Authorized translation from the Russian by **Herbert Lashinsky**
*University of Maryland*

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# TRANSLATOR'S PREFACE

In the interest of speed and economy the notation of the original text has been retained so that the cross product of two vectors **A** and **B** is denoted by [**AB**], the dot product by (**AB**), the Laplacian operator by $\Delta$, the curl by rot, etc. It might also be worth pointing out that the temperature is frequently expressed in energy units in the Soviet literature so that the Boltzmann constant will be missing in various familiar expressions. In matters of terminology, whenever possible several forms are used when a term is first introduced, e.g., magnetoacoustic and magnetosonic waves, "probkotron" and mirror machine, etc. It is hoped in this way to help the reader to relate the terms used here with those in existing translations and with the conventional nomenclature. In general the system of literature citation used in the bibliographies follows that of the American Institute of Physics "Soviet Physics" series; when a translated version of a given citation is available only the English translation is cited, unless reference is made to a specific portion of the Russian version. Except for the correction of some obvious misprints the text is that of the original.

We wish to express our gratitude to Academician Leontovich for kindly providing the latest corrections and additions to the Russian text, and especially for some new material, which appears for the first time in the American edition.

A paper on plasma turbulence by B. B. Kadomtsev that appeared in Volume 4 of the Russian series is not included here since a translation has already been published by Academic Press (London, 1965).

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# COOPERATIVE PHENOMENA AND SHOCK WAVES IN COLLISIONLESS PLASMAS

R. Z. Sagdeev

## §1. Cooperative Phenomena in a Plasma

It is well known that relaxation processes in a highly nonequilibrium collisionless plasma frequently involve cooperative plasma oscillations that result from various plasma instabilities. These oscillations can have an important effect on plasma transport phenomena and it is this aspect of the phenomenon which is of greatest interest from a practical point of view, an immediate example being the "anomalous" diffusion of hot plasma in magnetic-confinement devices [1]. Various aspects of this complicated problem, which is related to the theory of stability with respect to small perturbations, have been investigated extensively and in some cases the foundations of a nonlinear approach have already been laid.

Another interesting example of cooperative plasma phenomena is furnished by shock waves. In ordinary gas dynamics the minimum thickness of a shock front is usually at least of the order of the mean free path of the molecules in the gas; on the other hand, as a result of cooperative phenomena there are plasma shock waves in which the thickness of the shock front is appreciably smaller than the mean free path. This means that even a highly rarefied plasma is more closely related to a gas-dynamic medium than to a Knudsen gas.

It is the purpose of this review to present an integrated description of the basic concepts and results of the theory of cooperative phenomena in collisionless plasmas. Our primary objective is to obtain a qualitative description of these phenomena and we shall be mainly concerned with the physical significance of the various approximate models that will be analyzed.

1. Because of the long range of electrical forces the interaction between particles in a plasma is not so much in the nature of a collision as it is a reflection of the effect of the so-called self-consistent field. A plasma that can be regarded as an ideal gas (the criterion for the application of the

"gas" approximation is $na^3 \gg 1$, where n is the particle number density and $a$ is the Debye radius) can be analyzed by kinetic-theory methods; this means that the distribution function for the ions (electrons) $f_{i,e}(\mathbf{v}, \mathbf{r}, \mathbf{t})$ satisfies the Boltzmann-Vlasov equation

$$\frac{\partial f}{\partial t} + [H, f] = \mathrm{St}(f), \tag{1}$$

where [H, $f$] is the Poisson bracket and St ($f$) is the collision term.

The self-consistent field in Eq. (1) includes terms containing the electric and magnetic fields, which satisfy Maxwell's equations. The charge density and the current density are written in the form $\varrho = \sum e_k \int f_k \, dv$ and $j = \sum e_k \int v f_k dv$, where the summation is taken over all particle species in the plasma and collisions are introduced by means of the collision integral St(f), the actual form of this integral being determined by the composition of the plasma. Evidently the effects of "close"collisions and the effects due to the self-consistent field must be entirely different as far as the plasma dynamics problem is concerned. Thus, collisions must provide a mechanism for relaxation processes (the establishment of a local Maxwellian distribution, the exchange of energy and momentum between ions and electrons, and so on) each of which can be characterized by some characteristic time $\tau$ (the collision time). On the other hand, the self-consistent field is evidently responsible for the dispersion properties of the plasma,i.e., this field determines the features of the characteristic oscillations and wave properties of the plasma. In the simplest case (no magnetic field) the basic dispersion parameter in a plasma is the electron Langmuir frequency $\omega_0$ ($\omega_0^2 = 4\pi ne^2/m$, e is the charge of electron, m is its mass, and n the density). In most cases of interest the plasma oscillation frequency is so high that $\omega\tau \gg 1$, that is to say, the plasma exhibits two different time scales $\tau$ and $T = 2\pi/\omega$ (the oscillation period $T \ll \tau$). Hence, if the oscillation processes are of primary interest then the close collisions can be neglected and the collision integral in Eq. (1) can be omitted. This approach to the problem, which is called the "collisionless" plasma theory, makes it possible to simplify a large number of problems in plasma dynamics. This theory describes phenomena which occur in times much smaller than the mean free time $\tau$, the point of departure being the Vlasov equation (self-consistent fields and no collision integral):

$$\frac{\partial f}{\partial t} + [H, f] = 0. \tag{2}$$



Since entropy is conserved in the absence of collisions (this follows from the H-theorem), it would appear that the collisionless plasma theory should be capable of describing isentropic processes only, and that it should not be applicable to irreversible relaxation processes in a plasma, i.e., phenomena such as the establishment of thermal equilibrium (randomization), and so on. However, it is found experimentally that relaxation processes do in fact occur in times much smaller than $\tau$, that is to say, under conditions for which the collisionless theory should apply. These anomalous dissipation properties of a collisionless plasma are reminiscent of the situation in the ordinary hydrodynamic theory of turbulence. The characteristic time associated with irreversible diffusion of velocity is of order

$$\tau_v \sim \frac{R^2}{\nu}, \tag{3}$$

where R is a characteristic dimension and $\nu$ is the kinematic viscosity. In point of fact, however, the actual relaxation time is found to be much smaller: the development of instabilities leads to tnrbulence, i.e., reduction of the characteristic scale sizes, with the attendant reduction in mixing time. Two factors play important roles here. First, there is the existence of a very large number of degrees of freedom — the so-called fluctuation scales in the theory of turbulence; these degrees of freedom interact with each other by virtue of nonlinear mixing effects and it is this interaction which is responsible for the time irreversibility that arises when one goes from the dynamic description to the statistical description of the system, that is to say, from the Navier-Stokes equations to the equations that characterize the averaged motion of the fluid.*

Second, as the energy associated with the motion is fed into smaller and smaller scale sizes the role of the viscous effects becomes more important because of the higher spatial gradients; the quantity R in Eq. (3) is then replaced by the characteristic scale size† of the fluctuation $l$ and when $l \ll R$ the velocity diffusion time is reduced sharply.

In developing the analogy between hydrodynamic turbulence and anomalous dissipative processes in a collisionless plasma one can distinguish two important classes of related effects.

*For example, the system consisting of the infinite number of coupled equations for the velocity moments.

†However, the fluctuation scale in hydrodynamics, $l$, is never smaller than the mean free path $\lambda$.

1. The collisionless theory describes various kinds of plasma oscillations and waves. Since a plasma is frequently unstable the amplitudes of these oscillations increase rapidly and the nonlinear interaction between various modes of oscillation corresponds to the interaction between the fluctuation scales in hydrodynamics. The number of different modes in a plasma can be very large* and it is then appropriate to use a statistical description rather than a dynamic description. Thus, as in hydrodynamic turbulence, irreversible processes are possible even in a collisionless plasma.

2. Electric and magnetic fields associated with the plasma oscillations cause pronounced local changes in the particle velocity distributions. These changes occur because any wave of the form exp $i(\omega t - kr)$ will interact strongly with the so-called "resonance" particles, i.e., particles whose velocities are approximately the same as the phase velocity of the wave $v \sim \omega/k$. This interaction results in the formation of large gradienrs which are in velocity space, rather than in ordinary physical space. Collisions between charged particles correspond to a collision term $\hat{D}\Delta_v f$, where $\hat{D}$ is the diffusion coefficient in velocity space; this term is reminiscent of the viscosity term in the Navier-Stokes equation (but in velocity space). In the present case the predominant collisions are characterized by small-angle deviations, i.e., small changes in velocity. Thus, although analogy can be established with hydrodynamic turbulence, this analogy is more or less formal since the viscosity in ordinary physical space (hydrodynamics) is the analog of a "viscosity" in velocity space (plasma).

The theory that describes these anomalous effects in a plasma is usually called the theory of cooperative phenomena since this designation emphasizes the fact that the basic role in these phenomena is played by plasma oscillations and waves, which are essentially "cooperative" motions of the plasma particles. Our analogy with hydrodynamic turbulence provides an indication of the nature and scale of the difficulties that can be expected in a theory of cooperative phenomena in a plasma. Indeed, the analysis of cooperative

* The quantity N, the number of degrees of freedom of the Langmuir oscillations in a plasma, can be estimated as follows: in a unit volume the number of modes is given by

$$\frac{N}{V} \sim \int^{k_{max}} k^2\, dk,$$

where k is the wave number.

It is well known that $k_{max} \sim 1/a$ for plasma oscillations. Consequently $N \sim v/a^3$, a number which, by definition, is much larger than unity.



phenomena in a plasma is complicated still more by the fact that the particle velocity distribution function does not depend on four variables (**r**, t) as in hydrodynamics, but on seven variables (**r**, **v**, t).

The fundamental problem in the theory of cooperative phenomena is that of formulating the kinetics of nonequilibrium processes, i.e., the processes by which thermodynamic equilibrium is established in the plasma. If the initial state of the plasma is far from equilibrium the transition to equilibrium is not monotonic, but is characterized by the strong excitation of plasma oscillations as a consequence of instabilities.*

A theory of cooperative plasma phenomena should be able to provide the characteristic times for these transition phenomena. The strong random oscillations characteristic of these transition processes have an effect on transport phenomena such as diffusion, thermal conductivity, etc., and it is this aspect of the problem which is of greatest interest as far as practical application is concerned. For example, investigations of controlled thermonuclear fusion reactions are, for the most part, based on the notion of thermal isolation of the plasma by magnetic fields. However, equilibrium plasma configurations in a magnetic field are frequently found to be unstable. The instabilities can cause a marked deterioration in the magnetic thermal isolation and a significant increase in the flow of heat and particles to the walls, as a result of cooperative phenomena. A large quantity of experimental data concerning these effects has been accumulated in the last few years. However, it should be noted that the "anomalous" loss of plasma to the walls is frequently not due to cooperative plasma phenomena such as those described here, but rather to ordinary magnetohydrodynamic instabilities. It is, in fact, difficult to draw a sharp line of demarcation between the cooperative plasma effects and turbulent effects of magnetohydrodynamic origin; the situation is even more complicated because sometimes a collisionless plasma (in which the mean free path is very long) can be described with good accuracy by equations that are reminiscent of the usual magnetohydrodynamic equations.

**2.** An examination of the literature concerned with the dynamics of collisionless plasmas indicates that a number of completely different mathematical models have been used to analyze this problem. The most general approach has been to use the kinetic equation with self-consistent electric and magnetic fields. However, this approach is rather complicated and the

*Even a small deviation from thermodynamic equilibrium is frequently sufficient to produce an instability.

"hydrodynamic" equations are frequently used to describe a plasma (separate equations for the electrons and ions, especially in the analysis of problems arising in connection with oscillations and stability). Although the concept of hydrodynamics in the absence of collisions is not easily justified, this approach has been found to yield results that are quite reasonable in many respects.

As an example, let us consider the propagation of a wave in a plasma in the absence of a fixed magnetic field. If the phase velocity satisfies the condition $\omega/k \gg (T/M)^{1/2}$ the thermal motion of the particles is unimportant and it can be assumed that all of the ions and electrons at a given point in space move with the same velocity. In this case one simply uses the equations of motion for each particle species. In the Eulerian coordinate system this formulation of the problem is the zero-temperature hydrodynamic approximation. However, if one is interested in corrections due to the small thermal velocity spread the correct results (i.e., results that coincide with the kinetic results) are obtained by adding terms in the hydrodynamic equations to take account of the pressure gradients $\nabla p$ (for the ions and electrons); p is assumed to be governed by an adiabatic relation with specific-heat ratio $\gamma = 3$. This choice is not unreasonable: If there are no collisions each degree of freedom is independent of the others so that $\gamma = 3$ as in the case of one-dimensional motion. The simplified hydrodynamic approach can also be improved in another particular case. Let us assume that the phase velocity $\omega/k$ is appreciably greater than the ion thermal velocity $(T_i/M)^{1/2}$, but much smaller than the electron thermal velocity $(T_e/m)^{1/2}$. As before, the ions are described by an equation of motion in which the thermal velocity spread is neglected. However, the picture is different as far as the electrons are concerned. Since the electrons move much more rapidly than the wave they see an electric field that is essentially static. If the electron velocity distribution is Maxwellian, $f \sim \exp(-mv^2/2T)$, at the point where the electric potential $\varphi$ is a maximum, the electron density at any other point will be described by the Boltzmann relation $n = n_0 e^{e\varphi/T}$. For wavelengths appreciably greater than the Debye radius $a$, the electric field can be eliminated from the equations by invoking the neutrality condition: $n_i = n_e = n_0 e^{e\varphi/T}$. The term containing the electric field in the ion equation of motion $-e\nabla\varphi$ is replaced by $(-T/n)\nabla n$. Thus, the ion motion is described by the hydrodynamic equations with $\gamma = 1$ (the isothermal feature is provided by the electrons, which can easily equalize the temperature since they move much faster than the wave). However, the hydrodynamic approximation is not capable of describing certain particular features associated with the existence of the thermal motion. For instance, effects due to resonance particles are lost in a hydrodynamic analysis since these resonance particles have velocities close to the propagation velocity of the wave. These particles are responsible for the collisionless damping of oscillations. If $\omega/k \gg (T/m)^{1/2}$, the number of resonance particles is exponentially small and the resonance damping is small [of order $\exp\{-(m/T) \times (\omega/k)^2\}$].



If the plasma is located in a magnetic field the situation is entirely different. Within certain limitations the kinetic description can justifiably be reduced to a hydrodynamic description even in the absence of collisions. The physical justification is the fact that the particles are "tied" to the lines of force of the magnetic field so that the mean macroscopic velocity of the particles is determined by the "motion" of the lines of force themselves. These approximate equations are obtained formally by expanding the kinetic equation in powers of the ratio of the mean Larmor radius to the characteristic scale length R. The expansion in $r_H/R$ is reminiscent of the usual hydrodynamic approximation from kinetic theory in which the expansion parameter is $\lambda/R$ ($\lambda$ is the mean free path). The expansion in the magnetic field case actually implies a description of the plasma in terms of an ensemble of quasi-particles or "Larmor circlets" (guiding centers). The hydrodynamic equations obtained in this way then contain two pressures: a longitudinal pressure, and a transverse pressure (with respect to the direction of the magnetic field). Under these conditions $\gamma = 2$ since the transverse motion is two-dimensional.

In the present review, in addition to using the kinetic equations we shall make use of the simplified hydrodynamic equations whenever these equations can be justified. The hydrodynamic equations facilitate the analysis of certain nonlinear problems, if only by providing a basis for forming analogies with the nonlinear motion of ordinary hydrodynamics.

**3.** It is clear that stability plays an important part in the theory of cooperative phenomena. The stability of a given state of a system can generally be investigated by perturbation theory. If an initial perturbation of the stationary state of the system grows with time the state is unstable with respect to this particular perturbation. In practice one always speaks of stability only with respect to small perturbations, that is to say, departures from the initial state such that the describing equations can be linearized; in this case the describing equations can be expanded in terms of the perturbation amplitude and all terms higher than first order can be neglected, as in the theory of small oscillations. The theory of stability of a collisionless plasma is, in many respects, similar to the theory of magnetohydrodynamic stability. This similarity follows from the fact that a collisionless plasma can frequently be described with good accuracy by the magnetohydrodynamic equations, as we have noted above. On the other hand, a collisionless plasma is also subject to certain kinds of instabilities that cannot be described within the framework of the magnetohydrodynamic equations. These instabilities and their growth rates can only be analyzed within the framework of a kinetic theory. The collision integral is generally neglected since it is assumed that the growth rates characteristic of the instability are much faster than the collision frequencies. In analyzing an instability associated with a local deviation from thermodynamic equilibrium in a plasma it is often convenient to assume a "background" (stationary state of the plasma) which is uniform and of infinite extent. The investigation of stability in cases of this kind reduces to the solution of the appropriate dispersion equation which relates the characteristic frequency $\omega$ and the wave vector k. Frequently the determination of stability with respect to various simple kinds of perturbations does not require complicated calculations; simple physical pictures are sufficient [2]. It is also possible to examine the stability of a "weakly" inhomogeneous plasma in which the ratio $\lambda/R$ is small ($\lambda$ is the wavelength of the perturbation and R is the characteristic scale length of the inhomogeneity).

Let us consider an instability with respect to a wave-like distortion of the lines of force of the magnetic field. It is well known that in an equilibrium plasma these initial distortions of the force lines are propagated in the form of magnetohydrodynamic (Alfvén) waves which can be regarded as oscillations of elastic bands (the lines of force of the magnetic field). To investigate stability we consider the forces that arise when the lines of force are distorted (Fig. 1). Since they are "tied" to the force lines, particles that move along the curved portion are subject to a centrifugal force

$$F_c = \int f \frac{mv_\parallel^2}{R}\, dv, \tag{4}$$

which tends to increase the curvature.

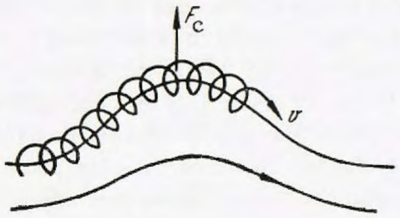


Fig. 1



Furthermore, since each "quasi-particle" has a magnetic moment $\mu$ oriented against the magnetic field H, in the inhomogeneous magnetic field these quasi-particles are subject to a force associated with the magnetization current

$$\mathbf{j}_\mu = c\nabla \times \int \mu f\, dv,$$

$$\mathbf{F}_\mu = \frac{[\mathbf{j}_\mu \times \mathbf{H}]}{c} = \left[\mathrm{rot} \int \mu f\, dv \times \mathbf{H}\right]. \tag{5}$$

This force and the tension in the magnetic force lines

$$F_t = -\frac{1}{4\pi}[\mathrm{rot}\,\mathbf{H} \times \mathbf{H}] \tag{6}$$

tend to restore the lines of force to the equilibrium position.

If $F_c > F_\mu + F_t$ the system moves away from the equilibrium configuration, that is to say, an instability arises. The instability criterion can be obtained easily from Eqs. (4)-(6):

$$p_\parallel - p_\perp > \frac{H^2}{4\pi}, \tag{7}$$

where

$$p_\parallel = \int mv_\parallel^2 f\, dv, \quad p_\perp = \int \mu H f\, dv, \quad \mu = \frac{mv_\perp^2}{2H}.$$

The velocity with which the plasma moves away from the equilibrium configuration can be found by equating the sum of the forces $F_c - F_\mu - F_t$ to the product of the mass of a unit volume of the plasma and the acceleration $\frac{dv}{dt} = \frac{d}{dt} c \frac{E_\sim}{H}$. It follows from Maxwell's equations that $E_\sim = H_\sim \omega/ck$, if the perturbation is written $\exp(\omega t - ikx)$. Thus, we find $\frac{dv}{dt} = i\omega^2 \frac{H_\sim}{kH}$. Substituting the values of the forces F, we have

$$\omega^2 = \frac{k^2}{\varrho}\left(\frac{H^2}{4\pi} + p_\perp - p_\parallel\right). \tag{8}$$

This instability is associated with the centrifugal force that arises in the motion of the particles along the curved line of force and is sometimes called the "firehose" instability by analogy with a rubber firehose which becomes contorted when water flows through it.

Similarly, in the other limiting case ($p_\perp > p_\parallel$) we obtain an instability criterion of the form

$$p_\perp \gg p_\parallel \left(1 + \frac{H^2}{8\pi p_\perp}\right). \tag{9}$$

The conditions in (7) and (9) show that a plasma becomes unstable when the particle velocity distribution exhibits a sufficiently strong deviation from isotropy; as H becomes smaller the instability can appear at a smaller anisotropy. However, at very small H the Larmor radii of the particles become very large and the notion of guiding centers no longer holds. Nevertheless, an instability due to anisotropy appears in the limit $H \to 0$. The instabilities being considered here are aperiodic, that is to say, the time dependence of these instabilities is of the form $\exp \gamma T$. Departure of the plasma from the state of thermodynamic equilibrium can also lead to the excitation of waves, i.e., the appearance of instability in the form of oscillations. The criterion for the excitation of this kind of instability, i.e., the criterion for the change of sign of $\omega_i$, the imaginary part of the frequency $\omega = \omega_r + i\omega_i$, can be determined by considering the energy balance between any plasma wave (that arises as a result of a fluctuation) and the plasma particles. If $\omega_i$ is very small ($\omega_i \ll \omega_r$), the wave characterized by the given $\omega$ and corresponding wave vector is almost periodic and ions (electrons) oscillating in the periodic wave field experience no change in energy on the average. The only exception arises for those particles in the velocity distribution which are in resonance with the wave. In the absence of a magnetic field the unperturbed plasma can exhibit a resonance only for particles whose velocity is close to the velocity of the wave $\omega/k$ (the resonance condition is $\omega - kv = 0$). In the presence of a constant magnetic field, however, an effective interaction with the wave is possible for particles which see a wave frequency $\omega' = \omega - kv$ (in their own characteristic coordinate system) which is close to the cyclotron frequency $\omega_H = eH/mc$ (or one of its harmonics $n\omega_H$, where $n = \pm 1, \pm 2, \ldots$); this frequency shift is a result of the Doppler effect. Particles whose velocity component along the magnetic field satisfies this condition will be accelerated continuously (or retarded) by the wave field much in the same way as ions are accelerated in a cyclotron. In the simplest case, in which there is no fixed magnetic field, a uniform plasma can only support the propagation of pure transverse waves or pure longitudinal waves. The transverse waves



need not be considered since their characteristic phase velocity is greater than the velocity of light ($\varepsilon = 1 - \omega_0^2/\omega^2$). However, the lower limit on the phase velocity of the longitudinal electron Langmuir waves is of the order of the electron thermal velocity (with corresponding minimum wavelength of the order of the Debye radius) and increases with increasing wavelength. Consider a Langmuir wave with frequency $\omega$ (and phase velocity $\omega/k$) in a coordinate system moving with respect to the laboratory with velocity $\omega/k$; in this coordinate system we have an electrostatic potential described by a fixed sinusoid of amplitude $\varphi_0$: The electrons see alternate potential wells and hills. Electrons with velocities appreciably different from $\omega/k$ will move freely in this periodic field, without experiencing any change in average energy. On the other hand, electrons whose velocity v differs from $\omega/k$ by an amount smaller than $\sqrt{2e\varphi_0/m}$ will be reflected from the potential hills. These electrons can be divided into two classes: the velocities characteristic of the first class are greater than $\omega/k$; the velocities in the other class are smaller than $\omega/k$. Electrons in the first class are reflected on reaching potential hills and give energy to the wave; electrons in the second class are carried along by the wave and acquire energy from it. A simple analysis of the energy balance for reflection of electrons from potential hills then provides an instability criterion, the instability mechanism being a kind of "inverse" Landau damping. The wave amplitude increases if energy is fed from the electrons into the wave; this is the case when the number of electrons in the first class is greater than the number in the second, i.e., when

$$\frac{df}{dv}\left(v = \frac{\omega}{k}\right) > 0. \tag{10}$$

In order for this condition to be satisfied the electron velocity distribution function must have at least one "extra" peak in the velocity region beyond the thermal velocity. On the other hand, if $df/dv < 0$ everywhere then $\omega_i < 0$ and the wave is damped (this is the well-known Landau damping phenomenon) [3].

By estimating the work done by the electric field in the wave we can also derive an instability criterion for the cyclotron resonance $\omega = \omega_H - kv$; this criterion is important for transverse waves propagating along a fixed magnetic field.

**4.** The most difficult problem in the theory of cooperative plasma phenomena is that of determining the ultimate fate of the plasma when instabilities arise: the exponentially growing perturbation must evidently sooner or later reach a magnitude at which the linear analysis no longer holds. In principle this problem can be solved in the case of oscillatory instabilities. In the oscillatory case the plasma with fully developed instabilities can be represented as a mixture of two ensembles: particles and waves. In particular, it is the interaction between the particles and the waves that is responsible for the instability. The wave—wave interaction, on the other hand, is strictly a nonlinear effect. If a wave exists for a time interval appreciably greater than its own characteristic period ($t \gg 2\pi/\omega$) it is legitimate to endow it with the properties of a "quasi-particle," in which case the ensemble of quasi-particles can be described by an appropriate distribution function in "quasi-momentum" space (the space characterized by values of the wave vector k); this distribution function will satisfy an appropriate kinetic equation. Formally the situation is very much like that encountered in the quantum theory of solids, where a mixture of two gases is also frequently considered: these gases are the electron gas and the phonon gas. However, the theory of plasma stability is much more complicated since the equations are fundamentally nonlinear because it is only meaningful to consider states which are far from thermodynamic equilibrium. In order to write an appropriate kinetic equation for a turbulent plasma one must know the form of the appropriate collision terms, i.e., the wave—particle and wave—wave interaction terms. The first of these is found from the so-called "quasilinear" theory, which takes account of small nonlinear effects in only one sense, i.e., the distortion of the distribution function due to the feedback effect of the waves [4-6].

In the quasilinear approximation the particle velocity distribution function is written as a sum of two parts: a slowly varying part $f_0(v,t)$ (which is called the "background") and a rapidly oscillating part $f_\sim(v, t)$. The slow change of the background due to the feedback effect of the oscillations on the particles is due to the averaged quadratic effects of the low-amplitude rapid oscillations; the situation is very similar to the familiar van der Pol analysis in nonlinear mechanics. On the other hand, the designation quasilinear means that the direct nonlinear coupling between different modes is not taken into account. Thus, the energy balance for the k-th mode is given in the same way as in the linear stability analysis:

$$\frac{d}{dt} E_k^2 = 2\nu E_k^2, \tag{11}$$

where $\nu$ is the imaginary part of the frequency.

Let us consider the derivation of the equations for the quasilinear approximation for longitudinal electron Langmuir oscillations in the one-dimensional case:



$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{eE}{m}\cdot\frac{\partial f}{\partial v} = 0, \quad \frac{\partial E}{\partial x} = 4\pi n_\sim e, \quad n_\sim = \int f_\sim dv. \tag{12}$$

The distribution function is separated into slowly and rapidly varying parts:

$$\begin{aligned} f_\sim &= \sum (f_k e^{i(kx-\omega_k t)} + \text{c.c.}), \\ E &= \sum (E_k e^{i(kx-\omega t)} + \text{c.c.}). \end{aligned} \tag{13}$$

The quantities $f_k$ and $E_k$ are connected by the usual relations of the linear theory

$$f_k = -i\frac{e}{m}\cdot\frac{1}{\omega_k - kv}\cdot\frac{\partial f_0}{\partial v}\cdot E_k. \tag{14}$$

The equation for the slowly varying part of the distribution function $f_0$ is obtained by averaging over the fast oscillations

$$\langle f\rangle = f_0. \tag{15}$$

In order to take this average we require that the plasma must simultaneously support many modes with different wave vectors and a random distribution of phases. The wave packets made up of these waves must be broad enough so that it is valid to neglect particle trapping in the potential wells associated with the individual modes in the packet. For example, in the case at hand (longitudinal Langmuir oscillations) the spread in phase velocities in the packet must be appreciably greater than the velocity with which a trapped particle would move in the potential well $e\varphi_0$: $\Delta(\omega/k) \gg (e\varphi_0/m)^{1/2}$. Taking $\langle Ef_0\rangle = \langle E\rangle f_0$, we obtain the following equation for $f_0$ from Eqs. (12) - (14):

$$\frac{\partial f_0}{\partial t} + \frac{\partial}{\partial v} D\frac{\partial f}{\partial v} = 0, \tag{16}$$

where D, the diffusion coefficient in velocity space, is proportional to the square of the electric field of the waves

$$-\frac{e^2}{m^2}\sum_{kk'}\Big\langle (E_{k'}\cdot e^{i(\kappa' x-\omega_{k'} t)} + \text{c.c.})\Big(\frac{E_k}{i(\omega_k - kv)}e^{i(kx-\omega_k t)} + \\ + \text{c.c.}\Big)\Big\rangle = -\frac{e^2}{m^2}2\pi\sum_k |E_k|^2 \operatorname{Im}(\omega_k - kv)^{-1}.$$

Equation (16) describes the feedback effect of the Langmuir oscillations on the particle distribution function. The applicability of the quasilinear equations is limited to cases in which the growth (damping) rates of the oscillations are much smaller than the frequency; if this condition is not satisfied the distribution function cannot be separated into rapidly varying and slowly varying parts.

It is clear from Eq. (16) for the averaged distribution function $f_0$ that the excitation of cooperative degrees of freedom (waves) gives rise to an additional diffusion in velocity space in addition to the usual collisional diffusion. In contrast with the original equation (12) we find that the present equation does not conserve entropy. This is not surprising since the averaging procedure used in going from Eq. (12) to Eq. (16) corresponds to going from a dynamical description to a statistical description. This same kind of approach, applied to the description of waves, leads to a kinetic equation for the quasi-particles [7-10]. Our statistical approach to the problem is essentially equivalent to the correlation method used in the theory of hydrodynamic turbulence. For wave-like instabilities, which are characterized by $\nu/\omega \ll 1$ (growth rate much smaller than the frequency), the coupled chain of equations for the correlation functions can be expanded in the small parameter $\nu/\omega$ [11]. However, instabilities of a non-wave-like natnre, which are characterized by $\nu \gg \omega$, cannot be considered this way since the problem does not contain a small parameter. In such cases cooperative phenomena can only be described by resorting to semiquantitative methods.

The present review is devoted primarily to the application of the theory of cooperative phenomena in the analysis of shock-wave thickness in a collisionless plasma.

By virtue of its collective properties one finds that a collisionless plasma can exhibit shock waves in which the thickness of the shock front is much smaller than the mean free path. At first glance this result might appear paradoxical. Let us consider a shock front (Fig. 2) whose thickness $\Delta$ is much smaller than the mean free path $l$. It would seem that the faster particles ($v > u$) from the region at the left (heated by the plasma shock wave) could move freely into the unperturbed plasma, thereby causing the transition region to expand to a thickness $l$ (the mean free path). We now ask for mechanisms that can prevent this expansion of the transition region.

1. The simplest case is that in which there is a magnetic field parallel to the plane of the front. The magnetic field turns the ions and electrons around in distances of the order of their respective Larmor radii $r_H$. Consequently one might reasonably expect $\Delta \sim r_H$. A sufficiently strong magnetic



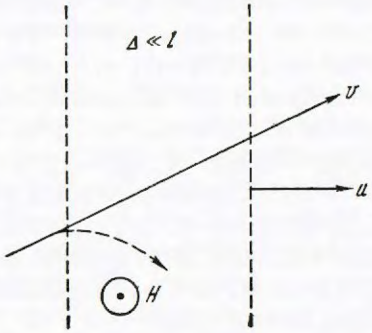


Fig. 2

field ($H^2/8\pi \gg nT$) will hinder expansion even if it does not lie in the plane of the front. This is due to the fact that when $H^2/8\pi \gg nT$ the shock-wave velocity is appreciably greater than the thermal velocity of the particles so that the fraction of ions (electrons) that overtake the wave is exponentially small. However, problems of this kind give rise to the following apparent paradox: The plasma states on the two sides of the shock front are presumably related by the appropriate conservation laws (Hugoniot adiabat), according to which the translational energy of the unperturbed plasma is transformed into internal plasma energy after passage of the shock wave. The question now arises as to what mechanism can provide dissipation if $\Delta \ll l$. The answer to this question is evidently that most of the internal energy in the perturbed plasma state behind the front resides in intense plasma oscillations. However, these nonlinear oscillations do not necessarily imply plasma instability. This question is closely related to the specific dispersion properties of the plasma. The second section of the present review is devoted specifically to the theory of nonlinear ordered plasma oscillations (the results are of interest independently of their relation to shock waves). The transient nonlinear motion of a plasma is extremely complicated and can only be analyzed in certain specific cases through the use of various simplifying assumptions. On the other hand, steady-state nonlinear oscillations can be analyzed fairly completely. Here it is interesting to note the useful analogy between nonlinear plasma waves and surface waves of finite amplitude in a heavy fluid in a channel of finite depth. In the theory of nonlinear plasma waves one also encounters "solitary" waves whose velocity depends on amplitude. Nonlinear waves in a plasma can be broken up as a consequence of various plasma instabilities and certain unstable nonlinear waves are considered at the end of § 2. One possible instability is the two-stream instability which appears in nonlinear waves in a magnetic field, being related to the electric current in the wave. If the ordered velocity of the electrons with respect to the ions is greater than the mean electron thermal velocity the wave energy is converted into energy associated with longitudinal electrostatic oscillations of the plasma as a consequence of the two-stream instability. There are also other kinds of instabilities which are inherent in nonlinear periodic waves: these are the so-called "decay" instabilities in which ordered waves decay, giving rise to a spectrum of irregular waves. This instability is reminiscent of the decay of collective excitations in the quantum theory of many-body systems. The combined effects of all of these mechanisms can be involved in the formation of the shock structure (§ 3).

2. When the magnetic field is small, or when there is no magnetic field, the mechanism which inhibits the expansion of a shock front is of a different nature. Let us assume that as a consequence of expansion some fast particles penetrate the unperturbed plasma in front of the shock (cf. Fig. 2). In this case the state of the plasma in this region is characterized by the unperturbed equilibrium distribution of the original particles plus that of the fast particles, that is to say, it becomes a nonequilibrium state since the particle velocity distribution is no longer Maxwellian. This nonequilibrium plasma is now unstable against the excitation of various instabilities and the fluctuating electric and magnetic fields arising as a consequence of these instabilities cause scattering of the ions and electrons. The essential point is that in the presence of fluctnating fields of this kind it is necessary to reexamine the notion of a mean free path. In a rarefied plasma the scattering on nonequilibrium random fluctuations can be much more important than the usual two-body Coulomb scattering.

## § 2. Nonlinear Plasma Oscillations

1. The most important role of nonlinear effects is to cause steepening of the leading edge of a wave. However, in plasma dynamics it is frequently found that the dispersion effects become significant as the steepness of the front increases. These two effects are responsible for some of the interesting features of the asymptotic motions that finally develop—the spontaneous production of intense oscillations as a consequence of the competition between nonlinearity and dispersion. The present section of this review is devoted to a systematic presentation of the theory of nonlinear undamped oscillations. We open our discussion with a general qualitative description of any initial perturbation.



In the linear theory the oscillatory motion of a plasma is described by a superposition of individual noninteracting modes [exp i $(\omega t - \mathbf{kr})$, where $\omega$ is the frequency and $\mathbf{k}$ is the wave vector] and, in general, a definite relation obtains between $\omega$ and $\mathbf{k}$: this is the dispersion relation $\omega = \omega(\mathbf{k})$. Any nonlinearity can evidently modify the pattern of motion described by the linear theory. It is instructive to consider the analogy with sound waves in ordinary gas dynamics. For example, sound waves, which are harmonic in the linear approximation, become distorted in the course of time because the wave amplitude becomes finite. This nonlinear deformation means, essentially, that portions of the wave profile characterized by high velocities tend to overtake portions characterized by low velocities so that a discontinuity is ultimately formed (provided the sound wave is not first damped).

Now let us trace the possible nonlinear distortion of the profile of a harmonic wave in a collisionless plasma. The tendency toward increasing steepness of the leading edge as a result of nonlinearity also operates in a collisionless plasma. [Transverse waves, for example, the magnetohydrodynamic Alfvén waves, are an exception. The equations describing these waves do not contain a nonlinear term of the form $(\mathbf{v}\nabla)\mathbf{v}$]. Now, in gas dynamics dissipative effects ultimately set a limit on the steepness of the front; in a collisionless plasma, however, the chief mechanism responsible for this function is dispersion. The competition between nonlinearity, which tends to "overturn" the wave, and dispersion can be illustrated as follows: The increasing steepness of the leading edge implies the generation of higher harmonics in the wave as a result of nonlinearity. In the first (linear) approximation any wave can be regarded as being pure harmonic [exp i$(\omega t - \mathbf{kr})$]; in the second approximation, however, the second harmonic must be included (as in the case of sound waves). In an expansion in terms of the wave amplitude the correction equation that arises in the second approximation is

$$\hat{L}_0 f_2 = \hat{l} f_1^2 \exp i\,(2\omega t - 2\mathbf{kr}). \tag{17}$$

Here, $f$ is the deviation of any field or plasma quantity from its equilibrium value (the subscripts 1 and 2 denote the first and second approximations, respectively); $L_0$ is a linear operator which characterizes the linear oscillations of the plasma according to some characteristic dispersion relation $\omega = \omega(\mathbf{k})$. The form of Eq. (17) is really that of an oscillator driven by a forcing function $\sim f_1^2$. It is clear that the second harmonic will be excited if this forcing function resonates with the characteristic frequency of the oscillator, i.e., if the original frequency multiplied by two corresponds (according to the dispersion relation) to a wave vector 2k. This resonance can be realized only if the dispersion relation $\omega = ck$ is linear, as is the case for ordinary sound waves. If the dispersion relation is an arbitrary one there will be no transfer of energy from the fundamental to the second harmonic if the driving force is far off resonance. This qualitative picture indicates that it is possible for periodic plasma waves to propagate without nonlinear distortion in a frequency region in which the dispersion exhibits an appreciable deviation from linearity. A knowledge of the dispersion relation $\omega(k)$ obtained from the linear theory can then provide certain general properties of the nonlinear behavior. For example, let us consider magnetoacoustic waves propagating across a magnetic field. At frequencies below $\omega_{Hi}$, the ion Larmor frequency, the characteristic phase velocity of these waves is

$$\frac{\omega}{k} = \left(\frac{\partial p}{\partial \varrho}\right)^{1/2} = \left(\frac{H_0^2}{4\pi\varrho_0} + 2\frac{p_0}{\varrho_0}\right)^{1/2}, \qquad (18)$$

where $H_0$ is the unperturbed magnetic field, $\varrho_0$ is the density, and $p_0$ is the pressure. As the frequency increases the phase velocity changes because of dispersion effects. In the general case the dispersion relation becomes very complicated even for these waves. Let us consider two limiting cases.

Low-Pressure Plasma ($p_0 \ll H_0^2/8\pi$). As the frequency $\omega$ increases the phase velocity diminishes from the value $H_0/\sqrt{4\pi\varrho_0}$ at low frequencies to zero at the frequency $(\omega_{Hi}\omega_{He})^{1/2}$ the so-called hybrid resonance frequency, $(e^2H^2/mMc^2)^{1/2}$; mM is the product of the electron and ion

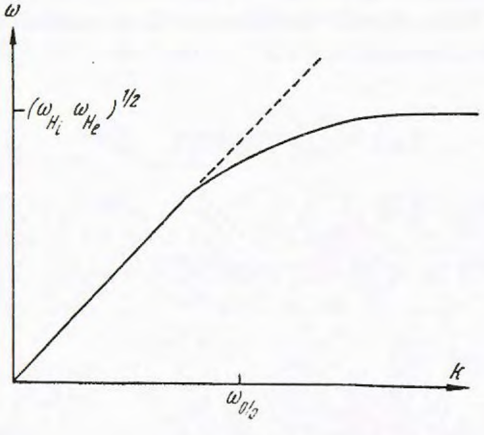


Fig. 3



masses. The dispersion curve for this wave is shown in Fig. 3. The corresponding dispersion relation is

$$\frac{\omega^2}{k^2} = \frac{H_0^2}{4\pi\varrho_0}\cdot\frac{\omega_0^2/c^2}{k^2+\omega_0^2/c^2}, \quad \omega_0^2 = \frac{4\pi ne^2}{m}.$$

The deviation from linearity in this dispersion relation becomes evident as $k \to \omega_0/c$. The quantity $c/\omega_0$ then determines the characteristic spatial scale for steady-state nonlinear magnetoacoustic waves. All of these considerations hold for the "weak"-field case in which $H^2/8\pi \ll nmc^2$. Under these conditions the plasma remains quasi-neutral as the magnetoacoustic wave propagates. On the other hand, in a very strong magnetic field

$$\frac{H_0^2}{8\pi} \gg n_0 mc^2,$$

the departure from neutrality becomes important at frequencies close to $\omega_{Hi}$. The dispersion relation for this case is (again neglecting thermal motion)

$$\frac{\omega^2}{k^2} = \frac{H_0^2}{4\pi\varrho_0}\,\frac{\dfrac{4\pi\varrho_0}{H_0^2}\,\Omega_0^2}{k^2+\dfrac{4\pi\varrho_0^2}{H_0^2}\,\Omega_0^2} \qquad \left(\Omega_0^2 = \frac{4\pi ne^2}{M}\right). \qquad (19)$$

The phase velocity now approaches zero at the ion Langmuir frequency $\omega \to \Omega_0$ and the characteristic length at which the departure from linearity becomes important is now $H_0M/4\pi\varrho_0 e$.

High-Pressure Plasma ($p_0 \gtrsim H^2/8\pi$). In this case dispersion effects become important when $\omega \to \omega_{Hi}$. At frequencies above $\omega_{Hi}$ the ion trajectory is only weakly distorted by the magnetic field in one oscillation period. In other words, the ion motion becomes one-dimensional rather than two-dimensional. Hence $\gamma$, the ion adiabaticity index (which characterizes the velocity of the magnetoacoustic wave) must be set equal to 3 when $\omega > \omega_{Hi}$ (rather than 2 as is the case when $\omega < \omega_{Hi}$). Consequently, the phase velocity of the wave increases when $\omega > \omega_{Hi}$. This means that the phase velocity increases with frequency (rather than decreasing) in a high-pressure plasma in this frequency range. The characteristic scale of the nonlinear waves is also different in this case, being of the order of the ion Larmor radius.

Up to this point we have been speaking of waves that propagate at precisely 90° with respect to the magnetic field. However, the linear small-oscillation theory shows that the dispersion relations change markedly even at small deviations of the direction of propagation from the perpendicular direction. The physical reason for the change is the fact that oblique waves possess an electric-field component along $H_0$. Under the influence of this electric field the electrons can move along $H_0$ much more rapidly than across $H_0$, thus producing strong modifications of the charge and current distributions in the wave. Let us again consider a cold plasma. At angles satisfying the condition $(m/M)^{1/2} \ll \theta \ll 1$, the dispersion relation $\omega = \omega(k)$ acquires the particularly simple asymptotic form [neglecting extremely short waves so that $\lambda \gg (c/\omega_0)\, 1/\theta$]

$$\frac{\omega^2}{k^2} \approx \frac{H_0^2}{4\pi\varrho_0}\left(1 + \frac{k^2\theta^2 c^2}{\Omega_0^2}\right). \tag{20}$$

The departure from linearity in $\omega = \omega(k)$ becomes important at wavelengths of order $(c/\Omega)\,\theta$. The phase velocity increases with increasing frequency at these wavelengths and one expects a change in the nature of the nonlinear motion.

Let us now consider the case in which there is no magnetic field. It is well known from the linear theory that ion-acoustic oscillations can propagate in a collisionless plasma only if the electron pressure is appreciably greater than the ion pressure $p_e \gg p_i$. This is the case, for example, in a two-temperature plasma in which the electron temperature is much higher than the ion temperature. If we simplify the analysis by assuming that the ions are cold ($T_i = 0$) the dispersion relation is similar to (19):

$$\left.\begin{aligned} \frac{\omega^2}{k^2} &= \frac{T_e}{M}\cdot\frac{\varkappa^2}{\varkappa^2 + k^2}, \\ \varkappa^2 &= \frac{\Omega_0^2 M}{T_e}. \end{aligned}\right\} \tag{21}$$

The characteristic scale length here is the Debye radius $1/\varkappa$.*

*Electron oscillations in a zero-temperature plasma with no magnetic field are characterized by the simple dispersion relation $\omega^2 = \omega_0^2$. There is no characteristic scale length in this case and steady-state nonlinear waves with any spatial period are possible.



All of the cases that have been considered above indicate that dispersion effects become important at short wavelengths or small scale lengths. In ordinary hydrodynamics dissipative effects also become important at small scale lengths and tend to limit the increasing steepness of the leading edge. In contrast with ordinary gas dynamics, however, in a collisionless plasma the limiting factor is dispersion and the difference between these two mechanisms is reflected in the mathematical structure of the original equations. Dissipative effects (viscosity, thermal conductivity, etc.) introduce irreversibility and increase the order of the derivatives by an odd number (for example, viscosity implies the addition of a term containing a second derivative in the Euler equation in gas dynamics). Dispersion effects, on the other hand, do not affect reversibility and increase the order of the derivatives in the equations by an even number. For example, let us consider the equations that describe the propagation of ion-acoustic waves when $T_e \gg T_i$. Under the assumptions made above one-dimensional motion is described by the equations:

$$\left.\begin{aligned} M\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) &= -e\frac{\partial \varphi}{\partial x}, \\ \frac{\partial n_i}{\partial t} + \frac{\partial (n_i v)}{\partial x} &= 0, \\ -\frac{\partial^2 \varphi}{\partial x^2} &= 4\pi e\left(n_i - n_0 e^{\frac{e\varphi}{T}}\right). \end{aligned}\right\} \tag{22}$$

Here M, v, and $n_i$ are, respectively, the ion mass, velocity, and density. The last equation in (22) contains the highest (second) derivative. For motion with a characteristic scale size appreciably greater than the Debye radius $(T/4\pi ne^2)^{1/2}$ it can be assumed that the plasma is quasineutral, $n_i = n_0 \exp(e\varphi/T)$, so that the $\partial^2\varphi/\partial x^2$ term in the last equation in (22) can be neglected. Eliminating the electric field from the remaining equations, we then then have

$$\left.\begin{aligned} M\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) &= -\frac{T}{n}\cdot\frac{\partial n}{\partial x}, \\ \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}nv &= 0. \end{aligned}\right\} \tag{23}$$

This system is of the same form as the equations for isothermal motion ($\gamma = 1$) in ordinary gas dynamics. In general, the front associated with any initial perturbation will become steeper and steeper in the course of time. This nonlinear distortion of the profile of the perturbation can be illustrated clearly in the particular case in which the initial velocity and density distributions are related functionally. In ordinary gas dynamics this case is amenable to the Riemann solution, which describes a so-called "simple" wave of arbitrary amplitude. The dependence of the velocity on time and coordinate in this solution is described by the implicit function

$$x = t\,[v \pm c] + \chi(v), \tag{24}$$

where c is the velocity of sound and $\chi(v)$ is a function that depends on the initial conditions. Equation (24) shows that the flow profile evolves in such a way that the solution must become triple-valued at some time. In ordinary gas dynamics (small mean free path) there is established under these conditions a steady-state flow characterized by a discontinuity (shock wave). On the basis of the described mathematical analogy one might then expect to find a collisionless shock wave in a collisionless plasma. In the plasma case, however, as soon as the leading edge of the perturbation becomes sufficiently steep the influence of dispersion effects becomes important [in Eq. (23), for example, these dispersion effects arise as a consequence of the departure from neutrality]. It is interesting to note that an analogous nonlinear wave is well known in the ordinary hydrodynamics of incompressible fluids; this is the nonlinear surface wave that propagates in a heavy fluid in a channel. If the channel is shallow the equation of two-dimensional motion reduces to an equation for one-dimensional motion (for the mean velocity of the fluid v at a given cross section and depth h):

$$\left.\begin{aligned} \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) &= -g\frac{\partial h}{\partial x}, \\ \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hv) &= 0, \end{aligned}\right\} \tag{25}$$

where g is the gravitational constant. These are the so-called shallow-wave equations, which represent the zeroth approximation for the asymptotic expansion of the exact equations of hydrodynamics of an incompressible fluid in a channel of finite depth in terms of the expansion parameter $h_0/L$, where $h_0$ is the channel depth and L is the characteristic scale size (for example, the wavelength). The shallow-wave equations are of the same form as the equations which describe the plane isentropic flow of a compressible gas (with



adiabaticity index $\gamma = 2$). For this reason, shallow water waves can be described by the Riemann simple-wave solutions. It follows from these solutions that an arbitrary initial profile of the fluid surface will eventually form a crest. We now note that the shallow-wave equations are exactly the same as the equations of motion for a collisionless plasma ($nT \ll H^2/8\pi$) moving across a magnetic field, if the wavelength is appreciably greater than $c/\omega_0$ (for example, if $H^2 \ll nmc^2$). The role of the channel depth h is played by the field H in the plasma equations. As the leading edge of the wave becomes steeper in the gas-dynamic case the dissipative effects becomes important; in contrast, in the collisionless plasma case, when the characteristic scale sizes approach $c/\omega_0$ the dispersion effects become important. Dispersion effects also play a role in the theory of shallow waves when L approaches $h_0$; Eq. (25) no longer holds. However, if $h_0/L$ is small, Eq. (25) can be improved by adding the higher-order terms in the expansion in $h_0/L$. These terms have the form of higher derivatives, corresponding to dispersion effects; the dispersion relation for low-amplitude waves then becomes

$$\frac{\omega^2}{k^2} = \frac{g}{k}\,\mathrm{th}\,(kh_0)$$

and at small values of $kh_0$ this relation can be written

$$\left(\frac{\omega}{k}\right)^2 \approx gh_0\left[1 - \frac{1}{3}\,(kh_0)^2 + \cdots\right. \tag{26}$$

The nonlinear oscillations of the surface of a heavy fluid have been investigated quite thoroughly; this is especially true for the so-called stationary waves, i.e., waves whose shape does not change in the course of time. In addition to finding periodic waves, characterized by wavelengths of order $h_0$, one also finds so-called "solitary" waves: these are essentially propagating isolated humps in the fluid level in the channel.

The analogy pointed out above indicates the possibility that a collisionless plasma might support similar periodic and solitary waves. However, because of the variety of dispersion relations that describe the various kinds of plasma oscillations one expects a greater variety of stationary waves. For instance, under certain conditions a plasma can support the propagation of solitary rarefaction waves (in the theory of surface waves in a fluid these would correspond to solitary depressions in the fluid level).

It will be shown in §3 that nonlinear waves of this kind in a collisionless plasma are intimately related with shock waves. Up to this point our discussion has been concerned with finite waves of low amplitude. However, the situation becomes completely different at high amplitudes. At high amplitudes the dispersion effects may not be sufficient to limit the increasing steepness of the wave and the front can "break" at some critical amplitude, producing a region of multivelocity flow (this obviously applies for a plasma that is initially cold).

**2.** In gas dynamics the asymptotic form (as $t \to \infty$) of any initial motion will, in general, be a shock wave. Let us now ascertain the nature of asymptotic motion in a collisionless plasma. It might be expected that a stationary wave pattern would be established as $t \to \infty$. Assuming that such a stationary motion exists, in the one-dimensional case, at least, we can analyze the problem without difficulty by solving the plasma dynamics equations directly. The standard procedure for obtaining the solutions is to choose a coordinate system that moves with the wave in the original equations. The time dependence disappears in this coordinate system and the problem reduces to the search for stationary flow, with the wave velocity u first being introduced into the problem as a free parameter. The solubility conditions then determine the limits within which u can change and also establish the relation between u and the wave amplitude. As far as the analysis of shock-wave thickness is concerned, primary interest attaches to the stationary nonlinear waves whose low-amplitude dispersion relation is linear at long wavelengths (sound) and in which dispersion effects appear at the short wavelengths.

We begin by considering waves propagating across a magnetic field. If the Larmor radius is small (drift approximation) the hydrodynamic equations can be used to describe the situation. The only stationary motion allowed by these equations is the trivial case of plane-parallel flow and in order to find the nontrivial stationary motion we must take account of the dispersion effects that appear at small scale lengths. These dispersion effects derive from the departure from neutrality and from electron inertia, the introduction of either one of these factors being sufficient to obtain stationary motion which is not a plane-parallel flow. These mechanisms are to be associated with two characteristic scale lengths.

Let us now examine the way in which dispersion effects lead to the formation of stationary waves in a cold plasma ($nmc^2 \gg H^2/8\pi \gg nT$). We neglect thermal motion so that the set of equations that describes the ion motion, the electron motion, and the field profile in the stationary wave is as follows ($m_i = M$, $m_e = m$; the wave propagates along the x-axis and the magnetic field is along the z-axis; Fig. 4):

$$m_{i,e} v'_{x_{i,e}} \left(v_{x_{i,e}} - u\right) = \pm eE_x \pm \frac{e}{c} v_{y_{i,e}} H, \tag{27}$$



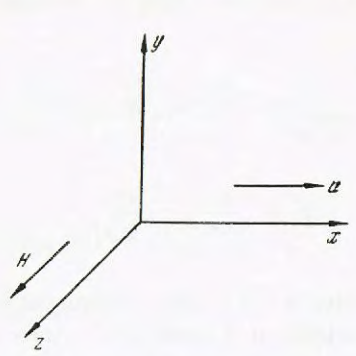


Fig. 4

$$m_{i,e} v'_{y_{i,e}} \left(v_{x_{i,e}} - u\right) = \pm eE_y \mp \frac{e}{c} v_{x_{i,e}} H, \tag{28}$$

$$\left.\begin{aligned} E_y &= \frac{u}{c}(H - H_0), \\ -H' &= \frac{4\pi}{c} ne\,(v_{y,i} - v_{y,e}), \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} n_{i,e} &= \frac{n_0 u}{u - v_{i,e}}, \\ n_i &= n_e. \end{aligned}\right\} \tag{30}$$

The last equation, which expresses the neutrality condition, implies that the velocity components in the x-direction are the same for the electrons and ions. Eliminating all variables except H from these equations we find (to accuracy of order m/M)

$$-\frac{mc^2}{4\pi n_0 e^2 u}\frac{d}{dx}\left[\frac{dH}{dx}\left(\frac{H^2 - H_0^2}{8\pi n_0 Mu} - u\right)\right]\left(\frac{H^2 - H_0^2}{8\pi n_0 Mu} - u\right)$$

$$= \left(\frac{H^2 - H_0^2}{8\pi n_0 Mu} - u\right)H + uH_0. \tag{31}$$

This equation determines the profile of H in the stationary wave. Integrating once we have

$$-a^2 H'^2 \left(\frac{H^2 - H_0^2}{8\pi n_0 Mu} - u\right)^2 = \frac{(H^2 - H_0^2)^2 - 16\pi n_0 Mu^2 (H - H_0)^2}{16\pi n_0 M} + C$$

$$\left(a^2 = \frac{mc^2}{4\pi n_0 e^2} = \frac{c^2}{\omega_0^2}\right). \tag{32}$$

Different values of C, which is the constant of integration, are to be associated with various kinds of solutions. It is instructive to trace the variation of the solution as C changes by plotting integral curves in the phase plane (H, H'). Curves of this kind are shown in Fig. 5.

The solntions of Eq. (32) must describe periodic waves of finite amplitude, with one exception; this is the solution for a special choice of the constant C:

$$C = 0.$$

This special choice gives $dH/dx = 0$ for $H = H_0$, in which case

$$\pm a \frac{dH}{dx} = \frac{(H - H_0)}{\frac{H^2 - H_0^2}{8\pi n_0 Mu} - u} \cdot (16\pi n_0 M)^{-1/2} \cdot \sqrt{16\pi n_0 Mu^2 - (H + H_0)^2}. \tag{33}$$

It is impossible to form a physically meaningful solution for H over the entire x-axis if a fixed sign is chosen in front of the radical in Eq. (33). How-

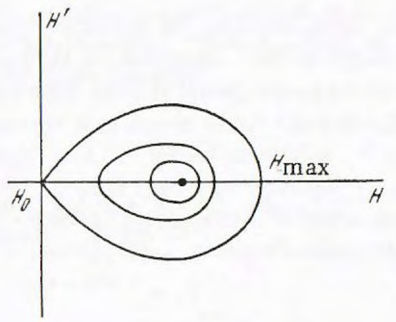


Fig. 5



ever, there are solutions which are everywhere continuous (up to the second derivative inclusively) for which the derivative H' changes sign at a certain $x = x_1$. H reaches its maximum value $H_{max}$ at this point. The equation $(dH/dx)(x_1) = 0$ relates the peak magnetic field $H_{max}$ to the velocity of propagation of the wave and plays a role analogous to that of a dispersion relation

$$16\pi n_0 Mu^2 - (H_{max} + H_0)^2 = 0. \tag{34}$$

Solving Eq. (34) for u, we find [12-15]

$$u^2 = \frac{(H_{max} + H_0)^2}{16\pi n_0 M}. \tag{35}$$

In the limiting case of low amplitudes ($H_{max} \to H_0$) Eq. (35) gives the so-called magnetic sound velocity and the propagation velocity increases with amplitude. Integration of Eq. (33) gives the profile of H in this wave, which is found to be symmetric with respect to $x = x_1$, and represents a single pulse of magnetic field with width of order

$$\delta \sim \frac{c}{\omega_0},$$

where

$$\omega_0 = \left(\frac{4\pi ne^2}{m}\right)^{1/2}$$

Thus, the solution of Eq. (33) is evidently the collisionless-plasma analog to a solitary wave. The magnetic-field profile in the solitary wave

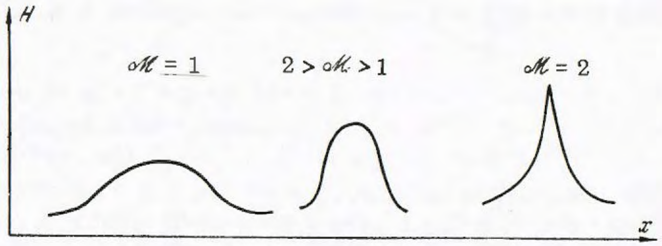


Fig. 6

at low amplitudes ($H_{max} - H_0 < H_0$) is given by a simple expression:

$$H = H_0\left\{1 + 2\left(\frac{u^2}{\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^2} - 1\right)\mathrm{sh}^2\frac{x}{\frac{c}{\omega_0}}\sqrt{\frac{u^2}{\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^2} - 1}\right\}. \quad (36)$$

The function $H = H(x)$ is shown in Fig. 6 for various values of the Mach number $\mathcal{M} = \frac{u}{H_0/\sqrt{4\pi n_0 M}}$.

Equation (17) does not have real solutions for arbitrarily large u and H. For example, solitary-wave solutions exist when $H_{max} \leq 3H_0$ [i.e., $u < 2(H_0/\sqrt{4\pi n_0 M})$]. As the wave amplitude approaches the critical value the ion (electron) density at the crest of the wave becomes infinite. This phenomenon can be described physically as follows: The solitary wave is essentially a hill in the electric potential $\varphi$ and in the coordinate system that moves with the wave the ion flux from $x = \infty$ impinges on this potential barrier with a velocity u. If the amplitude is not too large, the initial kinetic energy of the ion $Mu^2/2$ is greater than the height of the potential barrier $e\varphi_{max}$ and the ions pass through the barrier, being retarded in the process. However, the solution shows that as the wave amplitude increases the potential barrier becomes so high that $e\varphi_{max} > Mu^2/2$. The situation $e\varphi_{max} \approx Mu^2/2$ corresponds to an amplitude $H_{max} = 3H_0$ (in other words, the critical Mach number is 2). Having lost velocity, the ions are trapped at the crest of the wave and the ion density increases without limit. At still higher amplitudes the ions are reflected from the barrier, but the motion corresponding to this case is not described within the framework of our original system of equations (27)-(30) since the reflection implies a multistreaming flow (interacting flows of incident and reflected ions).

Thus it is evident that dispersion effects may not be sufficient to prevent breaking of the wave in a cold plasma if the amplitude is sufficiently large.

On the other hand, if the thermal spread in the ion velocity is taken into account, some ions are reflected from the barrier even at low amplitudes (these are the ions with small relative velocities $u - v_x$). The ions with low relative velocities are those that were originally moving in the direction of propagation of the wave with a velocity approximately equal to u; these ions are essentially trapped and extract energy, causing the wave to be damped. For the time being, however, we shall neglect damping, in which case it is an



easy matter to find a solution for the solitary wave in a more general form; the thermal spread is taken into account by introducing the ion velocity distribution function.

A closely related class of nonlinear oscillations can also be realized in a plasma in the absence of an external magnetic field; these are the nonlinear ion-acoustic oscillations. Linear theory indicates that ion-acoustic oscillations can only be excited when $T_e \gg T_i$ so that our analysis will be limited to this case.

If it is assumed that all quantities depend on x and t only in the form $x - ut$, Eq. (22) can be reduced to a single second-order differential equation for the potential:

$$\varphi'' = 4\pi n_0 e\cdot\left\{\frac{u}{\sqrt{u^2 - \frac{2e\varphi}{M}}} - \exp\left(\frac{e\varphi}{T}\right)\right\} \quad (37)$$

Integrating Eq. (37) once we have

$$-\frac{1}{2}(\varphi')^2 = 4\pi n_0 e\left(-\frac{uM}{e}\sqrt{u^2 - \frac{2e\varphi}{M}} - \frac{T}{e}\exp\frac{e\varphi}{T}\right) + C. \quad (38)$$

Various periodic waves can now be found depending on the choice of the integration constant C (see the integral curves on the phase plane in Fig. 7). A special case is represented by the value of C given by the condition $\varphi' \to 0$ when $\varphi \to 0$, i.e., $C = 4\pi n_0(Mu^2 + T)$. This case is treated specially in the phase plane and corresponds to a solitary wave (Fig. 8) which is a symmetric potential hill.

The velocity of propagation of this wave u, as a function of $\varphi_{max}$ the peak potential, is found from Eq. (38) by writing $\varphi' = 0$ when $\varphi = \varphi_{max}$ [4],

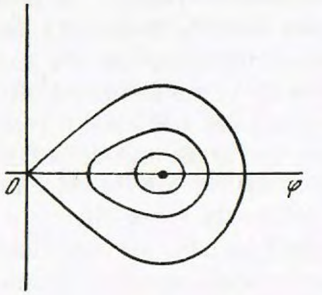


Fig. 7

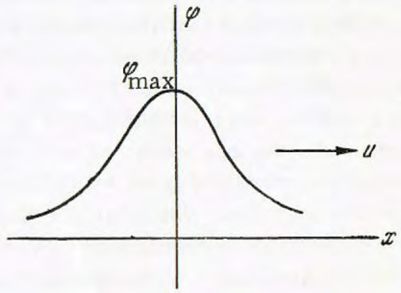


Fig. 8

$$u^2 = \frac{T}{2M} \cdot \frac{\left[\exp\left(\frac{e\varphi_{max}}{T}\right) - 1\right]^2}{\exp\left(\frac{e\varphi_{max}}{T}\right) - 1 - \frac{e\varphi_{max}}{T}}. \qquad (39)$$

In the limiting case of low amplitudes ($e\varphi_{max} \ll T$), u approaches the velocity of isothermal sound $\sqrt{T/M}$. At low amplitudes the profile potential in the solitary wave is given by

$$\varphi \approx \frac{3}{2}\frac{T}{e}\left(1 - \frac{T}{Mu^2}\right)\mathrm{sh}^2\left\{\frac{\sqrt{\pi n_0 e}}{\sqrt{T}}\sqrt{1 - \frac{T}{Mu^2}} \cdot x\right\}. \qquad (40)$$

Ion waves also exhibit an upper bound on amplitude, beyond which propagation is impossible. This is the point at which the motion becomes multivalued, because the ions can no longer get across the potential barrier. The critical amplitude of the solitary wave is given by $e\varphi_{max} = Mu^2/2$. If the wave velocity u is fonnd from Eq. (39), the quantity $e\varphi_{max} \approx 1.3$ T. This value then represents the critical amplitude for the solitary wave and corresponds to a Mach number $\mathcal{M} = \frac{u}{\left(\frac{T}{M}\right)^{1/2}} \approx 1.6.$

In both of the cases considered above (magnetoacoustic waves and ion waves) we have observed a similar pattern for the steady-state nonlinear motion. The only essential difference is in the magnitude of the characteristic scale lengths. This result is not surprising inasmuch as the corresponding linear dispersion relations are very much the same for these cases [cf. Eqs. (19) and (21)]. Solitary waves are of great interest since they represent a special kind of steady-state nonlinear motion. Whereas periodic waves are compatible with an arbitrary dispersion relation (so long as it is nonlinear), solitary waves require a very special kind of dispersion relation. This requirement follows from the fact that the spectral expansion of the profile of a solitary wave is a continuous spectrum whereas the spectral expansion of a periodic wave contains discrete values of $\omega$ and k only. For the former the discussion given at the beginning of this section does not hold. It is clear that the existence of a stationary solitary wave requires that the high-amplitude parts of the profile must propagate with velocities smaller than the velocity given by the linear theory. When this situation obtains the strong effect of nonlinearity in the high-amplitude part can, roughly speaking, be compensated by the reduction in $\partial\omega/\partial k$. This kind of plasma dispersion relation (decreasing $\partial\omega/\partial k$ with increasing k) is characteristic of the magnetoacoustic wave and



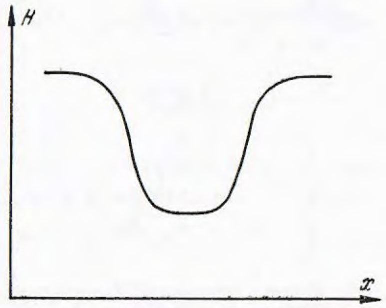


Fig. 9

the ion-acoustic wave when $T_e \gg T_i$. It is also characteristic of the dispersion relation for surface waves of a heavy fluid in a channel of finite depth.

Waves characterized by the inverse kind of dispersion relation, i.e., waves for which $\partial\omega/\partial k$ increases with increasing k, must form solitary rarefaction waves in the nonlinear case (Fig. 9), in contrast with the compression wave described above. This kind of dispersion relation is characteristic of the propagation of waves across a magnetic field in a high-temperature plasma. It is not difficult to find the profile for such waves and the relation between the phase velocity and the peak magnetic field. However, we have already examined the physical features of undamped nonlinear waves and now wish to investigate possible damping mechanism.

**3.** "Ordinary" damping mechanisms are to be associated with the conversion of energy of ordered motion into heat as a result of particle collisions; in a rarefied plasma, however, there is another possibility – "collisionless damping." This phenomenon is related to the presence of trapped particles, i.e., particles whose velocities are approximately the same as the phase velocity. We shall illustrate this effect using the simple example of electron Langmuir oscillations, but the essential qualitative features of the phenomenon are the same for any kind of wave. It was shown by Landau [3] that waves of extremely low amplitude are damped if the distribution function describing the resonance particles has a negative slope, i.e., if $df/dv < 0$ at $v = \omega/k$. This damping is due to the fact that the faster particles are retarded by the wave while the slower particles are accelerated. If there are fewer fast particles than slow particles at resonance the wave is damped. In actual fact the linear theory becomes inapplicable very rapidly because the form of $f(v)$ is changed by the damping process itself. For instance, in the "quasi-linear" theory the distortion of the distribution function is described by the equation

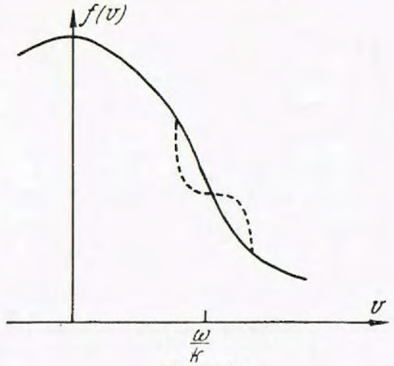


Fig. 10

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial v}D\frac{\partial f}{\partial v}, \qquad (41)$$

where $D \sim E^2$ and is especially large for the resonance particles.

According to Eq. (41) the resonance particles are redistributed and a plateau $df/dv \to 0$ is formed on the distribution function (Fig. 10) [Eq. (41) is analogous to the heat-conduction equation in an inhomogeneous medium]. When this happens a saturation point is reached and the damping process is terminated. However collisions, although rare, will still gradually "round off" the edge of the plateau and establish a quasistationary state in which $df/dv$ is slightly different from zero ($df/dv < 0$). In order to find the magnitude of this slope and the associated damping it is necessary to introduce a collision term in Eq. (41). Clearly, the larger the amplitude of the oscillations the stronger will be the feedback effect on the distribution function in the region $v \approx \omega/k$. Thus, it is reasonable to assume that the damping coefficient $[(1/\varepsilon)\, d\varepsilon/dt]$ ($\varepsilon$ is the wave energy), which is proportional to $df/dv$ ($v = \omega/k$), will diminish as $\varepsilon$ increases. The stationary slope $df/dv$ is found from the equation

$$\frac{d}{dv}D\frac{df}{dv}=\mathrm{St}\,(f), \qquad (42)$$

where the expression for the quasilinear diffusion coefficient D can be simplified for the resonance particles ($v \approx \omega/k$)

$$D\,(v)=\sum_k \frac{e^2}{m^2}E_k^2\,\mathrm{Im}\,(\omega+kv)^{-1}\sim\frac{e^2E^2}{m^2\omega}. \qquad (43)$$



In the expression for St (f) we retain the term containing the second derivative

$$\mathrm{St}\,(f)\approx \nu_{\mathrm{coll}}\frac{T}{m}\frac{d^2\,(f_0-f)}{dv^2},$$

where $f_0$ is the Maxwellian distribution function. This simplified form of the "collision integral" allows the reestablishment of the local equilibrium distribution. Integrating Eq. (42) once, we have

$$\frac{df}{dv}=\frac{df_0}{dv}\cdot\frac{1}{1+\dfrac{e^2E^2}{m\omega T\nu_{\mathrm{coll}}}}. \qquad (44)$$

It is clear from this expression that for a low-amplitude wave $e^2E^2/m\omega T\nu_{\mathrm{coll}} < 1$ the damping factor

$$\nu=\frac{\pi\omega_0}{2}\cdot\left(\frac{\omega}{k}\right)^2\frac{df}{dv}\left(v=\frac{\omega}{k}\right)\to\frac{\pi\omega_0}{2}\cdot\left(\frac{\omega}{k}\right)^2\frac{df}{dv}\left(v=\frac{\omega}{k}\right)\equiv\nu_0$$

approaches the Landau damping factor $\nu_0$. However, the linear theory does not apply at amplitudes such that $e^2E^2/m\omega T\nu_{\mathrm{coll}} > 1$. As indicated by Eq. (44), the damping in this case must diminish as the amplitude increases, varying as $E^{-2}$.

The damping analysis given above applies only for broad wave packets, since we have made use of the quasilinear theory. The relation in Eq. (44) would not hold for a monochromatic wave (with one $\omega$ and k), which requires special consideration. We shall limit ourselves here to a semiquantitative estimate in order to establish the dependence of the damping on amplitude. Equation (44) can be interpreted by writing it in the form $\nu = \nu_0/(1+\tau_1/\tau_2)$. Here, $\nu_0$ is the damping found in the linear approximation (Landau damping); $\tau_1$ is the characteristic time required to establish a local Maxwellian distribution; $\tau_2$ is the characteristic time required for distortion of the distribution function by the wave packet. If $\tau_1 \ll \tau_2$ (in which case the Maxwellian distribution function is established by collisions) then the usual Landau damping is obtained. As the wave amplitude increases the distortion due to the interaction with the wave becomes so large that collisions can no longer establish a Maxwellian distribution function and the damping rate is diminished. By extending this analysis it is possible to estimate the absorption for a monochromatic wave if the values of $\tau_1$ and $\tau_2$ are chosen properly. Let $\varphi$ be the potential in the wave; in this case particles with velocities (with respect to the wave) of order $\pm\sqrt{e\varphi/m}$ will cause absorption. This means that the distribution function will be affected primarily in a region $\Delta v$ with width of order $\pm\sqrt{e\varphi/m}$. Small-angle Coulomb collisions establish local equilibrium in this same region in a time $\tau_1 \sim e\varphi/\nu_{coll}T$. The time required for the nonlinear distortion due to the interaction with the wave is of order $\tau_2 \sim \lambda/\sqrt{e\varphi/m}$, where $\lambda$ is the wavelength. Finally, we find [4]

$$\nu = \frac{\nu_0}{1 + \frac{(e\varphi)^{3/2}}{T\lambda\nu_{coll}\sqrt{m}}}. \tag{45}$$

This result means that the damping goes as $E^{-3/2}$ for a monochromatic wave. A rigorous analysis verifies this conclusion [16].

4. The results given above lead to the conclusion that nonlinear waves in a collisionless plasma will be damped very slowly if the distribution of particles responsible for the damping is subject to a "relaxatiou" effect. This, however, does not guarantee that nonlinear steady-state waves can continue to exist once they are produced. It is still necessary to determine whether or not the waves are stable against various kinds of random disturbances; if they are unstable the energy of the nonlinear wave motion goes into some other kind of plasma motion, possibly random turbulent motion, and this process is equivalent to a damping process. It is clear that the propagation of a strong perturbation in a collisionless plasma implies a significant departure of the plasma from thermodynamic equilibrium; in turn this departure implies the possibility of instability.

For example, let us consider a steady-state solitary wave propagating across the magnetic field in a cold plasma ($nT \ll H^2/8\pi$). We shall again be interested in the motiou of the plasma ions and electrons in this wave. If $H^2/8\pi \ll nmc^2$, the plasma is quasineutral. The ions and electrons move with the same velocity in the directiou of propagation of the wave. However, the electric current in the direction perpendicular to the wave velocity and the magnetic field is due to the electrons alone. It is well known that the existence of an appreciable relative motion between the ions and electrons in a uniform plasma can result in the so-called two-stream instability. It is clear that an analogous effect might be expected here. The problem is simplified if we neglect terms that take account of the unperturbed motion of the plasma in the x-direction in analyzing small deviations from the stationary pattern of the solitary wave. This procedure is valid if the instability growth time is significantly smaller than the time required for the solitary wave to move through a given region. This time is of order $\delta\sqrt{4\pi nM}/H$, where $\delta$ is the



"width" of the wave. The problem can be solved easily if the perturbed motion of the ions and electrons is analyzed in terms of the two-fluid theory with adiabatic pressure variations. The effect of the magnetic field on the perturbed motion is neglected since we limit ourselves to oscillation frequencies much greater than the electron Larmor frequency $\omega_{He}$. In this approximation the equations for the perturbed quantities $v_e$ (electron velocity), $v_i$ (ion velocity), $n_e$ and $n_i$ (electron and ion densities), and $\varphi$ (electric potential) are

$$\left.\begin{aligned} i(\omega + kv_0)\,\mathbf{v} &= \nabla\frac{e}{m}\varphi - \nabla\frac{T}{mn_0}n, \\ i\omega\mathbf{V} &= -\nabla\frac{e}{M}\varphi; \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} i(\omega + kv_0)\,n_e + ikn_0v_y + \frac{d}{dx}(n_0v_x) &= 0, \\ i\omega n_i + \frac{d}{dx}(n_0V_x) + ikn_0V_y &= 0, \\ -k^2\varphi + \varphi'' &= 4\pi e(n_e - n_i). \end{aligned}\right\} \tag{47}$$

The equations in (46) are the electron and ion equations of motion; the equations in (47) are the electron and iou equations of continuity and Poisson's equation. We have taken the perturbed quantities to be of the form $\varphi(x)\,e^{i(\omega t + ky)}$. The quantities $v_0$, $T$, and $n_0$ appearing in the equations are the x-dependent unperturbed mean velocity of the electrons (y-direction), the electron temperature (we assume that the ions are cold), and the plasma density. Under the assumption that the x-derivatives of the perturbed quantities are much greater than the x-derivatives of the unperturbed quantities (semiclassical approximation) this system of equations reduces to a second-order differential equation for $n_e$:

$$\frac{T}{m}\frac{d^2n_e}{dx^2} + \left[(\omega + kv_0)^2 - \frac{T}{m}k^2 - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}}\right]n_e = 0. \tag{48}$$

The stability investigation reduces to the problem of finding the characteristic values for Eq. (48). To satisfy the boundary conditions we choose solutions that fall off in both directions going away from the solitary wave.

Let us investigate the behavior of the function

$$F(x, \omega, k) = (\omega + k v_0)^2 - \frac{T}{m} k^2 - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}}.$$

In a uniform plasma this function would be independent of x and the dispersion relation relating $\omega$ and k would be of the form

$$F(\omega, k) = 0. \tag{49}$$

This equation yields unstable solutions when $v_0^2 > T/m$, that is to say, when the mean relative velocity of the ions and electrons is greater than the electron thermal velocity. If k is not too large $[k^2 \ll (\omega_0^2/T)m]$ this equation can be written approximately as

$$F(\omega, k) \approx k^2\left(v_0^2 - \frac{T}{m}\right) - \frac{\omega_0^2}{1 - \frac{\Omega_0^2}{\omega^2}} = 0.$$

Thus

$$\omega^2 = \frac{\Omega_0^2 k^2\left(\frac{T}{m} - v_0^2\right)}{\omega_0^2 - k^2\left(v_0^2 - \frac{T}{m}\right)},$$

and $\omega$ becomes imaginary (instability) when $v_0^2 > T/m$. Now, returning to the inhomogeneous problem, let us consider the spatial behavior of the function

$F(x, \omega, k) \approx k^2\left(v_0^2 - \frac{T}{m}\right) - \frac{\omega_0^2}{1 - \Omega_0^2/\omega}$ (in this approximation it is sufficient to consider real $\omega^2$). In Fig. 11 we show the profiles of $v_0$ and $\omega_0^2$ in the solitary wave as a function of x. $n_e$ is described by an oscillatory solution where $v_0^2 > T/m$, i.e., where $F(x, \omega, k) > 0$. On the other hand, far from the solitary wave we have $F(x, \omega, k) < 0$, corresponding to exponentially damped solutions. These solutions are connected at the turning points, at which $F(x, \omega, k) = 0$. Thus, the required localized solutions always exist and the instability appears if there is a region in which $v_0^2 > T/m$ inside the solitary wave. The growth rate for this instability is known to be of order $\Omega_0$ [in a "zero-temperature" plasma the maximum growth rate is still higher, being $\Omega_0 (M/m)^{1/6}$]. Several simplifying assumptions have been made in obtaining these results. The problem is somewhat more complicated in the general case: the equation corresponding to Eq. (48) is a fourth-order equation and the connecting points move into the complex plane of x. However, the instability condition $v_0^2 > T/m$ remains unchanged.



The peak value of $v_0$ in the solitary wave increases as the Mach number $\mathcal{M}\left(\mathcal{M} = \frac{u\sqrt{4\pi n_0 M}}{H_0}\right)$ increases. There is a critical value of the Mach $(\mathcal{M} = \mathcal{M}^*)$ at which $v_0$ exceeds the mean thermal velocity of the electrons so that the wave becomes unstable. Using the solution for the profile of the solitary wave given earlier, we can show easily that for a cold plasma $(nT \ll H^2/8\pi)$

$$\mathcal{M}^* \approx 1 + \frac{3}{8}\left(\frac{8\pi n_0 T}{H_0^2}\right)^{1/3}. \tag{50}$$

Essentially this result means that by taking some care in analyzing inhomogeneous problems we can use the same criterion for the two-stream instability as in a uniform plasma. Let us now verify the original assumption that the growth time for the instability is appreciably smaller than the time for the solitary wave to move through the plasma region $\frac{c}{\omega_0}\cdot\left(\frac{H_0}{\sqrt{4\pi n_0 M}}\right)^{-1} \gg \nu^{-1}$. Substituting $\nu \sim \Omega_0$, we find $H_0^2 \ll 4\pi n_0 mc^2$.

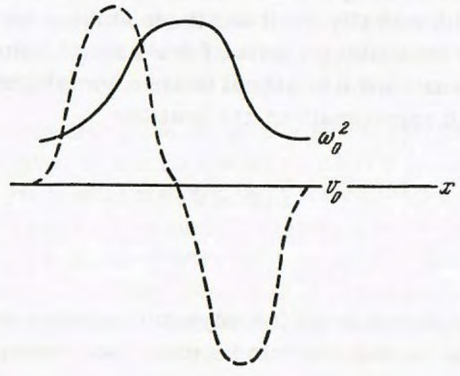


Fig. 11.

The development of the two-stream instability has an effect on the initial nonlinear wave by causing an effective damping; the energy associated with the ordered electron motion in the nonlinear wave is converted into the energy of random electron Langmuir oscillations. In this sense the net effect of the instability can be regarded as a kind of "collective" frictional force between the electrons and ions [4, 17, 18].

Although we have been interested in the instability of a solitary steady-state wave it is clear that similar considerations hold for other plasma waves in a magnetic field. The two-stream instability pertains to only one particular class of problems. The so-called "decay" instability, which can be observed in periodic nonlinear waves [19], is of a more general nature.

We start our analysis of the "decay" mode with some general remarks. In investigating the stability of stationary nonlinear waves (which for brevity will be called the "background") it is convenient to transform to a coordinate system that moves with the wave. In this coordinate system the coefficients in the linearized equations that describe the small deviation from the background are independent of time and the time dependence of the solutions can be written in the form $e^{i\omega t}$. The problem then reduces to that of solving a system of equations which can be written symbolically in the form

$$\hat{L}\varphi = 0, \tag{51}$$

where $\hat{L}$ is a linear differential operator. The actual form of the operator depends on the background and the characteristic frequency $\omega$, the determination of which represents the essence of the stability investigation. The operator $\hat{L}$ can be written as a sum of terms $\hat{L}_0$ and $\hat{L}_1$; $\hat{L}_0$ is a differential operator with constant coefficients while $\hat{L}_1$ is a differential operator that goes to zero together with the infinitesimally small amplitude of the perturbation used to test the stability of the stationary wave. For a wave of finite (but small) amplitude $\hat{L}_1$ will be small and it is natural to use a perturbation-theoretic treatment. In the zeroth approximation* the equation

$$\hat{L}_0\varphi = 0 \tag{52}$$

*If dissipation is neglected in the hydrodynamic approximation $\hat{L}_0$ is a self-adjoint operator and its characteristic functions must correspond to undamped waves.



describes the oscillations of a uniform plasma with characteristic functions proportional to $e^{ikr}$ and characteristic values of $\omega$ that satisfy the dispersion equation $\omega = \omega(k)$. In the first approximation in $\hat{L}_1$ we have diagonal matrix elements $\langle \varphi_{\omega'} | \hat{L}_1 | \varphi_\omega \rangle$ in which the spatial dependence of $\hat{L}_1$ is given by the factors $e^{\pm ik_0 r}$. It is clear that the matrix elements will vanish if each value of the frequency $\omega$ corresponds to only one value of the modulus of the wave vector k. The first perturbation-theoretic approximation gives a nonvanishing contribution only when there are degenerate states for which one value of $\omega$ corresponds to at least two wave vectors ($k_1$ and $k_2$). In this case the quantities $k_1$ and $k_2$ must satisfy the relation

$$k_1 = k_0 + k_2, \tag{53}$$

and the fact that they correspond to the same frequency can be written in the form $\omega_1 = \omega_2$. If we now convert from the wave coordinate system to the laboratory coordinate system the frequencies $\omega_1$ and $\omega_2$ will be different. The following condition will be satisfied:

$$\Omega_1 = \Omega_0 + \Omega_2, \tag{54}$$

where $\Omega_0$ is the oscillation frequency of the background ($\Omega_0 = k_0 u$) while $\Omega_1$ and $\Omega_2$ are frequencies corresponding to the wave vectors $k_1$ and $k_2$ ($\Omega_1 = \omega_1 + k_1 u$, $\Omega_2 = \omega_2 + k_2 u$). The conditions in (53) and (54) can be regarded as couservation laws for the quasi-energy and quasi-momentum in the interaction (decay) of the quasi-particles that represent the waves. Hereinafter we shall call these the decay conditions, and the instability that arises will be called the decay of a wave with frequency $\Omega_0$ and wave vector $k_0$ into waves with frequencies $\Omega_1$ and $\Omega_2$ and wave vectors $k_1$ and $k_2$. The decay conditions are not necessarily satisfied for arbitrary dispersion relations $\omega(k)$. Curves corresponding to various kinds of spectra are shown in Fig. 12.

It is evident that decay can occnr only for the spectra denoted by 1 and 4. Waves characterized by spectra similar to 2 and 3 are stable against decay. However, if there are several branches in a spectrum, waves characterized by spectra similar to 2 can be unstable against decay into waves which belong to a different branch. To put the matter more precisely: decay is possible when three points exist, corresponding to waves $\Omega_0$, $\Omega_1$, and $\Omega_2$ (in general these three points can lie on different branches), such that it is possible to draw a curve similar either to curve 1 or to curve 3 (in certain cases transitions between different branches are "forbidden" by polarization conditions). The

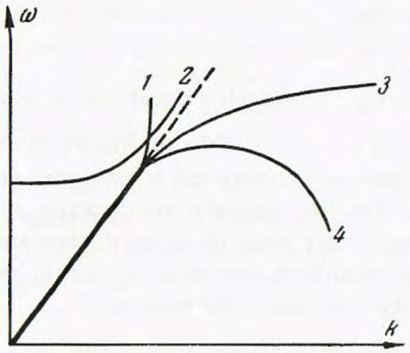


Fig. 12

fact that the decay conditions are satisfied does not necessarily mean that an instability will exist. If the correction to the frequency is computed in the first perturbation-theoretic approximation it will be found that either the frequence is imaginary, i.e., there actually is an instability, or that it is real, i.e., that the frequency is merely shifted. It is necessary to make a specific investigation to determine what actually happens in a given case. The quantities $A_1$, which characterize the background, can be written in the form

$$A_i = A_{i\,0} + 2\delta\,(k_0)\sin k_0 r + O\,(\delta A_i^2),$$

where $k_0$ is the wave vector. In what follows we shall neglect the $O(\delta A_i^2)$ term, that is to say, we shall investigate the stability of the fundamental taking account of the interaction of the fundamental with small deviations from the background. For this purpose we must first consider possible cases in which the decay conditions can be satisfied.

Let us first consider the simplest case, a plasma with no magnetic field. In such a plasma there are two branches: longitudinal electron waves and ion-acoustic waves ($T_e \gg T_i$). The electron waves are characterized by a spectrum corresponding to curve 2 (cf. Fig. 12) while the ion waves are characterized by a spectrum corresponding to curve 3; consequently these waves are stable when not coupled. However, coupled decays of the following form are allowed:

electron wave < electron wave, ion wave

ion wave < electron wave, electron wave



The decay of the longitudinal electron wave into a longitudinal electron wave and a longitudinal ion wave represents one of the simplest examples of a decay instability. The equations for small perturbations are of the form

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right) v_i - \frac{e}{M}E = -2\delta v_i \frac{\partial}{\partial x}\,(v_i \sin k_0 x),$$

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right) n_i + n_0\,\frac{\partial v_i}{\partial x} = -2\,\frac{\partial}{\partial x}\,\{(n_i\delta v_i + v_i\delta n_i)\sin k_0 x\}, \tag{55}$$

$$eE + \frac{T}{n_0}\,\frac{\partial n_e}{\partial x} = 2\,\frac{T}{n_0^2}\cdot\frac{\partial}{\partial x}\,(n_e\delta n_e \sin k_0 x)$$

(this system of equations describes the ion wave). Here, $v_i$ and $n_{i,e}$ are the ion velocity and the ion (electron) density.

The hydrodynamic forms of the electron equations are

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right) v_e + \frac{e}{m}\,E + \frac{1}{mn_0}\,\frac{\partial p_e}{\partial x} = -\,2\delta v_e\,\frac{\partial}{\partial x}\,(v_e \sin k_0 x) +$$

$$+\,\frac{\gamma p_0}{mn_0^3}\cdot 2\delta n_e \frac{\partial}{\partial x}\,(n_e \sin k_0 x),$$

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right) n_e + n_0\,\frac{\partial v_0}{\partial x} = -\,2\,\frac{\partial}{\partial x}\,\{(n_e\delta v_e + v_e\delta n_e)\sin k_0 x\}, \tag{56}$$

$$\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)\left(p_e - \gamma\,\frac{p_0}{n_0}\,n_e\right) = 2\gamma\,(\gamma - 1)\,\frac{p_0\delta n_e}{n_0^2}\left(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\right)\times$$

$$\times\,(n_e \sin k_0 x),$$

$$\frac{\partial E}{\partial x} = -\,4\pi e n_e,$$

where $v_e$ is the electron velocity; $\gamma$ is the adiabaticity index for the electrons, which can be set equal to 3 in the one-dimensional case; $p_e$ is the electron pressure; $\delta v$ and $\delta n$ are the amplitudes of the velocity and density of the electrons and ions in the original electron wave whose stability is being investigated. For reasons of simplicity we are treating a one-dimensional case, i.e., it has been assumed that quantities describing the small deviations from the background depend only on the time and the x-coordinate ($k_0$ is in the x-direction).

In accordance with the stability scheme proposed above we now seek a perturbation in the form of a superposition $C_i e^{i\,(\omega + k_1 x)} + C_e e^{i\,(\omega + k_1 x)}$ of ion and electron waves. These wave are independent in the zeroth approximation; however, coupling is provided by the right sides of Eqs. (55) and (56). Using the conditions that must be satisfied in order to solve the equations for $C_i$ and $C_e$, after some simple but rather tedious calculations we obtain the following expression for the square of the imaginary part of the frequency:

$$\nu^2 \approx \left(\frac{\delta v_e}{u}\right)^2 \frac{k_2 u}{4} \frac{\Omega_1}{\Omega_2} \left\{\Omega_2 + (\gamma - 2)\, k_2 v_T \frac{v_T}{u}\right\}, \tag{57}$$

where $v_T = \sqrt{\gamma p_0/n_0 m}$ and $\Omega_1$ and $\Omega_2$ are the ion and electron wave frequencies in the laboratory coordinate system.

The decay conditions are written in the following form:

$$\begin{gathered} \pm |k_1| = k_0 \pm |k_2|, \\ \pm k_1 u_i = \sqrt{\omega_0^2 + k_2^2 v_T^2} - \sqrt{\omega_0^2 + k_0^2 v_T^2}, \end{gathered} \tag{58}$$

where $u_i = \sqrt{p_0/n_0 M}$.

Using Eqs. (57) and (58) we can show that $-\nu^2 > 0$, that is to say, the electron longitudinal waves are unstable against decay into an electron longitudinal wave and an ion longitudinal wave. The most unstable waves are the short waves ($k_0 \lessgtr \varkappa$; $1/\varkappa$ is the Debye radius). In this case

$$\nu_{\max} \sim \frac{\delta v}{u} \sqrt[4]{\frac{m}{M}} \cdot \omega_0. \tag{59}$$

All of the above calculations have been carried out in the hydrodynamic approximation, i.e., the electron thermal motion has been introduced only by including the electron pressure. It is well known, however, that thermal motion can also cause wave damping. This damping of the electron waves can be neglected if $k \ll \varkappa$. The damping is not, however, exponentially small for the ion waves and instability of the background actually requires that the inequality $\nu > \nu_i$ must be satisfied, where $\nu_i$ represents the damping of the ion waves. It is well known (cf., for example, [20]) that when $p_i \ll p_e$

$$\nu_i \cong \sqrt{\frac{\pi}{8}}\, \Omega_1 \sqrt{\frac{m}{M}}. \tag{60}$$

Comparing this expression with Eq. (59) we find that waves whose amplitudes



satisfy the inequality $\delta v/u > (m/M)^{3/4}$ will be unstable. Further, we note from Eqs. (57) and (58) that decay leads to electron oscillations with frequencies smaller than the frequency of the background.

Carrying out a similar analysis for the second case of coupled decay we can show that the ion waves are stable: the coupling between $C_1$ and $C_2$ only leads to a frequency shift. Proceeding in the same way as in the first case we can investigate the stability of various kinds of nonlinear periodic low-amplitude plasma waves in a magnetic field. An investigation of this kind has been carried out for the case of Alfvén waves [21]. It is well known in magnetohydrodynamics (not necessarily only for an incompressible fluid, but for a gas as well) that the Alfvén waves are exact solutions for the nonlinear equations. Hence it might be thought that these waves could exist indefinitely without change of form. Analysis shows, however, that the Alfvén wave decays into two waves: an Alfvén wave and a slow magnetoacoustic wave (or a fast magnetoacoustic wave and a slow magnetoacoustic wave). The growth rates for the decay instability are proportioual to the first power of the amplitude of the initial nonlinear wave; thus, a low-amplitude wave can persist for a long time without decaying.

A more exact investigation of the equations $\hat{L}_0 \varphi = \hat{L}_1 \varphi$ which arise in the analysis of the decay instability shows that the form of the dispersion relation provides a means of telling whether the correction to the frequency will be imaginary (instability) or real (frequency shift). If $|\Omega_0| > |\Omega_1|$, $|\Omega_2|$, and if the decay conditions (53) and (54) are satisfied the initial wave characterized by frequency $\Omega_0$ is unstable. It will be shown below that the decay instability plays an important role in the theory of collisionless shock waves, a subject to which we now turn our attention.

## § 3. Shock Waves in Collisionless Plasmas

A survey of the voluminous literature on collisionless shock waves that has appeared in the last several years indicates the existence of completely different, and even contradictory, opinions on this subject. As a first approximation we might divide these opinions into two classes, each class being characterized by an opposite point of view:

1. Shock waves in which the thickness of the shock front is appreciably smaller than the mean free path do, in fact, exist and all phenomena that occur within the front can, in principle, be described within the framework of the laminar theory, i.e., ordered nonlinear oscillations;
2. The anomalous dissipation in a shock front is related to plasma turbulence.

In addition, there is a third and negative point of view, i.e., that collisionless shock waves do not exist at all. The arguments advanced by the proponents of the various theories contain many weak points so that it is difficult to make a choice between them. For example, in the turbulence approach the instability mechanism responsible for the transition to the turbulent state is not clearly indicated. On the other hand, the laminar theory is not supported by unambiguous results; indeed, it appears that contradictory results have been obtained in many cases. One feels that a natural and reasonable approach to the theory of collisionless shock waves should start with a laminar theory, based on the notion of regular oscillations (this step makes use of the development in the preceding section). The stability of the solutions obtained in this way would then be examined. Finally, in the unstable cases (and when no laminar solutions exist) the turbulence question would be examined.

**1.** The laminar analysis can be formulated quite easily: it is sufficient to take account of the effect of damping on the steady-state nonlinear waves. In the absence of damping these waves imply reversible motion. Thus, the state of the plasma after the passage of a solitary wave is found to be the same as it was before. It is clear that taking account of dissipation must violate reversibility so that the plasma state after the passage of the shock wave must be different from what it was before. If the nonlinear motion is described by the equations of mass, momentum, and energy conservation, in the steady-state these equations must, by definition, connect states governed by the equations of the Hugoniot adiabat. If damping is neglected the plasma states before and after the passage of the solitary wave satisfy the Hugoniot conditions trivially. We now ask how the form of the solitary wave changes if dissipation is included. The state following the passage of the solitary wave must be different from the original state, and this difference is obviously determined by the dissipation mechanism and the magnitude of the dissipative effects.

On the other hand, the Hugoniot conditions do not depend on dissipation. In the analysis of the thickness of a shock front in ordinary gas dynamics this apparent paradox is resolved by saying that the shape of the transition layer (thickness) depends on the viscosity, thermal conductivity, and so on. In a collisionless plasma, however, the "thickness" of a solitary wave (for small dissipation) is specified independently of the Hugoniot adiabat, by the dispersion properties. The resolution of this apparent paradox lies in the fact that the plasma is in a "perturbed" state after the passage of the solitary wave: the plasma supports intense oscillations whose contribution to the momentum and energy flux must be taken into account. This picture implies that regular oscillations of finite amplitude must grow spontaneously within the shock front. It is well known that the thickness of a weak shock front in ordinary gases is



appreciably greater than the mean free path. Because of this circumstance it is possible to investigate the structure of the shock front using the gas-dynamic equations with dissipative effects included.

We start our analysis with shock waves in a plasma in a magnetic field. In a collisionless plasma in a magnetic field in which the mean free path is appreciably greater than the mean ion Larmor radius the formal gas-dynamic description applies (for motion across the lines of force) within spatial regions smaller than the mean free path. The only requirement is that all quantities must not vary significantly over distances of the order of the Larmor radius. In analyzing the structure of a shock front propagating across the magnetic field in a collisionless plasma we shall assume that the Larmor radius is small compared with any characteristic dimension in the front, noting that this condition imposes a limitation on the wave amplitude. Consider a cold plasma ($p \ll H^2/8\pi$). The first damping mechanism we shall examine is Joule heating due to collisions between ions and electrons (as we shall see below, the actual magnitude of the damping is of purely academic interest in the present case). Our problem now is that of finding a set of differential equations for the quantities that characterize the plasma and the self-consistent electromagnetic fields within the shock front and solving this set. We introduce a coordinate system in which the wave front is at rest; the magnetic field is along the z-axis and the zy-plane is the plane of the front. The electric current is carried by the electrons in the y-direction (Fig. 13) and the electron inertia will turn out to have an important effect on the structure of the front. For reasons of simplicity we assume that the neutrality condition is satisfied

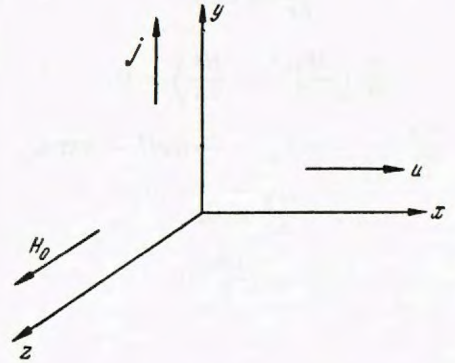


Fig. 13

inside the front $n_i = n_e$, where $n_{i,e}$ is the number density of the ions (electrons); this assumption is in accordance with the analogous problem on undamped nonlinear waves propagating across a magnetic field that was considered in §2.

The quantities that define the plasma and the fields are as follows: n; H; v is the plasma velocity in the direction of propagation of the wave; $v_y$ is the velocity of the electrons carrying the current; $E_y$ is the electric field along the y-axis (the $E_x$ component of the field does not appear in the equations because of the neutrality condition). For these six unknown quantities we have six equations: a) the equation for the conservation of particle flux; b) the equation for the conservation of momentum flux; c) the equation for the conservation of energy flux; d) the equation of motion for the electrons in the direction in which electrical current is transported, the y-axis; e) and f) the Maxwell equations for the appropriate components of curl E and curl H. The original system of six equations can, after some simple transformations, be reduced to a second-order differential equation for one of the variables, say H. However, since the gas-dynamic approximation itself only holds for weak shock waves, the equations can be simplified at the outset. In a weak wave propagating in a cold plasma the plasma pressure differential will be negligibly small compared with the magnetic pressure differential $p/H_0^2 \ll (H-H_0)/H_0$. The equation for conservation of momentum flux can then be used to express the plasma velocity v directly in terms of H and there is no subsequent need for using the energy-flux conservation equation since p does not appear in the remaining equations (the initial set of equations has been separated). When these approximations are introduced the equations become

$$\left.\begin{aligned} &\frac{d}{dx} nv = 0, \\ &\frac{d}{dx}\left(\frac{Mnv^2}{2} + \frac{H^2}{8\pi}\right) = 0, \\ &mnv\frac{dv_y}{dx} = -enE_y + \frac{e}{c} nvH - \bar{v}mnv_y, \\ &\frac{dE_y}{dx} = 0, \\ &\frac{dH}{dx} = \frac{4\pi ne}{e} v_y. \end{aligned}\right\} \tag{61}$$

The last term on the right side of the electron equation of motion denotes the friction force excited on the electrons by the ions ($\bar{\nu}$ is the mean electron–ion



collision frequency $n < v_e\sigma >$, where n is the ion number density, $\sigma$ is the collision cross section, and $v_e$ is the relative electron–ion velocity; $\bar{\nu}$ can be regarded as approximately constant within the front in a weak shock); M and m are the ion and electron masses.

Eliminating all variables except H we reduce the set in (61) to the following differential equation:

$$-a^2\frac{d^2H}{dx^2} = H_0 - H + H\frac{H^2 - H_0^2}{8\pi M n_0 u^2} + \frac{a^2}{u}\nu\frac{dH}{dx}. \tag{62}$$

Here, $H_0$ is the magnetic field in the plasma before passage of the shock wave (for $x \to \infty$); $n_0$ is the unperturbed ion number density (electron); u is the velocity of the shock wave with respect to the unperturbed plasma;

$$a^2 = \frac{mc^2}{4\pi ne^2} = \frac{c^2}{\omega_0^2}.$$

If the friction term is eliminated this equation is reminiscent of Eq. (31). The only difference is that the present equation is limited to low amplitudes. Equation (62) describes an anharmonic oscillator with friction; H plays the role of the generalized coordinate and x plays the role of the time.

The shape of the potential well is given by

$$V(H) = \frac{1}{2}(H - H_0)^2\left[\frac{(H + H_0)^2}{16\pi n_0 M u^2} - 1\right]. \tag{63}$$

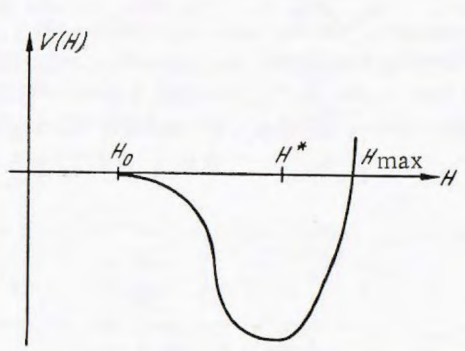


Fig. 14

Figure 14 shows the form of V(H). When

$$H = H^* = -\frac{H_0}{2} + \sqrt{8\pi n_0 M u^2 + \frac{H_0^2}{4}}$$

V(H) reaches a minimum. Using the analogy with the oscillator case it is a simple matter to establish the profile of H inside the shock front; H oscillates about the value $H^*$, with an amplitude that is damped until $H = H^*$, which corresponds to the magnetic field behind the shock front. If $H_0$ is to correspond to the minimum magnetic field in the wave, i.e., in order for V(H) to have the form shown in Fig. 14, the condition $u^2 > H_0^2/4\pi n_0 M$ must be satisfied. When $\bar{\nu} \to 0$ the maximum amplitude, reached at the end of the first half cycle, is

$$H_{\max} = 4u\sqrt{\pi n_0 M} - H_0.$$

The explicit form of H(x) cannot be found; however, if the damping per period is small it is possible to use a simple approximate method—the so-called method of slowly varying amplitudes, in which averages are taken over the fast oscillations. In the absence of friction the motion of the "particle" in the potential well is determined by the single constant c, which represents the total energy of the particle (cf. Fig. 14). Then, the inverse functional dependence of x on H reduces to the quadrature

$$\int \frac{dH}{\sqrt{(H-H_0)^2\left[1-\frac{(H+H_0)^2}{16\pi n_0 M u^2}\right]+C}} = \pm\frac{x}{a}. \tag{64}$$

Suppose that the solution of the frictionless problem is $H = \Phi(x, C)$. Using the method of slowly varying amplitudes we can seek a solution (taking account of friction) in the form $H = \Phi(x, C_x)$, where C is now assumed to be slowly diminishing function of x (as a consequence of the "dissipation" of energy). When averages are taken the dependence of C on x is given by the equation:

$$\frac{dc}{dx} = \frac{\frac{\bar{\nu}}{u}\int\limits_{\Phi_1}^{\Phi_2}\sqrt{(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+c}\,d\Phi}{\int\limits_{\Phi}^{\Phi_2}\left(\sqrt{(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+c}\right)^{-1}d\Phi}. \tag{65}$$



Here, $\Phi_{1,2}$ represents the two positive roots of the equation

$$(\Phi-H_0)^2\left[1-\frac{(\Phi+H_0)^2}{16\pi n_0 M u^2}\right]+C=0, \tag{66}$$

which are larger than $H_0$. The problem is thus reduced to solving Eqs. (64) and (65). When $x \to \infty$ we have as a boundary condition $H \to H_0$, $dH/dx \to 0$, i.e., $C \to 0$. Both equations have simple asymptotic solutions for small C. Thus, $\Phi(x,0)$ is of the form

$$\Phi(x, 0) \approx H_0\left[1 + 2(\mathscr{M}^2-1)\,\mathrm{sh}^2\frac{x}{a}\sqrt{\mathscr{M}^2-1}\right],$$

which obviously coincides with the profile of the low-amplitude solitary wave (cf. §2) where $\mathscr{M} = \left(\frac{4\pi M n_0 u^2}{H_0^2}\right)^{1/2}$, the magnetic Mach number. When $C \to 0$, Eq. (65) becomes

$$\frac{dC}{dx} \approx -\frac{4}{15}\frac{\bar{\nu}}{u}\cdot\frac{H_0^2 8(\mathscr{M}-1)^3}{\ln\frac{\sqrt{-c}}{H_0\sqrt{\mathscr{M}^2-1}}}, \tag{67}$$

whence we find

$$C\ln\frac{\sqrt{-c}}{H_0\sqrt{\mathscr{M}^2-1}} \approx -\frac{4}{15}\frac{\bar{\nu}}{u}8H_0^2(\mathscr{M}-1)^3x + \text{const.} \tag{68}$$

When C is large, in which case the amplitude of the oscillations is reduced appreciably compared with the initial oscillation, the solution is a damped sinusoid

$$H - H^* \sim e^{\frac{\bar{\nu}}{u}x}\sin\left(\sqrt{\mathscr{M}-1}\,\frac{x}{a}\right).$$

The profile of H inside the shock front can be described as follows (Fig. 15). There first appears in the unperturbed plasma a solitary wave, at the crest of which the magnetic field reaches its maximum value; as a result of irreversible dissipation (friction) the state of the plasma after the passage of this wave is somewhat different from the initial state. At a distance of order

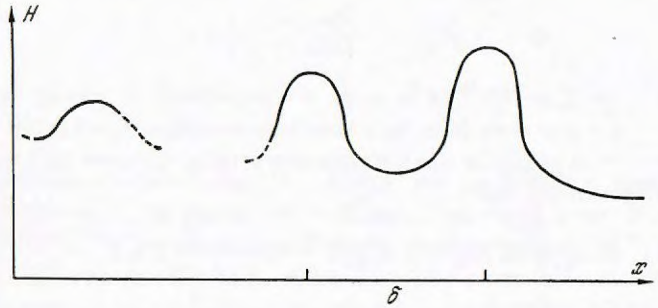

Fig. 15

$$\delta \approx \frac{a}{\sqrt{\mathcal{M}-1}} \ln \frac{u}{\nu a} \sqrt{\mathcal{M}-1} \tag{69}$$

behind the first wave there is a second wave, and so on. If one is not interested in the exact structure of the oscillations in the shock front and considers averages over distances greater than $\delta$, the quantity $\delta$ can be taken as the effective thickness of the shock front which connects the two plasma states; the unperturbed state (before the passage of the wave) and the perturbed state (modulated by intense oscillations); obviously the contribution of those oscillations must be taken into account in computing the conservation relations at the jump. In this sense the damping is really academic since the expression for $\delta$ (69) (width of the shock front) contains the damping in the argument of the logarithm [22].

The damping of the nonlinear waves behind the shock front proceeds in the following way. The amplitude diminishes gradually in the sequence of solitary waves and the spacing between neighboring peaks in the magnetic field is reduced to $\frac{a}{\sqrt{\mathcal{M}-1}}$, with the sequence of peaks and valleys becoming a damped sinusoid. The total damping length is of order $\Delta$

$$\Delta \sim \frac{u}{\overline{\nu}}. \tag{70}$$

This formula does not apply unless the Mach numbers are close to unity since $\overline{\nu}$ will be changed within the shock front. However, $\Delta$ can be estimated using the simple expression $u/\langle \nu \rangle$, where $\langle \nu \rangle$ is the mean frequency of electron–



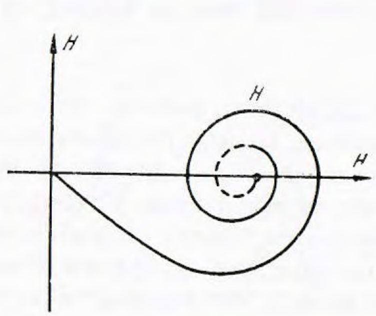

Fig. 16

ion collisions. Expressing quantities in terms of the mean free path $\lambda = \overline{v}_e/\overline{\nu}$ ($\overline{v}_e$ is the mean relative velocity of the electrons with respect to the ions) we find

$$\Delta \sim \sqrt{\frac{H^2}{8\pi nT} \cdot \frac{m}{M}}\, \lambda, \tag{71}$$

whence it is evident that the collisional damping length corresponding to the present approximation can be appreciably smaller than the mean free path (if $\frac{H^2}{8\pi nT}\frac{m}{M} \ll 1$). This result is a reasonable one because the electron temperature is increased by the Joule heating and the electron relaxation time is generally much smaller than the ion relaxation time because the electron velocity is higher. The ions and electron temperatures will be equalized

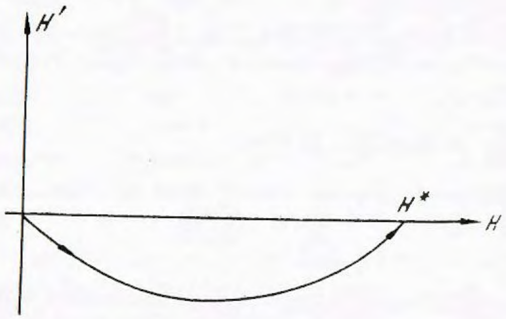

Fig. 17

when the oscillations inside the front are damped, at distances of order $\lambda(M/m)^{1/2}$.

There is another limitation, in addition to those noted above, when high Mach numbers are considered. Although the plasma may be cold in the unperturbed state, it ultimately becomes so hot that the electron Larmor radius becomes comparable with the characteristic wavelength $c/\omega_0$. This situation arises when the electron pressure becomes comparable with the magnetic pressure $nT \sim H^2/8\pi$. On the other hand, the behavior in the initial stages will be of the same general nature as that described above for reasonably large Mach numbers. The oscillatory solution for the profile within the shock front as seen in the phase plane (H', H) will exhibit the pattern shown in Fig. 16 (this is to be compared with the corresponding integral curves in the absence of damping shown in Fig. 5).

It is instructive to establish the relation between the solution obtained above for a collisionless plasma and the familiar expression for the shock front obtained in plasma magnetohydrodynamics for the analogous case of a weak wave propagating across a magnetic field

$$\Delta \sim \frac{\eta_m}{u(\mathcal{M}-1)}, \tag{72}$$

where $\eta_m$ is the so-called magnetic viscosity ($\eta_m = c^2/4\pi\sigma$, $\sigma = ne^2/m\nu$). The point $H = H^*$ is a singular point of the equation (62). Up to now, in treating collisionless plasmas we have been tacitly assuming that the damping [last term in Eq. (62)] is small and that the point $H^*$ is automatically a focus. However, in a dense plasma the singularity at $H^*$ becomes a node (Fig. 17) when

$$\frac{c}{\omega_0} < \frac{c^2 m\nu \cdot 1}{4\pi n e^2 u \cdot \sqrt{6}} \cdot \frac{H_0^{1/2}}{(H^* - H_0)^{1/2}}. \tag{73}$$

In contrast, in the limiting case in which

$$\frac{c}{\omega_0} \ll \frac{c^2 m\bar{\nu}}{4\pi n e^2 u}$$

we obtain the familiar hydrodynamic profile, determined by the magnetic viscosity. The thickness of the shock front is then given by Eq. (72).



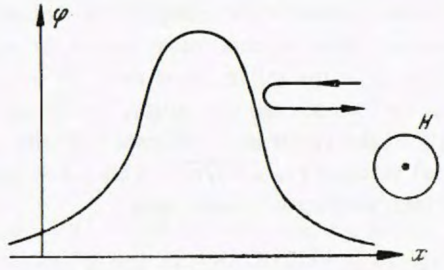

Fig. 18

2. What effects can be expected from the small "collisionless" damping due to those particles whose velocities are approximately the same as the velocity of propagation of the shock wave [14, 23]? The damping is essentially due to the acquisition of energy by the ions reflected from the potential barrier in the shock front (Fig. 18). The magnetic field is not important in such a reflection if the ion Larmor radius is appreciably greater than the characteristic scale size of the wave $\frac{c}{\omega_0} \cdot \frac{1}{\mathcal{M}-1}$.

The greatest number of reflections occur at the first solitary wave (if collisions and the "turning" effect of the magnetic field are neglected, reflections occur only at the first solitary wave). It is difficult to treat the effect of ion reflections quantitatively and we shall not attempt to do so (this calculation is given below for the simpler case of a wave in the absence of a magnetic field).

We wish to point out a curious acceleration mechanism that operates on certain ion bunches in such a shock wave. Ions whose velocities are very close to the velocity of the shock wave will have small Larmor radii. Upon being reflected from the potential barrier they are immediately "turned" by the magnetic field and reflected again; this process occurs several times. After several reflections (Fig. 19) these ions acquire a very high velocity in the y-direction (in the plane of the front and transverse to H). However, this velocity cannot become arbitrarily large because as $v_y$ increases the Lorentz force $(e/c)\, v_y H$ becomes important in the region of the barrier; ultimately this force becomes greater than the "reflection" force $-e\nabla\varphi$, and the ion passes through the barrier. The maximum energy of such an ion is of order $(M/m) \cdot Mu^2/2$, where $Mu^2/2$ is the mean energy of the ordered motion executed by an ion iu these oscillations.

Another possible mechanism for collisionless damping is represented by any instability that tends to convert the energy of the ordered oscillations into energy of random motion. Here we shall make use of the results of the earlier sections. The most obvious instability candidate for a nonlinear wave in a magnetic field is the two-stream instability, which can arise when the mean ordered velocity of the electrons (with respect to the ions) is greater than the mean thermal velocity ($v_0 > \sqrt{T/m}$). This condition is satisfied for waves in which the Mach number is greater than

$$\mathcal{M}^* \approx 1 + \frac{3}{8}\left(\frac{8\pi n T}{H^2}\right)^{1/3}$$

[cf. Eq. (50)]. Physically the instability means that electrons moving with respect to the ions are not only retarded by ordinary collisions [the last term in Eq. (62)], but are also retarded by a specialized frictional force of collective nature—the coherent emission of plasma oscillations as a consequence of the instability. A rough estimate of the magnitude of this effect can be made on the basis of the following considerations: in the expression for the electrical conductivity $\sigma_{\text{eff}} \sim ne^2/m\nu$ the quantity $\nu$ is now taken to mean the reci-

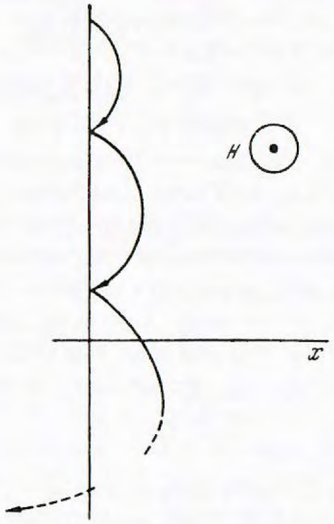


Fig. 19



procal time for the loss of electron energy by virtue of the instability. This can reasonably be of the order of the effective growth time for the instability, that is to say,

$$\sigma_{\text{eff}} \sim \frac{ne^2}{m\Omega_0}. \tag{74}$$*

If the condition $\mathcal{M} > \mathcal{M}^*$ is satisfied the behavior at the leading edge of the shock is determined specifically by this damping effect. In the picture of the effective potential well V(H) the structure of the front will be qualitatively that shown in Fig. 20. The sharp retardation of the particle at the beginning is due to the effect of the instability. Then, as the amplitude of the fluctuation decreases (as the temperature rises) the instability is suppressed and further retardation is inhibited [17]. The main point here is that the damping of the oscillations in this region can be anomalous because of the decay instability.

3. Up to this point we have been considering the structure of a wave propagating in a cold plasma at precisely right angles with respect to the magnetic field. The earlier analysis can now easily be generalized to the case in which propagation is not exactly perpendicular to H. Dispersion effects are extremely sensitive to the direction of propagation. If the wave does not propagate exactly perpendicularly the dispersion equation relating $\omega$ and k is of the form given by Eq. (20) of §2 and the characteristic dispersion length is

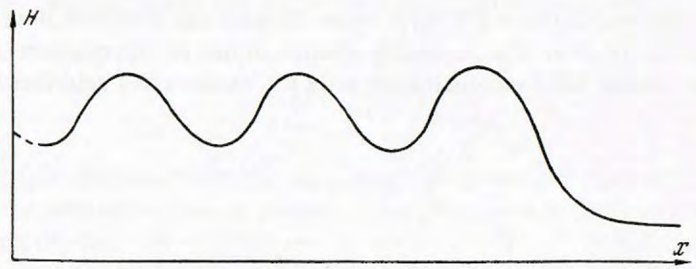


Fig. 20

* This means that when the instability condition $v_0 > \sqrt{T/m}$ is satisfied we have an anomalous electrical resistance leading to anomalous dissipation. This effect has, in fact, been observed experimentally in high-amplitude waves in a plasma in a magnetic field [24].

$(C/\Omega_0)\,\theta$ (when $\sqrt{m/M} \ll \theta \ll 1$). The electron inertia is not important in these waves, but it is now important to take account of the fact that the plasma is a gyrotropic medium. The starting equations for this case

$$M\frac{d\mathbf{V}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{V}\cdot\mathbf{H},$$

$$\frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{V} = 0,$$

$$-e\mathbf{E} + \frac{e}{c}\mathbf{v}\times H = 0, \quad \frac{\partial n}{\partial t} + \operatorname{div} n\mathbf{v} = 0,$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$

$$\operatorname{rot}\mathbf{H} = \frac{4\pi e n}{c}(\mathbf{V} - \mathbf{v})$$

can be reduced to the form

$$\left.\begin{aligned} \varrho\frac{d\mathbf{V}}{dt} &= \nabla\frac{H^2}{8\pi} + \frac{(\mathbf{H}\nabla)\mathbf{H}}{4\pi}, \\ \frac{\partial \varrho}{\partial t} &+ \operatorname{div}\varrho\mathbf{V} = 0, \\ \frac{\partial \mathbf{H}}{\partial t} &= \operatorname{rot}\mathbf{V}\times\mathbf{H} - \frac{Mc}{e}\operatorname{rot}\frac{d\mathbf{V}}{dt}. \end{aligned}\right\} \qquad (75)$$

The term $(Mc/e)\operatorname{rot}(d\mathbf{V}/dt)$ is responsible for the deviation from linear dispersion at large k. The stationary solution of this set of equations (in which we must include the Joule dissipation as in the earlier case) describes the

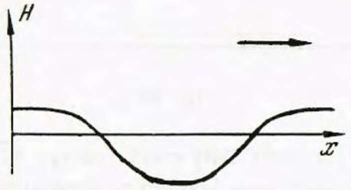


Fig. 21



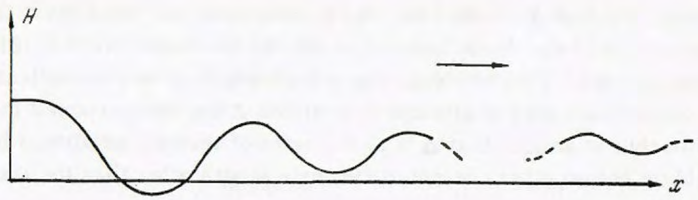


Fig. 22

profile of the shock wave. When $\sqrt{m/M} \ll \theta \ll 1$, the equation describing the magnetic field profile in the wave is [17, 25]

$$\frac{c^2}{\Omega_0^2}\theta^2\frac{d^2H}{dx^2} = H\left\{1 + \frac{H_0^2}{8\pi\varrho_0 u^2} - \frac{H^2}{8\pi\varrho_0 u^2}\right\} - H_0 + \alpha\frac{dH}{dx}. \qquad (76)$$

Here, both the dissipation length$[c/\omega_0$ is replaced by $(c/\Omega_0)\Theta]$ and the nature of the dispersion ($\omega/k$ increases as k increases) are changed. Comparison of Eq. (76) with Eq. (62) shows that the sign of the "effective mass" has been changed. If the damping term $\alpha\, dH/dx$ is neglected Eq. (76) describes a nonlinear periodic steady-state wave. The solitary wave (Fig. 21) is a particular solution, but in this case this solitary wave is a rarefaction wave. The profile of the shock front will be of the general form shown in Fig. 22. It is curious to note that the magnetic field inside the shock now approaches a minimum value which is smaller than the magnetic field in the unperturbed plasma.

The damping length due to the usual frictional force is of order

$$\Delta \sim \lambda\theta\left(\frac{H^2}{8\pi nT}\right)^{1/2}. \qquad (77)$$

The principle difference from the preceding case is the fact that the leading edge of the oscillation front is not sharp. For this reason it might appear that we are not dealing with a collisionless shock because Eq. (77), which gives $\Delta$ (the damping length), contains $\lambda$, the mean free path. However, the dispersion relation for these oscillations $\omega(k)$ is precisely of the class in which the nonlinear periodic waves are unstable against decay [26] (cf. § 2). As a consequence of the decay instability the nonlinear ordered oscillations are damped much more rapidly than is indicated by Eq. (77) since their energy is converted into the energy associated with a broad noise spectrum. The damping length $\Delta$ obtained in this way can be identified with the thickness of the shock front. To find $\Delta$ we must be able to determine the noise level that arises as a result of the decay instability and the feedback effect of this noise on the background. This problem, which is obviously a very complicated one, has not yet been solved (an attempt to estimate $\Delta$ has been reported in [8]). It is reasonable to assume that $\Delta$ is of the order of several oscillation lengths (the problem has no other characteristic scale length other than the wavelength of the oscillations).

Thus, the analysis of the laminar structure of nonlinear oscillations inside a shock front reduces to two different cases (Fig. 23): 1)The case in which the dispersion curve $\omega(k)$ is of the form denoted by 1 (waves perpendicular to H in a cold plasma); in this case the leading edge of the front is sharp (all phenomena start with solitary waves)and one can speak of a collisionless shock wave even in the laminar theory; 2) the case in which the short waves have a higher propagation velocity than the long waves (the curve denoted by 2). In this case the leading edge of the wave front becomes smeared out because the short waves outrun the front. Anomalous damping is required in order for a collisionless shock to exist in this case. The origin of this damping can be the decay instability (which is inherent in a spectrum of the type denoted by 2). The plasma becomes turbulent as a result of the development of this instability. The evaluation of the shock thickness is simple in the first case; in the second case, however, a quantitative analysis is extremely complicated. Nevertheless the mechanisms which are important in this case are already qualitatively clear.

There are other examples which can be related to one or the other of the two cases that have been analyzed. For instance, a shock wave propagating across a magnetic field in a high-pressure plasma ($p \gtrsim H^2/8\pi$) is to be as-

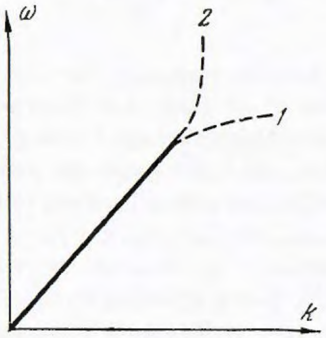


Fig. 23



sociated with the second case noted above since the corresponding dispersion relation (cf. §2) can be classified as a type 2 relation. The ion-acoustic wave in a two-temperature plasma ($T_e \gg T_i$) is characterized by a type 1 spectrum so that the question of a collisionless shock wave can be resolved within the laminar formulation.

**4.** In order to establish the profile of the shock front in the ion-acoustic case we proceed by analogy with the preceding analysis. In the absence of a magnetic field nonlinear steady-state oscillation can be excited when $T_e \gg T_i$. If damping is neglected the equation that describes the potential profile of $\varphi$ in this wave [cf. Eq. (37)] is

$$\frac{d^2\varphi}{dx^2} = 4\pi n_0 e\left(\frac{u}{\sqrt{u^2 - \frac{2e\varphi}{M}}} - e^{\frac{e\varphi}{T}}\right) = -\frac{dV(\varphi)}{d\varphi}, \qquad (78)$$

where $V(\varphi)$ is the effective potential energy. We shall asume that the usual dissipation due to the ion–ion collisions is absent but shall take account of the reflection of ions from the leading edge of the front; this process plays the role of a collisionless dissipation mechanism.

The structure of the collisionless shock wave that arises under these conditions can be described by the following simplified picture. In the absence of any dissipation we have a solitary wave, which is represented by a symmetric potential barrier. However, there always are ions that are reflected from the moving potential barrier (even if the number of such ions is arbitrarily small) causing an asymmetry to arise; beyond the barrier there are periodic oscillations. The net result is a peculiar kind of shock wave which connects two different plasma states: the unperturbed state (in front of the shock) and a state with intense ordered oscillations (behind the front). A correct "shock adiabat" must take account of the additional contributions to the energy and momentum fluxes associated with these ordered oscillations behind the front. It should be noted, however, that the distribution of energy between the thermal motion and the oscillations depends on the actual collisionless dissipation mechanism. The shock profile can be determined if the number of reflected particles is small. The potential profile in the wave is shown in Fig. 24. In the absence of dissipation $\varphi_1 = \varphi_2$ and $\lambda = \infty$ and we have the symmetric solitary wave.

If ion reflections are taken into account the potential in region I (cf. Fig. 24) is described by an equation which differs from Eq. (78) by the presence of additional terms on the right side:

$$-4\pi n_0 e f(\varphi_1)\frac{u}{\sqrt{u^2-\frac{2e\varphi}{M}}}+2\cdot 4\pi n_0 e f(\varphi).$$

The first term corresponds to subtraction of the reflected ions from the total number of ions $n_0$; the second term represents the contribution of the reflected ions. The quantity $n_0 f(\varphi)$ is the total density of the reflected ions at a point characterized by the potential $\varphi$ (the actual form of $f$ can be found easily if the unperturbed ion velocity distribution is known).

The potential jump $\varphi_1$ is associated with ions that are reflected from the potential barrier and escape to infinity; in the case being considered here, in which the number of reflected particles is small ($f \ll 1$), the potential jump $\varphi_1$ is proportional to $f$. However, $\varphi_2$ will be proportional to the square root of the number of reflected particles so that $\varphi_1 \ll \varphi_2$. The plasma state behind the front (region II) is characterized by the quantities $\varphi_{max}$ and $\varphi_2$ which determine the amplitude of the oscillations and their wavelength $\lambda$; Eq. (78) holds in this region.

By solving the potential equation in regions I and II taking account of the continuity requirements on $\varphi$ and $d\varphi/dx$ we can find the potential profile. If the analogy with the motion of a particle in a potential well $V(\varphi)$ is again invoked it can be shown that the effect of the reflected ions is essentially to make the total energy C negative. This leads to periodic motion (a periodic structure behind the shock front).

The reduction energy C is proportional to the number of reflected ions

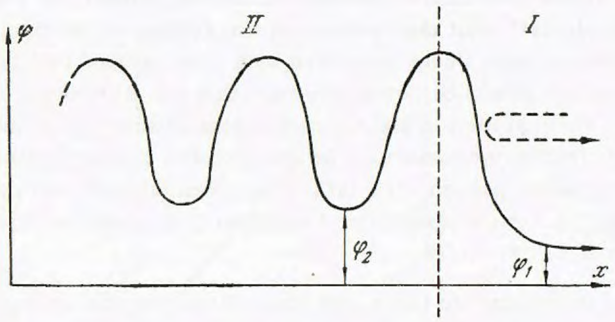


Fig. 24.



$$-C \sim \int_0^{\varphi_{max}} f(\varphi)\, d\varphi.$$

Since the potential energy $V(\varphi)$ varies quadratically at small $\varphi$ the turning point $\varphi_2$ is proportional to the square root of the energy $-C$

$$\varphi_2 \sim \sqrt{-C},$$

and the oscillation period increases logarithmically as the energy is reduced

$$\lambda \sim \ln\frac{1}{-C}.$$

Thus, the minimum value of the potential behind the front $\varphi_2$ is

$$\varphi_2=\frac{2\mathcal{M}}{\sqrt{\mathcal{M}^2-1}}\left(\frac{T}{e}\int_0^{\varphi_{max}} f(\varphi)\,d\varphi\right)^{1/2}\quad\left(\mathcal{M}^2=\frac{u^2}{\frac{T}{M}}\right). \tag{79}$$

The value of $\varphi_{max}$ is very close to the corresponding value in a solitary wave with the same Mach number.

The wavelength at the front is [27]

$$\lambda=\frac{A}{\sqrt{\mathcal{M}^2-1}}\left(\frac{T}{\pi n_0 e^2}\right)^{1/2}\ln\frac{\varphi_{max}}{\varphi_2}, \tag{80}$$

where $A \sim 1$.

5. We have not yet encountered cases in which a laminar analysis of the shock front is not applicable. This situation arises in the examples that have already been considered if the amplitudes become so high that steady-state nonlinear waves cannot be sustained. We first consider the case in which the shock wave propagates across a strong magnetic field in a cold plasma. At low values of the Mach number we know that ordered oscillations are sustained within the shock front. The ordered oscillatory structure is destroyed when the magnetic field in the wave becomes twice as large as the initial magnetic field. Indeed, it follows from § 2 that a solitary wave is not formed at these amplitudes (which correspond to Mach numbers greater than 2); furthermore it is impossible to formulate a steady-state nonlinear flow with a unique velocity, that is to say, a flow in which the ion velocity has a single value at a given point in space. Physically this means that the wave breaks as soon as the amplitude reaches a cricital value ($H_{max} = 3H_0$). There is then a point in space at which the fast ions overtake the slow ions (Fig. 25), and the velocity profile becomes triple-valued at this point.

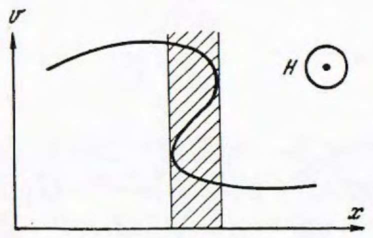


Fig. 25

It is interesting to note that the analogous effect has been studied quite throughly in the theory of finite-amplitude waves on the surface of a heavy liquid in a channel of finite depth. The latter case also gives rise to nonlinear steady-state solitary waves and periodic waves and these waves also break at high amplitudes. It is clear that any rigorous mathematical analysis of the breaking phenomenon would be extremely difficult. We shall content ourselves with a qualitative analysis of some of the important features which are analogous to those in waves in a liquid.

The basic problem is to ascertain whether breaking means that the plasma motion enters a steady-state regime or whether the transition region (shown cross-hatched in Fig. 25) continues to spread without limit, as is the case in an ordinary collisionless gas. In the case of surface waves in water, breaking is followed by a steady-state flow called a water surge or "bore," characterized by a transition region of finite thickness; this is usually replaced by an idealized mathematical surface which divides the two plane parallel flows. The appropriate conservation laws must be satisfied across this surface. In some sense the bore is the analog of the shock wave. The stationarity of the width of the transition layer arises physically because those parts of the profile which move ahead in breaking ultimately describe an arc and fall under the effect of gravity, becoming "mixed" with the portions that are at rest. In the plasma the role of gravity is played by the magnetic field, which forces the ion to gyrate. As long as the ion velocity distribution is far from Maxwellian the plasma states on both sides of the transition region can be connected by the conservation laws for mass, momentum, and energy; by the energy of the thermal motion we are to understand $(M/2)\ \overline{(v-\overline{v})^2}$ (the bar means an average over the velocity distribution). The width of the transition region can be estimated as the radius of curvature of the ions after breaking occurs in the magnetic field [22]. Inasmuch as the peak velocity $v \gtrsim H/\sqrt{4\pi\rho}$, in a wave characterized by a Mach number greater than 2 the width of the transition layer (the width of the collisionless shock wave) is of order



$$\delta \sim \frac{vMc}{eH} \sim \frac{c}{\Omega_0}, \quad \left(\Omega_0^2 = \frac{4\pi ne^2}{M}\right). \tag{81}$$

The multivelocity flow with velocities perpendicular to the magnetic field, which arises after breaking, is necessarily unstable. For simplicity let us consider a double-humped ion distribution with a velocity difference between the humps greater than $\sqrt{T/M}$; this distribution results in an instability with excitation of waves characterized by wave vectors almost parallel to the beam velocity. A bore also exhibits an instability of similar nature (opposed flows); this is simply the instability due to the tangential discontinuity between the incident jet and the surface of a liquid at rest.

If the characteristic dimensions of the regions of multivelocity motion are appreciably greater than the wavelengths of the instabilities that arise it is valid to make use of the stability analyses for a uniform plasma. For example, in the case of two opposed ion flows moving across a magnetic field with velocities $v_0$ and $-v_0$ the dispersion equation is [28]

$$\frac{2}{\omega_{H_i}\omega_{H_e}} = \frac{1}{(\omega - kv_0)^2} + \frac{1}{(\omega + kv_0)^2}. \tag{82}$$

The maximum growth rate is of order $(\omega_{Hi}\,\omega_{He})^{1/2}$. When $v_0 \lesssim H/\sqrt{4\pi nM}$ the characteristic wavelength of the instability is of order $c/\omega_0$. Thus multivelocity motion across a magnetic field is unstable.

On the other hand, if the plasma is hot it is necessary to take account of the velocity spread and the dispersion equation given in (82) no longer holds. When $v_{Ti} \sim H/\sqrt{4\pi\rho}$ the maximum growth rate for the instability is of order $\omega_{Hi}$ and the corresponding wavelength is of order $c/\Omega_0$. These quantities then characterize the thickness of a shock front in a strong magnetic field.*

* A numerical calculation for one-dimensional, high-amplitude plasma motion across a magnetic field in which the flow becomes multivalued (and unstable) appears in [29]. In particular, when $\mathcal{M} = 5.8$ the effective "mixing length" (the thickness of the front) is found to be $3.4\ r_i$.

Kantrowitz and Petschek [7] have formulated a phenomenological theory for the turbulent structure of a shock front propagating across a magnetic field. These authors assume that some unknown plasma instability gives rise to a broad spectrum of waves at the very beginning and that the interaction between the various modes is responsible for the transport of energy and momentum.

6. In the foregoing we have considered the question of collisionless turbulent shock waves in a plasma with propagation occuring across a strong magnetic field. A magnetic field parallel to the plane of the shock front confines the hotter particles and inhibits the expansion of the transition layer between the unperturbed cold plasma (in front of the shock wave) and the heated plasma behind the wave. A number of authors have also discussed the possibility of collisionless shock waves in a plasma with no magnetic field. It is proposed that the mechanism responsible for inhibiting the expansion of the transition region in this case is the two-stream instability [30]. This approach, however, does not take account of the thermal spread within each of the beams. A more rigorous analysis, which includes the thermal motion, does not give an instability for Mach numbers ranging from unity up to approximately $(M/m)^{1/2}$ if the electron temperature is comparable to or smaller than the ion temperature (M is the ion mass and m is the electron mass; cf., for example, [31]).

This problem does not arise in a two-temperature plasma ($T_e \gg T_i$) since it is possible to formulate a laminar analysis. However, another approach is needed when $\mathcal{M} > 1.6$, because of the breaking phenomenon. One possible method is based on the familiar velocity anisotropy instability. When the faster ions from the region behind the front enter the unperturbed plasma in front of the shock the ion velocity distribution in this region becomes anisotropic. This plasma state is known to be unstable and random fluctuations of the electric and magnetic fields arise. The thickness of the shock front in this case is then a quantity of the order of the mean free path of the ions with respect to scattering on these nonequilibrium fluctuations. To the degree that "rigor" can be achieved in the theory of turbulence, it can be said that this problem has been resolved in [27]. However, here we shall be content with some very qualitative physical estimates.

Assume that H = 0 in the unperturbed plasma. Let us now try to understand the physical meaning of the anisotropic instability in this case. We consider a plasma in which the mean particle energies are different in the x and y directions [$\varepsilon_{x,y} = M\overline{(v - \overline{v})^2}$, with $\varepsilon_y > \varepsilon_x$] and introduce a perturbation in the form of an arbitrarily small fluctuation of the magnetic field, assuming that the field is in the z direction (Fig. 26). The anisotropy in the distribution can cause this perturbation to grow. Consider particles moving along the y-axis close to the point $x_0$, where the magnetic field changes sign. These particles are subject to a Lorentz force $F_x = (e/c)\, Hv_y$. Particles for which $v_y > 0$ will be pushed toward $x_0$ and particles for which $v_y < 0$ will be pushed away from $x_0$. Thus, a concentration of particles with $v_y > 0$ tends to build up near $x_0$. This implies the appearance of an electric current $j_y$. The



direction of this current flow is such as to increase the original fluctuation in the magnetic field and the ultimate result is an instability. However, we have not taken account of the stabilizing effect of the thermal motion along the x-axis, which tends to inhibit the concentration of particles with the same sign of $v_y$. In general, there is no instability if $\varepsilon_x = \varepsilon_y$. However, when $\varepsilon_y > \varepsilon_x$ this inhibiting effect cannot quench the instability if the wavelength is long enough. The critical wavelengths for these perturbations can be estimated easily. We need only consider two forces: the Lorentz force, which tends to move the system away from equilibrium, and the counteracting force which, for estimation purposes, can be taken to be the pressure gradient along x. If the instability is to occur the following condition must be satisfied:

$$\frac{e}{c} v_y n_0 \delta H > \operatorname{grad} M\overline{v_x^2}\delta n, \tag{83}$$

where δH and δn are the fluctuations in magnetic field and density. On the other hand, δH and δn are related by the Maxwell equation

$$\operatorname{rot} \delta H \sim \frac{4\pi}{c} e v_y \delta n. \tag{84}$$

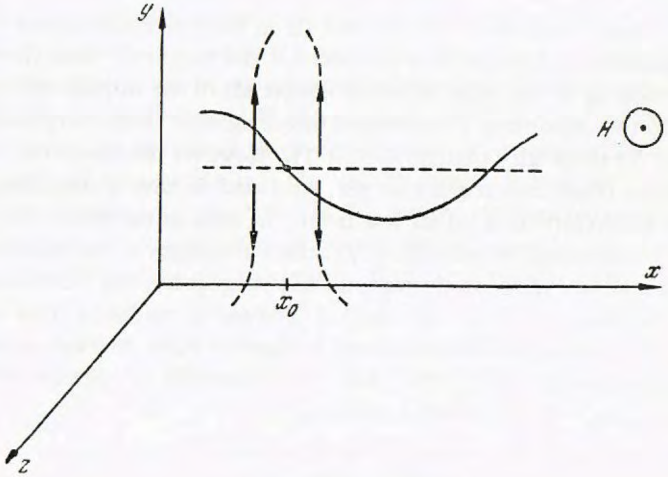


Fig. 26

Writing $\delta H$ and $\delta n$ in a form $\sim e^{ikx}$ and using (84) we can write (83) as

$$\frac{\Omega_0^2}{c^2} v_y^2 > kv_x^2,$$

whence, assuming that $v_x^2$ and $v_y^2$ are of the same order, we find the characteristic wave number k

$$k^2 < \frac{\Omega_0^2}{c^2}. \tag{85}$$

where $\Omega_0^2 = 4\pi ne^2/M$. The entire effect is obviously associated with the ions, that is to say, the ions carry the basic energy.

Assume that a perturbation arises in some region of a collisionless plasma. In the absence of any confinement mechanism the perturbation will spread in the course of time because of the gradual loss of the faster particles. However, when these particles arrive in new regions an anisotropy is produced in the velocity distribution and this leads to an instability; the disordered magnetic field that results obviously scatters particles in much the same way as collisions and there is a possibility of propagation of a nonspreading perturbation which is of the same nature as a shock wave, as in ordinary gas dynamics.

Now let us estimate the fluctuation $\delta H$ in the nonlinear regime of the growing instability. It might be expected that the magnetic-field fluctuation would increase up to the point at which almost all of the surplus ion energy $n\Delta\varepsilon$ (due to the anisotropy) is converted into magnetic field energy$(\delta H^2)/8\pi$ (we assume for simplicity that $\Delta\varepsilon \sim \varepsilon \sim T$). However the electrons, which have not been taken into account as yet, are found to have a quenching effect which holds $(\delta H)^2$ to a rather low level. As soon as the mean electron Larmor radius becomes of order $\lambda \sim 1/k$, the wavelength of the perturbation which characterizes the spatial magnetic inhomogeneity, the electrons are frozen in the magnetic field. Any further increase in magnetic field requires an enormons increase in electron energy because of the conservation of the adiabatic invariant $\mu = mv_\perp^2/2H$. Thus, it is reasonable to estimate $\delta H$ from the condition $r_{He} \sim 1/k$, thereby obtaining

$$\frac{(\delta H^2)}{8\pi} \sim \frac{m}{M} nT. \tag{86}$$



The ion scattering in this magnetic field is of a diffusional nature. The diffusion coefficient (in velocity space) can be estimated easily:

$$D \sim \frac{e^2}{M^2c^2} \cdot \frac{(\delta H)^2}{k} \bar{v}_i. \tag{87}$$

Thus, we find the ion "scattering" time $r \sim v_i^2/D$ and the corresponding mean free path $l \sim r \cdot v_i \sim \frac{M}{m} \cdot \frac{c}{\Omega_0}$. These quantities then determine the order of magnitude of the shock thickness

$$\Delta \sim \frac{M}{m} \cdot \frac{c}{\Omega_0}. \tag{88}$$

The "rigorous"theory leads to the following results:

$$\Delta \sim \frac{c}{\Omega_0} \cdot \frac{M}{m} \cdot \frac{1}{(\mathcal{M} - 1)^2} \quad \text{for } M - 1 < 1; \tag{89}$$

in this theory the dependence on wave amplitude is also taken into account.

A similar analysis can be carried out for a plasma in which there is initially a weak magnetic field ($H^2/8\pi \ll nT$). Starting at $H \sim \delta H$, the shock front is compressed with increasing H [27, 32].

**7.** It thus appears that the general methods of shock-wave analysis can be applied far beyond the bounds of ordinary gas-dynamic theory, which is based on the notion of a mean free path with respect to two-particle collisions; in a rarefied plasma the primary feature is the existence of cooperative phenomena—plasma oscillations. At the present time there does not exist a unified theory for plasma shock waves from which the results for particular cases can be obtained automatically. The variety of effects associated with collective phenomena is far too large. In this review we have only summarized various limiting cases and approaches with which it possible to understand some of these new ideas and to compare them with ordinary gas dynamics: some of these important features are the dispersion effects, the microscopic instabilities, collisionless damping,etc. Correspondingly, various limiting cases give rise to "scale" lengths which characterize the thickness of the shock front [the Debye radius, Larmor radius$(C/\Omega_0)$ M/m, and so on].

Unfortunately, at the present time almost no systematic laboratory experiments on shock waves in collisionless plasmas have been reported. However, individual effects which constitute the basic ingredients of the theory of collisionless shock waves have been experimentally verified in recent years.

One indirect verification of the theory is furnished by the rapid initial phase of geomagnetic storms. As far back as 1955 Wilde concluded that the rapid rise of the earth's magnetic field (several minutes) in the first phase of a magnetic storm could only be explained by assuming that solar flares generate shock waves in the interplanetary gas. Assuming that the ion density in the interplanetary plasma $n \sim 10^2\ cm^{-3}$ and using Eq. (88) we obtain a shock front thickness of order $10^9-10^{10}$ cm which, for a velocity $10^8$ cm/sec, yields a characteristic time of one minute.

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