$1/f$ noise from nonlinear stochastic differential equations

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We consider a class of nonlinear stochastic differential equations, giving the power-law behavior of the power spectral density in any desirably wide range of frequency. Such equations were obtained starting from the point process models of $1/f^{\beta}$ noise. In this article the power-law behavior of spectrum is derived directly from the stochastic differential equations, without using the point process models. The analysis reveals that the power spectrum may be represented as a sum of the Lorentzian spectra. Such a derivation provides additional justification of equations, expands the class of equations generating $1/f^{\beta}$ noise, and provides further insights into the origin of $1/f^{\beta}$ noise.

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I. INTRODUCTION

Power-law distributions of spectra of signals, including $1/f$ noise (also known as $1/f$ fluctuations, flicker noise, and pink noise), as well as scaling behavior in general, are ubiquitous in physics and in many other fields, including natural phenomena, human activities, traffics in computer networks, and financial markets. This subject has been a hot research topic for many decades (see, e.g., a bibliographic list of papers by Li [1], and a short review in Scholarpedia [2]).

Despite the numerous models and theories proposed since its discovery more than 80 years ago [3,4], the intrinsic origin of $1/f$ noise still remains an open question. There is no conventional picture of the phenomenon and the mechanism leading to $1/f$ fluctuations are not often clear. Most of the models and theories have restricted validity because of the assumptions specific to the problem under consideration. A short categorization of the theories and models of $1/f$ noise is presented in the introduction of the paper [5].

Until recently, probably the most general and common models, theories and explanations of $1/f$ noise have been based on some formal mathematical description such as fractional Brownian motion, the half-integral of the white noise, or some algorithms for generation of signals with scaled properties [6–14] and the popular modeling of $1/f$ noise as the superposition of independent elementary processes with the Lorentzian spectra and a proper distribution of relaxation times, e.g., a $1/\tau_{\mathrm{relax}}$ distribution [15–21]. The weakness of the latter approach is that the simulation of $1/f^{\beta}$ noise with the desirable slope β requires finding the special distributions of parameters of the system under consideration; at least a wide range of relaxation time constants should be assumed in order to allow correlation with experiments [22–28].

Nonlinear stochastic differential equation with linear noise and nonlinear drift, was considered in Ref. [9]. It was found that if the damping is decreasing with increase in the absolute value of the stochastic variable, then the solution of such a nonlinear stochastic differential equation (SDE) has long correlation time. Recently nonlinear SDEs generating signals with $1/f$ noise were obtained in Refs. [29,30] (see also recent papers [5,31]), starting from the point process model of $1/f$ noise [27,32–39].

The purpose of this article is to derive the behavior of the power spectral density directly from the SDE, without using the point process model. Such a derivation offers additional justification of the proposed SDE and provides further insights into the origin of $1/f$ noise.

II. PROPOSED STOCHASTIC DIFFERENTIAL EQUATIONS

Starting from the point process model, proposed and analyzed in Refs. [27,32–38], the nonlinear stochastic differential equations are derived [5,29,30]. The general expression for the SDE is

$$dx=\sigma^2\left(\eta-\frac{\nu}{2}\right)x^{2\eta-1}dt+\sigma x^{\eta}dW. \tag{1}$$

Here, x is the signal, η is the exponent of the multiplicative noise, ν defines the behavior of stationary probability distribution, and W is a standard Wiener process.

SDE (1) has the simplest form of the multiplicative noise term, $\sigma x^{\eta}dW$. Multiplicative equations with the drift coefficient proportional to the Stratonovich drift correction for transformation from the Stratonovich to the Itô stochastic equation [40] generate signals with the power-law distributions [5]. Equation (1) is of such type and has probability distribution of the power-law form $P(x)\sim x^{-\nu}$. Because of the divergence of the power-law distribution and the requirement of the stationarity of the process, the SDE (1) should be analyzed together with the appropriate restrictions of the diffusion in some finite interval. For simplicity, in this article, we will adopt reflective boundary conditions at $x=x_{\min}$ and $x=x_{\max}$. However, other forms of restrictions are possible. For example, exponential restriction of the diffusion can be obtained by introducing additional terms in Eq. (1),

$$dx=\sigma^2\left(\eta-\frac{\nu}{2}+\frac{m}{2}\left(\frac{x_{\min}}{x}\right)^m-\frac{m}{2}\left(\frac{x}{x_{\max}}\right)^m\right)x^{2\eta-1}dt+\sigma x^{\eta}dW. \tag{2}$$

Here, m is some parameter.

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Equation (1) with the reflective boundary condition at x_{min} and x_{max} can be rewritten in a form that does not contain parameters σ and x_{min}. Introducing the scaled stochastic variable $x \to x/x_{\mathrm{min}}$ and scaled time $t \to \sigma^2 x_{\mathrm{min}}^{2\eta-2} t$ one transforms Eq. (1) to

$$dx = \left(\eta - \frac{\nu}{2}\right) x^{2\eta-1} dt + x^{\eta} dW. \tag{3}$$

The scaled Eq. (3) has a boundary at $x=1$ and at

$$\xi = \frac{x_{\mathrm{max}}}{x_{\mathrm{min}}}. \tag{4}$$

Further, we will consider Eq. (3) only. In order to obtain $1/f^{\beta}$ noise we require that the region of diffusion of the stochastic variable x should be large. Therefore, we assume that $\xi \gg 1$.

III. POWER SPECTRAL DENSITY FROM THE FOKKER-PLANCK EQUATION

According to Wiener-Khintchine relations, the power spectral density is

$$S(f) = 2\int_{-\infty}^{\infty} C(t) e^{i\omega t} dt = 4\int_{0}^{\infty} C(t)\cos(\omega t) dt, \tag{5}$$

where $\omega = 2\pi f$ and $C(t)$ is the autocorrelation function. For the stationary process, the autocorrelation function can be expressed as an average over realizations of the stochastic process,

$$C(t) = \langle x(t')x(t'+t)\rangle. \tag{6}$$

This average can be written as

$$C(t) = \int dx \int dx' x x' P_0(x) P_x(x',t|x,0), \tag{7}$$

where $P_0(x)$ is the steady-state probability distribution function and $P_x(x',t|x,0)$ is the transition probability (the conditional probability that at time t the signal has value x' with the condition that at time $t=0$ the signal had the value x). The transition probability can be obtained from the solution of the Fokker-Planck equation with the initial condition $P_x(x',0|x,0) = \delta(x'-x)$.

Therefore, for the calculation of the power spectral density of the signal x we will use the Fokker-Planck equation instead of stochastic differential Eq. (3). The Fokker-Planck equation corresponding to the Itô solution of Eq. (3) is [41,42]

$$\frac{\partial}{\partial t} P = -\left(\eta - \frac{\nu}{2}\right) \frac{\partial}{\partial x} x^{2\eta-1} P + \frac{1}{2} \frac{\partial^2}{\partial x^2} x^{2\eta} P. \tag{8}$$

The steady-state solution of Eq. (8) has the form

$$P_0(x) = \begin{cases} \dfrac{\nu-1}{1-\xi^{1-\nu}} x^{-\nu}, & \nu \neq 1, \\[2ex] \dfrac{1}{\ln \xi} x^{-1}, & \nu = 1. \end{cases} \tag{9}$$

The boundary conditions for Eq. (8) can be expressed using the probability current [42]

$$S(x,t) = \left(\eta - \frac{\nu}{2}\right) x^{2\eta-1} P - \frac{1}{2} \frac{\partial}{\partial x} x^{2\eta} P. \tag{10}$$

At the reflective boundaries $x_{\mathrm{min}} = 1$ and $x_{\mathrm{max}} = \xi$ the probability current $S(x,t)$ should vanish, and, therefore, the boundary conditions for Eq. (8) are

$$S(1,t) = 0, \quad S(\xi,t) = 0. \tag{11}$$

A. Eigenfunction expansion

We solve Eq. (8) using the method of eigenfunctions. An ansatz of the form

$$P(x,t) = P_{\lambda}(x) e^{-\lambda t} \tag{12}$$

leads to the equation

$$-\left(\eta - \frac{\nu}{2}\right) \frac{\partial}{\partial x} x^{2\eta-1} P_{\lambda} + \frac{1}{2} \frac{\partial^2}{\partial x^2} x^{2\eta} P_{\lambda} = -\lambda P_{\lambda}(x), \tag{13}$$

where $P_{\lambda}(x)$ are the eigenfunctions and $\lambda \geq 0$ are the corresponding eigenvalues. The eigenfunctions $P_{\lambda}(x)$ obey the orthonormality relation [42]

$$\int_1^{\xi} e^{\Phi(x)} P_{\lambda}(x) P_{\lambda'}(x) dx = \delta_{\lambda,\lambda'}, \tag{14}$$

where $\Phi(x)$ is the potential, associated with Eq. (8),

$$\Phi(x) = -\ln P_0(x). \tag{15}$$

It should be noted that the restriction of diffusion of the variable x by x_{min} and x_{max} ensures that the eigenvalue spectrum is discrete. Expansion of the transition probability density in a series of the eigenfunctions has the form [42]

$$P_x(x',t|x,0) = \sum_{\lambda} P_{\lambda}(x') e^{\Phi(x)} P_{\lambda}(x) e^{-\lambda t}. \tag{16}$$

Substituting Eq. (16) into Eq. (7) we get the autocorrelation function

$$C(t) = \sum_{\lambda} e^{-\lambda t} X_{\lambda}^2. \tag{17}$$

Here,

$$X_{\lambda} = \int_1^{\xi} x P_{\lambda}(x) dx \tag{18}$$

is the first moment of the stochastic variable x evaluated with the λ-th eigenfunction $P_{\lambda}(x)$. Such an expression for the autocorrelation function has been obtained in Ref. [43]. Using Eqs. (5) and (17) we obtain the power spectral density

$$S(f) = 4\sum_{\lambda} \frac{\lambda}{\lambda^2 + \omega^2} X_{\lambda}^2. \tag{19}$$

This expression for the power spectral density resembles the models of $1/f$ noise using the sum of the Lorentzian spectra [15–18,27,28,44,45]. Here, we see that the Lorentzians can arise from the single nonlinear stochastic differential equation. Similar expression for the spectrum has been obtained in Ref. [14] where reversible Markov chains on finite state spaces were considered [Eq. (34) in Ref. [14] with $-\gamma_{k,m}$ playing the role of λ].

A pure $1/f^{\beta}$ power spectrum is physically impossible because the total power would be infinity. It should be noted that the spectrum of signal x, obeying SDE (3), has $1/f^{\beta}$ behavior only in some intermediate region of frequencies, $f_{\mathrm{min}} \ll f \ll f_{\mathrm{max}}$, whereas for small frequencies $f \ll f_{\mathrm{min}}$ the spectrum is bounded. The behavior of spectrum at frequencies $f_{\mathrm{min}} \ll f \ll f_{\mathrm{max}}$ is connected with the behavior of the autocorrelation function at times $1/f_{\mathrm{max}} \ll t \ll 1/f_{\mathrm{min}}$. Often $1/f^{\beta}$ noise is described by a long-memory process, characterized by $S(f) \sim 1/f^{\beta}$ as $f \to 0$. An Abelian-Tauberian theorem relating regularly varying tails shows that this long-range dependence property is equivalent to similar behavior of autocorrelation function $C(t)$ as $t \to \infty$ [46]. However, this behavior of the autocorrelation function is not necessary for obtaining required form of the power spectrum in a finite interval of the frequencies which does not include zero [47–49].

From Eq. (19), it follows that if the terms with small λ dominate the sum, then one obtains $1/f^2$ behavior of the spectrum for large frequencies f. If the terms with $1/f^{\beta}$ (with $\beta < 2$) are present in Eq. (19), then at sufficiently large frequencies those terms will dominate over the terms with $1/f^2$ behavior. Since the terms with small λ lead to $1/f^2$ behavior of the spectrum, we can expect to obtain $1/f^{\beta}$ spectrum in a frequency region where the main contribution to the sum in Eqs. (17) and (19) is from the large values of λ. Thus we need to determine the behavior of the eigenfunctions $P_{\lambda}(x)$ for large λ. The conditions when eigenvalue λ can be considered as large will be investigated below.

B. Eigenfunctions of the Fokker-Planck equation

For $\eta \neq 1$, it is convenient to solve Eq. (13) by writing the eigenfunctions $P_{\lambda}(x)$ in the form

$$P_{\lambda}(x) = x^{-\nu} u_{\lambda}(x^{1-\eta}). \tag{20}$$

The functions $u_{\lambda}(z)$ with $z = x^{1-\eta}$ obey the equation

$$\frac{d^2}{dz^2}u_{\lambda}(z) - (2\alpha - 1)\frac{1}{z}\frac{d}{dz}u_{\lambda}(z) = -\rho^2 u_{\lambda}(z), \tag{21}$$

where the coefficients α and ρ are

$$\alpha = 1 + \frac{\nu - 1}{2(1-\eta)}, \quad \rho = \frac{\sqrt{2\lambda}}{|\eta - 1|}. \tag{22}$$

The area of diffusion of the variable $z = x^{1-\eta}$ is restricted by the minimum and maximum values z_{min} and z_{max},

$$z_{\mathrm{min}} = \begin{cases} \xi^{1-\eta}, & \eta > 1, \\ 1, & \eta < 1, \end{cases} \qquad z_{\mathrm{max}} = \begin{cases} 1, & \eta > 1, \\ \xi^{1-\eta}, & \eta < 1. \end{cases} \tag{23}$$

The probability current $S_{\lambda}(x)$, Eq. (10), rewritten in terms of functions u_{λ}, is

$$S_{\lambda}(z) = \frac{1}{2}(\eta - 1) z^{(\nu-\eta)/(\eta-1)} \frac{\partial}{\partial z} u_{\lambda}(z). \tag{24}$$

Therefore, the boundary conditions for Eq. (21), according to Eq. (11) are $u_{\lambda}'(1)=0$ and $u_{\lambda}'(\xi^{1-\eta})=0$. Here, $u_{\lambda}'(z)$ is the derivative of the function $u_{\lambda}(z)$. The orthonormality relation (14) yields the orthonormality relation for functions $u_{\lambda}(z)$,

$$\frac{1-\xi^{1-\nu}}{(\nu-1)(1-\eta)} \int_1^{\xi^{1-\eta}} z^{(\eta-\nu)/(1-\eta)} u_{\lambda}(z) u_{\lambda'}(z) dz = \delta_{\lambda,\lambda'}. \tag{25}$$

The expression (18) for the first moment X_{λ} of the stochastic variable x evaluated with the λ-th eigenfunction becomes

$$X_{\lambda} = \frac{1}{1-\eta} \int_1^{\xi^{1-\eta}} z^{1+\eta-\nu/1-\eta} u_{\lambda}(z) dz. \tag{26}$$

C. Solution of the equation for eigenfunctions

The solutions of Eq. (21) are [50]

$$u_{\lambda}(z) = z^{\alpha}[c_1 J_{\alpha}(\rho z) + c_2 Y_{\alpha}(\rho z)], \tag{27}$$

where $J_{\alpha}(z)$ and $Y_{\alpha}(z)$ are the Bessel functions of the first and second kind, respectively. The coefficients c_1 and c_2 needs to be determined from the boundary and normalization conditions for function $u_{\lambda}(z)$. The asymptotic expression for the function $u_{\lambda}(z)$ is

$$u_{\lambda}(z) \approx c_{\lambda} z^{\alpha - 1/2} \rho^{-1/2} \cos(\rho z + a), \quad \rho z \gg 1. \tag{28}$$

Here, a is a constant to be determined from the boundary conditions and c_{λ} is the constant to be determined from the normalization (25).

The behavior of the power spectral density in Eq. (19) as $1/f^{\beta}$ can be only due to terms with large λ. Therefore, we will consider the values of λ for which at least the product ρz_{max} is large, $\rho z_{\mathrm{max}} \gg 1$. The first moment of the variable x in the expression for the autocorrelation function (17) is expressed via integral (26). If the condition $\rho z \gg 1$ is satisfied for all z then the function $u_{\lambda}(z)$ has frequent oscillations in all the region of the integration, and the integral is almost zero. Consequently, the biggest contribution to the sum in Eq. (17) makes the terms corresponding to those values of λ, for which the condition $\rho z \gg 1$ is not satisfied for all values of z between z_{min} and z_{max}. Therefore, we will restrict the values of λ by the condition $\rho z_{\mathrm{min}} \ll 1$. Thus we will consider eigenvalues λ satisfying the conditions

$$1/z_{\mathrm{max}} \ll \rho \ll 1/z_{\mathrm{min}}. \tag{29}$$

Explicitly, we have conditions $1 \ll \rho \ll \xi^{\eta-1}$ if $\eta > 1$ and $1/\xi^{1-\eta} \ll \rho \ll 1$ if $\eta < 1$.

The derivative of the function $u_\lambda(z)$, Eq. (27), is

$$u_\lambda'(z)=\rho z^\alpha[c_1 J_{\alpha-1}(\rho z)+c_2 Y_{\alpha-1}(\rho z)]. \tag{30}$$

Since we consider the case $\rho z_{\mathrm{min}} \ll 1$, then, using Eq. (30), instead of the boundary condition $u_\lambda'(z_{\mathrm{min}})=0$ we can approximately take the condition

$$\lim_{y\to 0}[c_1 J_{\alpha-1}(y)+c_2 Y_{\alpha-1}(y)]=0.$$

If $\alpha>1$, then we get $c_2=0$; if $\alpha<1$ then $c_2=-c_1\tan(\pi\alpha)$. Using those values of the coefficient c_2, we obtain the solutions of Eq. (21)

$$u_\lambda(z)\approx\begin{cases} c_\lambda z^\alpha J_{-\alpha}(\rho z), & \alpha<1,\\ c_\lambda z^\alpha J_\alpha(\rho z), & \alpha>1. \end{cases} \tag{31}$$

From approximate solution (31), using asymptotic expression for the Bessel functions, we can determine the parameter a in the asymptotic expression (28). We obtain that the parameter a depends on α and does not depend on ρ.

D. Normalization

Taking $\lambda=\lambda'$ from Eq. (25) we get the normalization condition. Using Eq. (31), we have

$$c_\lambda^2\frac{1-\xi^{1-\nu}}{(\nu-1)(1-\eta)}\int_1^{\xi^{1-\eta}} zJ_{\pm\alpha}^2(\rho z)dz = \frac{1-\xi^{1-\nu}}{(\nu-1)}\frac{c_\lambda^2}{|1-\eta|\rho^2}\int_{\rho z_{\mathrm{min}}}^{\rho z_{\mathrm{max}}} yJ_{\pm\alpha}^2(y)dy\approx 1.$$

Taking into account the condition $\rho z_{\mathrm{min}}\ll 1$ and replacing the lower limit of integration by 0, we obtain that the integral is approximately equal to

$$\int_0^{\rho z_{\mathrm{max}}} yJ_{\pm\alpha}^2(y)dy\approx\frac{\rho z_{\mathrm{max}}}{\pi}.$$

Here, we assumed that $\rho z_{\mathrm{max}}\gg 1$. Therefore, the normalization constant c_λ is

$$c_\lambda\approx\sqrt{\frac{|1-\eta|}{z_{\mathrm{max}}}\frac{\nu-1}{1-\xi^{1-\nu}}\pi\rho}. \tag{32}$$

IV. CALCULATION OF THE POWER SPECTRAL DENSITY

A. Estimation of the first moment X_λ of the stochastic variable x

The expression (17) for the autocorrelation function contains the first moment X_λ of the variable x, expressed as an integral of the function u_λ, Eq. (26). Using Eq. (31), we get

$$X_\lambda\approx\frac{c_\lambda}{1-\eta}\int_1^{\xi^{1-\eta}} z^{\beta-1}J_{\pm\alpha}(\rho z)dz = \frac{c_\lambda}{|1-\eta|\rho^\beta}\int_{\rho z_{\mathrm{min}}}^{\rho z_{\mathrm{max}}} y^{\beta-1}J_{\pm\alpha}(y)dy.$$

Here,

$$\beta=1+\frac{\nu-3}{2(\eta-1)} \tag{33}$$

and "+" sign is for $\alpha>1$, while "−" is for $\alpha<1$.

If $\pm\alpha+\beta>0$, taking into account that $\rho z_{\mathrm{min}}\ll 1$, we can replace the lower limit of the integration by 0, $X_\lambda\approx\frac{c_\lambda}{|1-\eta|}\frac{1}{\rho^\beta}\int_0^{\rho z_{\mathrm{max}}} y^{\beta-1}J_{\pm\alpha}(y)dy$. We get that the integral in the expression for X_λ approximately does not depend on the lower limit of integration ρz_{min}. We can integrate the integral by parts and use the properties of the Bessel functions to obtain

$$X_\lambda\approx\frac{c_\lambda}{|1-\eta|\rho^\beta}\left(\mp y^{\beta-1}J_{\pm(\alpha-1)}(y)\Big|_{\rho z_{\mathrm{min}}}^{\rho z_{\mathrm{max}}} \pm(\beta+\alpha-2)\int_{\rho z_{\mathrm{min}}}^{\rho z_{\mathrm{max}}} y^{\beta-2}J_{\pm(\alpha-1)}(y)dy\right). \tag{34}$$

Using expression (31) for the function $u_\lambda(z)$, the boundary conditions $u_\lambda'(z_{\mathrm{min}})=0$ and $u_\lambda'(z_{\mathrm{max}})=0$ leads to

$$J_{\pm(\alpha-1)}(\rho z_{\mathrm{min}})=0,\quad J_{\pm(\alpha-1)}(\rho z_{\mathrm{max}})=0. \tag{35}$$

Therefore, the first term in the expression (34) for X_λ is zero. If $\beta<\frac{5}{2}$, taking into account that $\rho z_{\mathrm{max}}\gg 1$, we can extend the upper limit of integration to $+\infty$. We get that the integral for X_λ approximately does not depend on the upper limit of integration ρz_{max}.

Therefore, the first moment X_λ of the variable x is proportional to the expression

$$\frac{c_\lambda}{|1-\eta|}\frac{1}{\rho^\beta}. \tag{36}$$

Now we are ready to estimate the power spectral density.

B. Power spectral density

Since $\rho z_{\mathrm{max}}\gg 1$, from the boundary condition $u_\lambda'(z_{\mathrm{max}})=0$ using the asymptotic expression (28) for the function $u_\lambda(z)$ we obtain the condition $\sin(\rho z_{\mathrm{max}}+a)=0$ and $\rho z_{\mathrm{max}}=\pi n-a$. Then

$$\lambda_n\approx\frac{(1-\eta)^2}{2z_{\mathrm{max}}^2}(\pi n-a)^2. \tag{37}$$

Equation (37) shows that the density of eigenvalues $D(\lambda)$ is proportional to $1/\sqrt{\lambda}$. Since the parameter a does not depend on λ, it follows that the density of eigenvalues and, consequently, the autocorrelation function do not depend on the parameter a.

In order to estimate the sum in expression (17) for the autocorrelation function, we replace summation by the integration,

$$C(t)\approx\int e^{-\lambda t}X_\lambda^2 D(\lambda)d\lambda \tag{38}$$

Such a replacement is valid when $\rho z_{\mathrm{max}}\gg 1$. Using the approximate expressions (32) and (36), we get the expression for the autocorrelation function

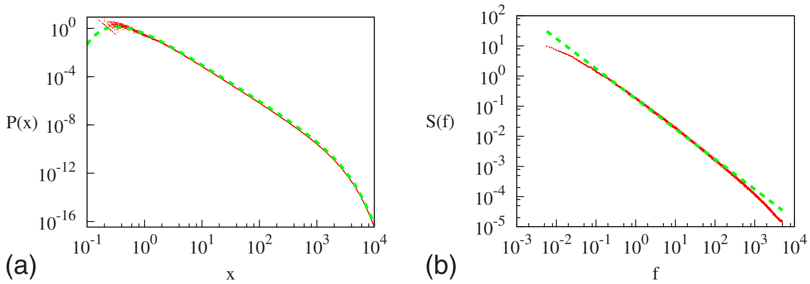

FIG. 1. (Color online) Probability distribution function $P(x)$ (left) and power spectral density $S(f)$ (right) for the stochastic process defined by the stochastic differential Eq. (44). Dashed green lines are analytical expression (9) for the steady-state distribution function $P_0(x)$ on the left and the slope $1/f$ on the right. Parameters used are $x_{\mathrm{min}}=1$, $x_{\mathrm{max}}=10^2$, and $\sigma=1$.

$$C(t) \sim \int_{z_{\mathrm{max}}^{-2}}^{z_{\mathrm{min}}^{-2}} \lambda^{-\beta} e^{-\lambda t} d\lambda = t^{\beta-1}[\Gamma(1-\beta, z_{\mathrm{max}}^{-2}t) - \Gamma(1-\beta, z_{\mathrm{min}}^{-2}t)]. \quad (39)$$

Here, $\Gamma(a,z)=\int_z^{\infty} t^{a-1}e^{-t}dt$ is the incomplete Gamma function. When $z_{\mathrm{min}}^2 \ll t \ll z_{\mathrm{max}}^2$ we have the following lowest powers in the expansion of the approximate expression (39) for the autocorrelation function in the power series of t:

$$C(t) \sim \begin{cases} \dfrac{z_{\mathrm{max}}^{2(\beta-1)}}{\beta-1} - \dfrac{t z_{\mathrm{max}}^{2(\beta-2)}}{\beta-2}, & \beta > 2 \\ z_{\mathrm{max}}^2 + (\gamma-1)t + t\ln(z_{\mathrm{max}}^{-2}t), & \beta = 2 \\ \dfrac{z_{\mathrm{max}}^{2(\beta-1)}}{\beta-1} + t^{\beta-1}\Gamma(1-\beta), & 1 < \beta < 2 . \\ -\gamma - \ln(z_{\mathrm{max}}^{-2}t), & \beta = 1 \\ \dfrac{1}{t^{1-\beta}}\Gamma(1-\beta), & \beta < 1 \end{cases} \quad (40)$$

Here, $\gamma \approx 0.577216$ is the Euler's constant. Similar first terms in the expansion of the autocorrelation function in the power series of time t has been obtained in Ref. [5].

Similarly, when $\rho z_{\mathrm{max}} \gg 1$, replacing in Eq. (19) the summation by the integration, we obtain the power spectral density

$$S(f) \approx 4\int \frac{\lambda}{\lambda^2+\omega^2} X_\lambda^2 D(\lambda) d\lambda. \quad (41)$$

Equation, similar to Eq. (41) has been obtained in Ref. [8] by considering a relaxing linear system driven by white noise [Eq. (27) in Ref. [8]]. Similar equation also has been obtained in Ref. [14] where reversible Markov chains on finite state spaces were considered. In both Refs. [8,14] the power spectral density is expressed as a sum or an integral over the eigenvalues of a matrix describing transitions in the system.

Using the approximate expressions (32) and (36), we get the equation

$$S(f) \sim \int_{z_{\mathrm{max}}^{-2}}^{z_{\mathrm{min}}^{-2}} \frac{1}{\lambda^{\beta-1}} \frac{1}{\lambda^2+\omega^2} d\lambda. \quad (42)$$

When $z_{\mathrm{max}}^{-2} \ll \omega \ll z_{\mathrm{min}}^{-2}$ then the leading term in the expansion of the approximate expression (42) for the power spectral density in the power series of ω is

$$S(f) \sim \begin{cases} \omega^{-\beta}, & \beta < 2, \\ \omega^{-2}, & \beta \geq 2. \end{cases} \quad (43)$$

The second term in the expansion is proportional to ω^{-2}. In the case of $\beta<2$, the term with $\omega^{-\beta}$ becomes larger than the term with ω^{-2} when $z_{\mathrm{max}}^{-2} \ll \omega$. Therefore, we obtain $1/f^{\beta}$ spectrum in the frequency interval $1 \ll \omega \ll \xi^{2(\eta-1)}$ if $\eta>1$ and the frequency interval $1/\xi^{2(1-\eta)} \ll \omega \ll 1$ if $\eta<1$. It should be noted that time t and frequency ω in our analysis are dimensionless.

Equation (41) shows that the shape of the power spectrum depends on the behavior of the eigenfunctions and the eigenvalues in terms of the function $X_\lambda^2 D(\lambda)$. This function $X_\lambda^2 D(\lambda)$ should be proportional to $\lambda^{-\beta}$ in order to obtain $1/f^{\beta}$ behavior. Similar condition has been obtained in Ref. [14]. Equation (13) for discrete time process and the unnumbered equation after Eq. (34) for a continuous time process in Ref. [14] are analogous to the condition $X_\lambda^2 D(\lambda) \sim \lambda^{-\beta}$ since the density of eigenvalues in Ref. [14] is proportional to $1/|\gamma'(x)|$. Equations (39) and (42) for the power spectral density and autocorrelation function resembles those obtained from the sum of Lorentzian signals with appropriate weights in Ref. [27].

V. NUMERICAL EXAMPLES

If $\nu=3$, we get that $\beta=1$ and stochastic differential Eq. (1) should give signal exhibiting $1/f$ noise. We will solve numerically two cases: $\eta=\frac{5}{2}>1$ and $\eta=-\frac{1}{2}<1$. For the numerical solution, we use Euler-Marujama approximation, transforming differential equations to difference equations. Equation (44) with $\eta=5/2$ was solved using variable step of integration, solution Eq. (45) with $\eta=-1/2$ was performed using a fixed step of integration.

When $\eta=\frac{5}{2}$ and $\nu=3$ then Eq. (1) is $dx=\sigma^2 x^4 dt + \sigma x^{5/2} dW$. Using exponential restriction of the diffusion region, we have the equation

$$dx = \sigma^2\left(1 + \frac{1}{2}\frac{x_{\mathrm{min}}}{x} - \frac{1}{2}\frac{x}{x_{\mathrm{max}}}\right)x^4 dt + \sigma x^{5/2} dW. \quad (44)$$

The equation was solved using the variable step of integration, $\Delta t_k = \kappa^2/x_k^3$, with $\kappa \ll 1$ being a small parameter. The steady-state probability distribution function $P_0(x)$ and the power spectral density $S(f)$ are presented in Fig. 1. We see a good agreement of the numerical results with the analytical expressions. The $1/f$ interval in the power spectral density in Fig. 1 is approximately between $f_{\mathrm{min}} \approx 2\times10^{-1}$ and $f_{\mathrm{max}} \approx 2\times10^{2}$. The width of this region is much narrower than

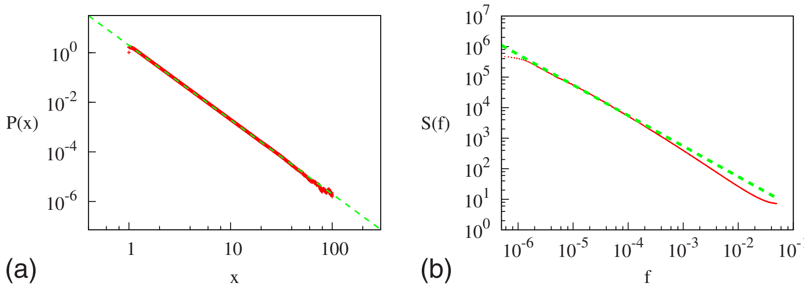

FIG. 2. (Color online) Probability distribution function $P(x)$ (left) and power spectral density $S(f)$ (right) for the stochastic process defined by the stochastic differential Eq. (45). Dashed green lines are analytical expression (9) for the steady-state distribution function $P_0(x)$ on the left and the slope $1/f$ on the right. Parameters used are $x_{\min}=1$, $x_{\max}=10^2$, and $\sigma=1$.

the width of the region $1 \ll 2\pi f \ll 10^6$ ($\xi = 10^2$) predicted in the previous section.

When $\eta=-1/2$ and $\nu=3$ then Eq. (1) is

$$dx = -2\frac{\sigma^2}{x^2}dt + \frac{\sigma}{\sqrt{x}}dW. \tag{45}$$

We used reflective boundary conditions at $x_{\min}=1$ and $x_{\max}=100$. The equation was solved with a constant step of integration. The steady-state probability distribution function $P_0(x)$ and the power spectral density $S(f)$ are presented in Fig. 2. The $1/f$ interval in the power spectral density in Fig. 2 is approximately between $f_{\min} \approx 10^{-6}$ and $f_{\max} \approx 2\times 10^{-4}$. The width of this region is much narrower than the width of the region $10^{-6} \ll 2\pi f \ll 1$ ($\xi=10^2$) predicted in the previous section.

Numerical solution of the equations confirms the presence of the frequency region for which the power spectral density has $1/f^{\beta}$ dependence. The width of this region can be increased by increasing the ratio between minimum and maximum values of the stochastic variable x. In addition, the region in the power spectral density with the power-law behavior depends on the exponent η: if $\eta=1$ then this width is zero; the width increases with increasing the difference $|\eta-1|$. However, the estimation of the width of the region, obtained in the previous section, is too broad, the width obtained in numerical solutions is narrower. Such a discrepancy can be explained as the result of various approximations, made in the derivation.

VI. DISCUSSION

In summary, we derived the behavior of the power spectral density from the nonlinear stochastic differential equation. In Refs. [29,30] only the values of the exponent of the multiplicative noise η greater than 1 has been used. Here, we showed that it is possible to obtain $1/f^{\beta}$ noise from the same nonlinear SDE for $\eta<1$, as well. The analysis reveals that the power spectrum may be represented as a sum of the Lorentzian spectra with the coefficients proportional to the squared first moments of the stochastic variable evaluated with the appropriate eigenfunctions of the corresponding Fokker-Planck equation. Nonlinear SDE, corresponding to a particular case of Eq. (1) with $\eta=0$, i.e., with linear noise and nonlinear drift, was considered in Ref. [9]. It was found that if the damping is decreasing with increase of $|x|$, then the solution of such a nonlinear SDE has long correlation time.

As Eq. (41) shows, the shape of the power spectrum depends on the behavior of the eigenfunctions and the eigenvalues in terms of the function $X_{\lambda}^2 D(\lambda)$, where $D(\lambda)$ is the density of eigenvalues. The SDE (3) considered in this article gives the density of eigenvalues $D(\lambda)$ proportional to $1/\sqrt{\lambda}$. One obtains $1/f^{\beta}$ behavior of the power spectrum when this function $X_{\lambda}^2 D(\lambda)$ is proportional to $\lambda^{-\beta}$ for a wide range of eigenvalues λ, as is the case for SDE (3). Similar condition has been obtained in Ref. [14].

One of the reasons for the appearance of the $1/f^{\beta}$ spectrum is the scaling property of the stochastic differential Eq. (1): changing the stochastic variable from x to $x'=ax$ changes the time-scale of the equation to $t'=a^{2(1-\eta)}t$, leaving the form of the equation invariant. From this property it follows that it is possible to eliminate the eigenvalue λ in Eq. (13) by changing the variable from x to $z=\lambda^{1/2(1-\eta)}x$. The dependence of the eigenfunction on eigenvalue λ then enters only via the boundary conditions. Such scaling properties were used estimating the norm of the eigenfunction and the first moment X_{λ} of the stochastic variable x evaluated with the λ-th eigenfunction. Other factor in obtaining the power-law spectrum is wide range of the region of diffusion of the stochastic variable x.

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