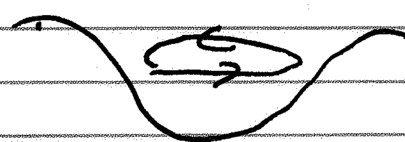
Homogenization and Single Wave Nonlinear Evolution

→ Recall, from Landau Damping discussion, one limitation on linear theory is:

$t < \tau_b$ → evolution time < bounce time



i.e. resonant particles suffer strong orbit distortion → linear trajectories invalid

$1/\tau_b \sim k\Delta v$

$\Delta v \sim (e\phi/m)^{1/2}$

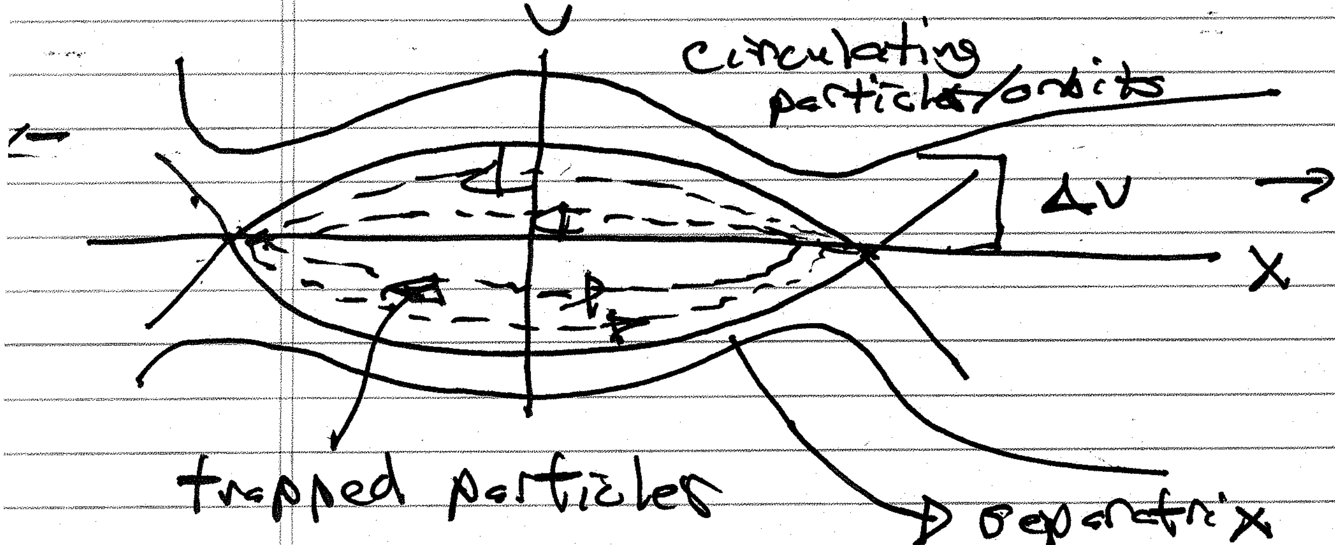
⇒ need $\gamma_{Landau} > 1/\tau_b$ for linear Landau damping calculation to be relevant.

→ What happens for $t > \tau_b$, or

$\gamma_{Landau} < 1/\tau_b$?

- basic orbits ~~for~~ of resonant particles strongly perturbed.
- need integrate evolution of wave using these

i.e. for trapped orbits



Trapped response $\Leftrightarrow$ calculate in $\gamma(t)\,\tau_b < 1$ ordering $\Rightarrow$ basic orbits are those of bounce, i.e. closed.

$\gamma(t)$: instantaneous growthrate

Many questions emerge:

i.) What is end state?

ii.) What is mixing, decay mechanism?

iii.) What is $\gamma_k(t)$?

For insight, consider simpler closely related problem → that of PV homogenization (Prandtl - Batchelor Theorem). (also relevant to flux expulsion)

Consider 2D fluid. Then potential vorticity evolves according to:

$$\partial_t q + \underline{v} \cdot \nabla q - \underline{\nabla} \cdot \nu \cdot \nabla q = 0$$

here $q = \nabla^2 \phi$ (PV = vorticity)

$\underline{v} = \underline{\nabla} \phi \times \vec{z}$, ($\underline{\nabla} \cdot \underline{v} = 0$, 2D)

$\nu \equiv$ usual viscosity → important!

so can re-write as:

$$\partial_t \nabla^2 \phi + \underline{\nabla} \phi \times \vec{z} \cdot \underline{\nabla} \nabla^2 \phi - \nu \nabla^2 \nabla^2 \phi = 0$$

- i.e. viscous 2D fluid.

Can extend to more general PV, i.e.

$$\partial_t q + \underline{\nabla} \phi \times \vec{z} \cdot \underline{\nabla} q - \nu \nabla^2 q = 0$$

obviously $q \Leftrightarrow$ charge density

System is obviously relevant to Vlasov, i.e.:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{M}E\frac{\partial f}{\partial v} = C(f) \to 0$$

$$\frac{df}{dt} = 0$$

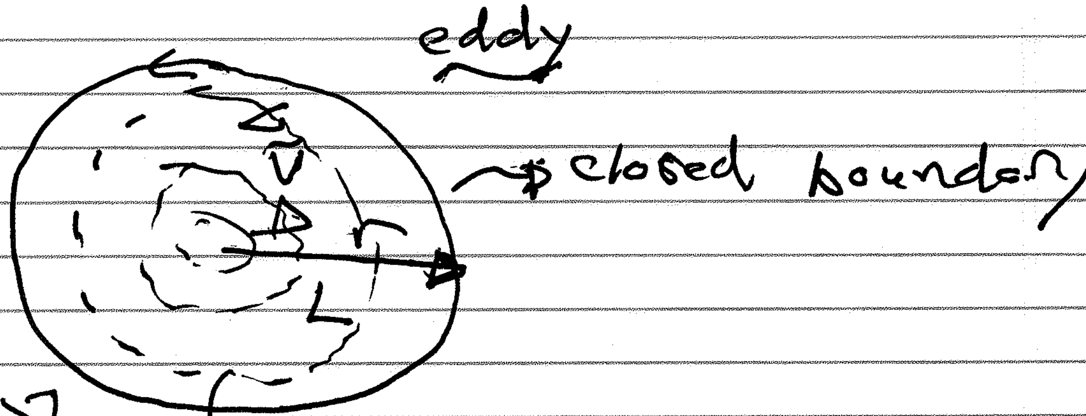
$$\frac{\partial q}{\partial t} + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla} q = \nu \nabla^2 q$$

$$\frac{dq}{dt} = \nu \nabla^2 q$$

- common: Hamiltonian structure
  conservation along trajectories

  up to: dissipation, coarse graining.
- different: viscosity vs $C(f)$
  $f \to 0 \Rightarrow$ coarse graining.

and consider set up of eddy with closed stream line as boundary i.e.



$Pe = \frac{\tilde{V}L}{\nu} \gg 1$

eddy flow – $\underbrace{\text{differential rotation}}$

Now, as for single wave:

→ What is ultimate distribution $\Omega(r)$? (N.B. Assume circle for simplicity)

→ timescales?

Observation re: viscosity,

Consider stationary state:

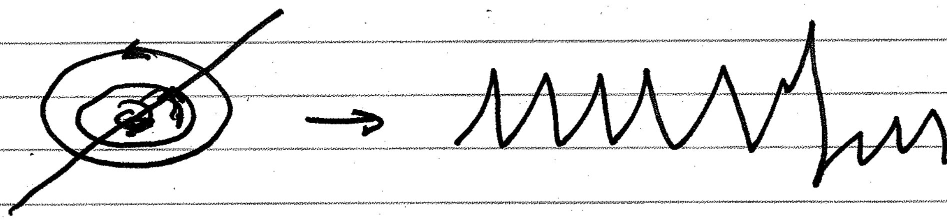
$$\cancel{\partial_t \Omega} + \underline{\nabla}\phi \times \hat{z} \cdot \nabla \Omega - \nu \nabla^2 \Omega = 0$$

if $\nu \to 0$

$$\nabla \phi \times \hat{z} \cdot \nabla \Omega = 0$$

so $\Omega(\phi) = \Omega$ is solution

i.e.



- can tag each streamline arbitrarily, generate non-differentiable "crinkly" solution.
- no smoothing of sharp gradients.
- unphysical!

"Not all solutions of the Navier-Stokes [N.B. really Euler] equations are realized in nature."

— Landau, Lifshitz (Fluid Mechanics)

But: - with $\nu \neq 0$ will show that $q(r) \to$ const is end state

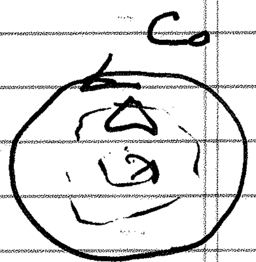
- PV homogenized! i.e. ∇q flattened.

Note: - $\nu = 0$ each streamline decouples

- $\nu \neq 0$, global solution.

Homogenization ⇒ Prandtl–Batchelor Theorem.

Theorem: Consider a region of 2D incompressible flow (i.e. vorticity advection), enclosed by a closed streamline C_0. Then if diffusive dissipation,



i.e. $\partial_t \zeta + \nabla\phi \times \hat{z} \cdot \nabla \zeta = \underline{\nabla} \cdot (\nu \underline{\nabla} \zeta)$

then $\zeta \to$ uniform (homogenization!) as $t \to \infty$, within C_0.

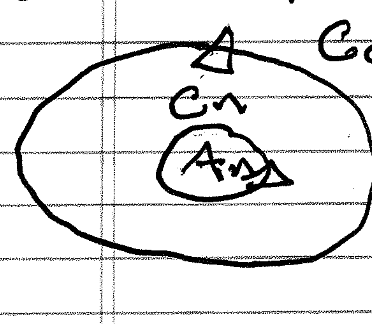
N.B. – finite ν crucial
– no comment on how long?

To show:

– $t \to \infty$, with ν finite:

$$\nabla\phi \times \hat{z} \cdot \nabla \zeta = \underline{\nabla} \cdot \nu \cdot \underline{\nabla} \zeta$$

– choose arbitrary closed C_n within C_0. C_n a streamline.



– simply connected flow.
– stationary ⇒ ω const along streamlines
– C_0 specified on ⊥ bndry. $C_0 \to$ B.C.

$\therefore \quad \omega \to \omega_0$ on C_0

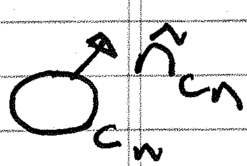
$\omega \to \omega_n$ on C_n

Now, for A_n enclosed by C_n:

$$\int_{A_n} d^2x \, \underline{v} \cdot \underline{\nabla} q = \int_{A_n} d^2x \, \underline{\nabla} \cdot (\nu \, \underline{\nabla} q)$$

but

$$\int_{A_n} d^2x \, \underline{v} \cdot \underline{\nabla} q = \int_{A_n} d^2x \, \nabla \cdot [\underline{v} \, q]$$



$$= \int_{C_n} dl \, \hat{n}_{C_n} \cdot (\underline{v} \, q)$$

normal to C_n

but $\underline{v}$ is streamline, along C_n, so

$$\int_{C_n} dl \, (\hat{n}_{C_n} \cdot \underline{v}) \, q = 0.$$

Thus have shown:

so
$$0 = \int_{A_n} d^2x \; \underline{\nabla} \cdot (\nu \, \underline{\nabla} \, q)$$

$$= \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla q$$

Now, stationary state must have $q \to$ const along streamline.

so
$$q = q(\phi)$$

$$q_{C_n} = q(\phi_n)$$

and
$$0 = \nu \int_{C_n} d\ell \; \hat{n}_{C_n} \cdot \nabla \phi_n \frac{\partial q}{\partial \phi_n}$$

$$= \nu \frac{\partial q}{\partial \phi_n} \int_{C_n} d\ell \, (\hat{n} \cdot \nabla \phi_n)$$

And: $\Gamma = \oint d\underline{\ell} \cdot \underline{v} \Rightarrow$ circulation

$$= \int d\underline{\ell} \cdot (\underline{\nabla} \phi \times \hat{z})$$

$$= \int (\hat{z} \times \hat{n}) \cdot \underline{\nabla} \phi \times \hat{z} = -\int d\ell \, (\nabla \phi \cdot \hat{n})$$

so $\quad 0 = \nu \dfrac{\delta q}{\delta \phi_n} \Gamma_n$

$\Gamma_n \neq 0 \quad \Rightarrow \quad \boxed{\delta q / \delta \phi_n = 0}$

as C_n arbitrary, $\delta q / \delta \phi_n = 0$ for all ϕ_n so:

$$\boxed{\delta q / \delta \phi = 0, \quad \text{all } \phi}$$

- no line-to-line variation
- $\boxed{q \text{ homogenized}}$ $\quad \nabla q \to 0$

Now,
- note order limits

first $t \to \infty$, then $\left\{ \begin{array}{l} q \doteq q(\phi) \to \text{concentric lines} \\ \nu \to \text{small} \end{array} \right.$

- expect ∇q large at boundary C_0

i.e. PV gradient steepening at bndry
$\Rightarrow$ barrier!

Further:

- Key assumption

  separatrix – closed, bounding streamline

  viscous dissipation ⇒ form matters.

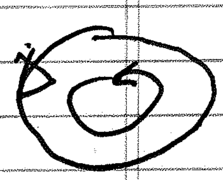
- large Pe

  i.e. $\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1$

  → establish concentric circulation

  → then diffuse accross to homogenize

Now, time scales:



- sheared, concentric flow
- viscous diffusion

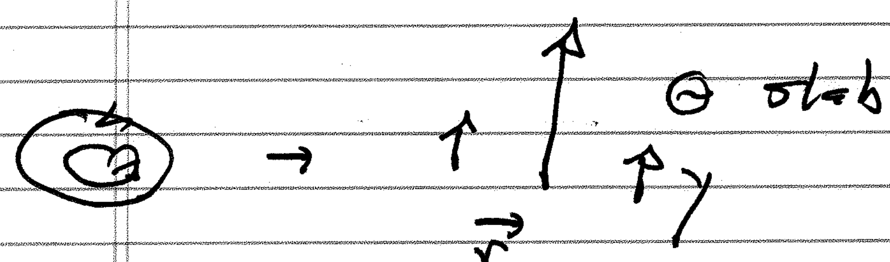
Need interact to homogenize on non-trivial time scale.

Look for synergism ⇒ shear dispersion!

(a) pure diffusion:

$$1/\tau_d \sim \nu/L^2$$

(b) diffusion + shear:



$$r\theta = y$$

$$\frac{dy}{dt} = V_y(r)$$

so

$$\frac{d\,\delta y}{dt} = \frac{\partial V_y}{\partial r}\,\delta r$$

$$\delta y = \int dt \left(\frac{\partial V_y}{\partial r}\right)\delta r$$

$$\langle \delta y^2 \rangle = \left(\frac{\partial V_y}{\partial r}\right)^2 \langle \delta r^2 \rangle\, t^2$$

$$\langle \delta r^2 \rangle = \nu t$$ → molecular diffusion.

so

$$\langle \delta y^2 \rangle \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \nu t^3$$

Mixing occurs when mean square excursion $\sim L_y^2 \sim L^2$ $(L \sim 2\pi R)$

$$\langle \delta y^2 \rangle / L^2 \sim \left(\frac{\partial V_{0y}}{\partial r}\right)^2 \frac{\nu}{L^2} t^3 \sim 1$$

$$\boxed{\tau_c^{-1} = 1/\tau_{mix} \sim \left[\left(\frac{\partial V_{0y}}{\partial r}\right)^2 \nu / L^2\right]^{1/3}}$$

hybrid time scale, of shear and ν.

$$\tau_{mix}^{-1} \sim \frac{V_0}{L} / P_e^{1/3}$$

$$\boxed{\tau_{mix} \sim \tau_{circ} P_e^{1/3}}$$

$\Rightarrow$ Longer time scale smoothing (on $\tau \sim L^2/\nu$) completes homogenization

For more on homogenization, see W2018 Phys. 218b Notes, Lecture 7 and references.

→ Return to Single Wave Problem.