then

$$W_k = \omega_k \frac{\partial \epsilon}{\partial \omega} \frac{|E_k|^2}{8\pi}$$

$$\epsilon = 1 - \frac{\omega_p^2}{(\omega - kV_0)^2}$$

$$\omega = \underset{\text{Doppler}}{\underset{\downarrow}{kV_0}} \pm \omega_p \qquad V_0 > 0$$

$$W_k = (kV_0 \pm \omega_p) \frac{2\omega_p^2}{(\pm\omega_p)^3} \frac{|E_k|^2}{8\pi}$$

$$= \frac{(kV_0 \pm \omega_p)}{\pm\omega_p} \frac{|E_k|^2}{4\pi}$$

$$= k \frac{(V_0 \pm \omega_p/k)}{+\omega_p} \frac{|E|^2}{4\pi}$$

so have:

(i) + root $\Rightarrow$ "fast" wave
$\omega = \omega_p + kV_0$

$$W = \left(\frac{kV_0 + \omega_p}{\omega_p}\right)(\ ) > 0$$

$\rightsquigarrow$ positive energy wave.

ii) − root → "slow" wave

$$\omega = -\omega_p + k V_0$$

$$W = \left(\frac{k V_0 - \omega_p}{-\omega_p}\right)\left(\quad\right)$$

$$= \frac{\omega_p - k V_0}{\omega_p}\left(\quad\right)$$

$W < 0$ for $\omega_p < k V_0$

→ $\boxed{\text{negative energy wave!}}$

What is a 'negative energy wave'?

− excited by extraction of energy from system

− contrast usual ⊕ → excited by input

− to excite by 'extraction' → "active" medium V_0

↓

motion → beam energy.

- active medium suggests free energy available for relaxation
  $\rightarrow$ instability

How tap ?
$\downarrow$
key question

$\rightarrow$ dissipation (extracts energy, $W<0$ $\Rightarrow$ excitation)

$\rightarrow$ couple to positive ($\ominus$ meets $\oplus$)

For destabilization by dissipation:

$$\partial_t W_k + \nabla \cdot S_k + Q_k = 0$$

if $\nabla \cdot S_k = 0$ (radiative damping can de-stabilize)

$$\Rightarrow \quad 2\gamma_k \cong -Q_k / W_k$$

$\downarrow$
growth damping rate

$W<0$, $Q>0$ $\Rightarrow$ $\gamma_k > 0$ growth!

example: add weak collisions to beam $\Rightarrow Q>0$

Dissipation destabilizes!!

<u>or/and</u>:

ii.) ⊕, ⊖ energy wave coupling

⇒ beam plasma $\frac{1}{2}$

To be continued.