Fluid

$$\frac{\partial}{\partial t} q + [\psi, q] = \nu \Delta q$$

Poisson bracket

$$[\psi, q] = \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x}$$

$$q = q(\psi)$$

ψ_0 on a closed boundary. Steady state:

$$\boldsymbol{v} \cdot \nabla q = \nu \Delta q$$

$$\int_A \boldsymbol{v} \cdot \nabla q d^2 x = \int_A \nabla \cdot (\boldsymbol{v} q)\, d^2 x = \int_\Gamma q \boldsymbol{v} \cdot \boldsymbol{n} dl \mathbf{= 0}$$

So we have

$$0 = \int_A \nu \Delta q d^2 x = \int_A \nabla \cdot (\boldsymbol{\nu \nabla} q)\, d^2 x = \nu \int_\Gamma \nabla q \cdot \boldsymbol{n} dl$$

$$\nabla q\left(\psi\right) = \frac{dq}{d\psi}\nabla\psi$$

but

$$\int_{\Gamma} \nabla\psi \cdot \boldsymbol{n} dl \neq 0$$

circulation in closed streamline.

$$\frac{dq}{d\psi} = 0$$

\ arbitrary Γ

$$\frac{dq}{d\psi} \equiv 0$$

in closed streamlines region

Particles

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = St\left(f\right) \sim \nu\frac{\partial^2 f}{\partial v^2}, \quad \nu \to 0$$

Here f (charge density) plays a role of vorticity, $q = \Delta\psi$, (ψ -stream function) while the particle Hamiltonian

$$H = \frac{p^2}{2m} + e\phi = \epsilon$$

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = \frac{\partial f}{\partial t} + \frac{\partial H}{\partial p}\frac{\partial f}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial f}{\partial p}$$

ϵ is a "stream function". If ϕ is independent of t, $H = \epsilon = const.$
Any stationary solution $f(x, p) = f[H(x, p)]$
Use $\epsilon = H$ as a new variable labeling each individual orbit (stream line). Better than ϵ, is the action variable

$$J = \oint p(\epsilon, x)\, dx, \qquad S(x) = \int^x p(\epsilon, x)\, dx$$

$$p = \sqrt{2m(\epsilon - e\phi)}$$

angle: $\alpha = \frac{\partial S}{\partial J}$, can be obtained through differentiation wrt ϵ. It is convinient to scale α mod2π. $f(\alpha + 2\pi) = f(\alpha)$. The transformation from x, p to α, J is canonical

$$\frac{\partial f}{\partial t}+\frac{\partial H}{\partial p}\frac{\partial f}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial f}{\partial p}=St\left(f\right)\sim\nu\frac{\partial^2 f}{\partial p^2},\quad \nu\to 0$$

becomes (Poisson bracket is an invariant of canonical transforms)

$$\frac{\partial f}{\partial t}+\frac{\partial H}{\partial J}\frac{\partial f}{\partial \alpha}-\frac{\partial H}{\partial \alpha}\frac{\partial f}{\partial J}=St\left(f\right)\sim\nu\frac{\partial^2 f}{\partial J^2},\quad \nu\to 0$$

but $H=H\left(J\right)$, and $\partial H/\partial J=\Omega\left(J\right)$

$$\frac{\partial f}{\partial t}+\Omega\frac{\partial f}{\partial \alpha}\sim\nu\frac{\partial^2 f}{\partial J^2},\quad \nu\to 0$$

For $\nu=0$ (following the logic of PB theorem except time-dependent $\nu\to 0$ solution),

$$f\left(\alpha,J,t\right)=f\left(\alpha-\Omega t,J,0\right)=F_0\left(J\right)+\sum_n F_n\left(J\right)e^{in\left(\alpha-\Omega t\right)}$$

$$F_0=\left\langle f\left(t=0\right)\right\rangle_\alpha$$

Phase mixing: $F_n\to 0$, $t\to\infty$, small ν.

$$\frac{\partial F_n}{\partial t} \sim -\nu \left(\frac{d\Omega}{dJ} \right)^2 n^2 t^2 F_n$$

All modes with $n \neq 0$, decay as

$$F_n \approx F_n^0 \exp \left\{ -\frac{\nu}{3} \left(\frac{d\Omega}{dJ} \right)^2 n^2 t^3 \right\}$$

Critical: $\Omega' \neq 0$.
The longest survivor F_1 spreads α at the rate

$$\delta\alpha^2 \sim \nu \left(\frac{d\Omega}{dJ} \right)^2 t^3$$

To show this consider an initial distribution as a "blimp" of size $\delta\alpha \ll 1$ it dissolves as follows:
All of F_n^0 are of ~ 1. E.g. for for an initial $f = \delta(\alpha)$, $F_n^0 = 1/2\pi$, recall Poisson formula:

$$\sum_n \delta(\alpha - 2\pi n) = \frac{1}{2\pi} \sum_n e^{in\alpha}$$

The slolution for

$$f(\alpha, J, t) - F_0(J) = \frac{1}{2\pi}\sum_n e^{in(\alpha-\Omega t)-\frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3}$$

many $n's$ may contribute, so

$$\sum_n \to \int dn$$

BUT: the phase function

$$\Phi = in(\alpha - \Omega t) - \frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3$$

has a critical point on a complex $n-$ plane. We may deform the integration path (analytic function under the integral) and pass it through the maximum of Φ on the new path (saddle point). This will be at

$$n = \frac{3}{2}\frac{i(\alpha - \Omega t)}{\nu\left(\frac{d\Omega}{dJ}\right)^2 t^3}$$

$$\int dn e^{in(\alpha-\Omega t)-\frac{\nu}{3}\left(\frac{d\Omega}{dJ}\right)^2 n^2 t^3} \sim \frac{1}{t^{3/2}} \exp\left[-\frac{3}{4}\frac{(\alpha-\Omega t)^2}{\nu\left(\frac{d\Omega}{dJ}\right)^2 t^3}\right]$$

Note that the solution conserves $\int d\alpha$ as long as it is narrow at $\alpha - \Omega t$, gives $\delta(\alpha)$ for $t \to 0$.
Recall now the diffusion

$$\frac{\partial f}{\partial t} = D\Delta f$$

and its solution for a point source $\delta(\boldsymbol{r})$ at t=0:

$$f = \frac{1}{\sqrt{4\pi D t^3}} \exp\left[-\frac{r^2}{4Dt}\right]$$

Here we have $\alpha - \Omega t$ instead of r (moving point at a speed Ω) but also spreading as

$$(\alpha - \Omega t)^2 \propto \nu \left(\frac{d\Omega}{dJ}\right)^2 t^3$$

See LN 11a, simpler derivation of a similar result:

stream line along y diffusion across, in r

$$\left\langle \delta y^2 \right\rangle \sim \nu \left(\frac{\partial V_y}{\partial r} \right)^2 t^3$$

Characteristic mixing time for trapped particles $\delta\alpha \sim 2\pi$

$$\tau_{mix}^{-1} \sim \left\{ \nu \left(\frac{\partial V_y}{\partial r} \right)^2 \right\}^{1/3}$$

Let us assume that the beam is monoenergetic initially, $V_b = V = V_0$. If the wave is growing slowly, one can write $m\left(V - \omega/k\right)^2 - 2e\phi = const$, so that $\delta V \approx (e\phi/m)\left(V_0 - \omega/k\right)^{-1}$. In course of time, while ϕ grows, particles with $V > V_0$ overtake the wave and get bunched in decelerating phase of the wave, which thus requires that initially $V_0 > \omega/k$. As a result, the beam is broken down into clusters near the decelerating phases of the wave which further increases the wave amplitude due to the increased beam modulation.
When the wave amplitude is high enough to trap the beam clusters into potential troughs, the wave ceases to grow. Indeed, as it

follows from the analysis of Landau damping, when bunched particles bounce off the decelerating phase of the wave, more particles will move slower than the wave and the latter will even decay for a while until the situation is reversed again. The particle phase mixing will suppress the bunching effect and the wave amplitude oscillations decay.

The condition for the beam particles to be trapped is that in the time when they cross the trapping area (wave length) the wave amplitude should grow significantly

$$k\left|V-\frac{\omega}{k}\right|\sim\gamma,$$

or

$$e\phi/m\sim\gamma^2/k^2.$$

The growth rate

$$\gamma\sim\omega_p\left(\frac{n_b}{n_0}\right)^{1/3}$$

The saturation wave energy thus amounts to (square and use the beam resonance condition) $kV_b = \omega_p$

$$\frac{E^2}{4\pi} \sim mV_b^2 n_0 \left(\frac{n_b}{n_0}\right)^{4/3} \sim mV_b^2 n_b \left(\frac{n_b}{n_0}\right)^{1/3} .$$

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