

THE KRAMERS PROBLEM: FIFTY YEARS OF DEVELOPMENT

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NORTH-HOLLAND

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Abstract:

In the last fifty years the seminal work by Kramers of 1940 has been greatly extended both by elaboration on new theoretical approaches and through applications to new experimental systems. The most interesting case turns out to be the regime of weak-to-medium damping, in which case the Fokker–Planck equation can be reduced to an equation or to a system of integral equations of the Wiener–Hopf type. Exact solutions can then be given for the escape rate from single- and double-well potentials. This general scheme can be naturally extended to include quantum penetration through a semiclassical barrier and the effect of quantum noise. Finally, we consider the Brownian motion in a tilted washboard potential using Josephson junctions as an illustrative example. In that context we calculate (i) fluctuation-induced voltage–current characteristics; (ii) the lifetime of a zero-voltage state; (iii) the lifetime of the running state; (iv) partial probabilities of the phase jumps by $2\pi n$ (n is an integer); and (v) retrapping current distribution in both the classical and quantum regimes.

1. Introduction

The aim of this article is to give a survey of the present state of the Kramers problem. Originally, Kramers considered the escape rate of a Brownian particle out of a deep potential well. His final results are applicable in the regimes of strong and extremely weak friction. The problem of bridging the region between these two limits remained unresolved for more than four decades. Meanwhile, great progress has been achieved in extending the Kramers approach to multidimensional systems. On the other hand, with lowering of the temperature the activated decay of metastable states is suppressed and the quantum tunneling becomes the dominant decay process. Interest in these problems was renewed in connection with the phenomenon of macroscopic quantum tunneling which found a physical realization in experiments on Josephson junctions. At sufficiently high temperature these devices also provide a laboratory system for studying the classical Brownian motion in tilted periodic potentials (the so-called washboard potential). Being qualitatively different from both single- and double-well potentials, the periodic potential displays a wide spectrum of new fluctuation-induced phenomena.

In the main body of this paper we will focus on deriving some rigorous results for a number of physically important systems. Since most of the work is original, and attempting to do justice to all those who contributed to the field is beyond our scope, it is reasonable at this point to give a short commentary on the history of the problems under consideration.

The modern paradigm of the theory of decay rates includes on an equal basis the concepts of the activation escapes [1] and the quantum-tunneling escapes [2]. Quantum-tunneling corrections to activated escape rates were first considered by Wigner [3]. Almost at the same time Eyring developed the transition state theory (TST) [4], an important conceptual step for chemistry. These considerations did not take into account effects of friction, and were thus applicable in a wide range of friction strength for which thermal noise is sufficiently strong to thermalize the escaping particles, but friction is still considered not to affect particle motion across the top of the potential barrier. The explicit role of friction was first discussed by Kramers who has shown that for sufficiently weak friction the escape rate is suppressed due to depletion of the well population, whereas for strong friction it is suppressed due to the slowing down of the particle motion at the top of the barrier [5]. At this stage of development the research in the field was summarized by Chandrasekhar [6].

An important breakthrough was achieved in the multidimensional Kramers problem when Landauer and Swanson calculated the escape rate in the limit of strong dissipation [7]. However, most influential in the field were the articles by Langer who reduced the calculation of escape rates to an analytical continuation of the free energy [8, 9]. Mathematical procedures developed in these papers have proved to be very useful in the solution of a large number of physical problems. An alternative approach to similar problems was proposed by Miller, who considered the semiclassical limit of quantum TST and substantially clarified the physical aspects of the problem [10].

For a rather long time the prospect of experimentally confirming these results seemed remote, a situation which changed with the discovery of the Josephson effect [11]. In the simplest model of a resistively shunted Josephson junction the behavior of the junction can be modelled in terms of the motion of a classical particle in a tilted washboard potential in the presence of friction and thermal noise [12]. From this time on, investigations of fluctuation-induced effects in Josephson junctions represent a major part of the further development of the Kramers problem.

Already in the Kramers work the limits of extremely weak and strong friction were investigated separately, by different mathematical approaches. Intuitively, it was clear that the solution for the weak-friction limit required a special technique. The first step in that direction was made by Iche and

Nozières who pointed out that in this regime the Fokker–Planck equation can be reduced to an integral equation [13]. In particular, they have investigated some properties of the solution of this equation (or a system of such equations) and succeeded to extend Kramers' weak-friction-limit result to a double-well potential. In their second paper these authors have written down a system of integral equations for the case of a washboard potential and discussed qualitatively the fluctuation-induced voltage–current characteristics of a Josephson junction [14]. Since the case of strong friction is much simpler, the calculations were first done in the limit of overdamped junctions [15]. The finite-friction case was investigated numerically by Risken and Vollmer using a matrix continued-fraction method [16]. They also found analytically the particle distribution function for the running state [17] (see also ref. [14]), formulated the problem of the lifetime of the running state and calculated the threshold current which separates the fluctuation-induced and deterministic branches of the voltage–current characteristic [18]. The problem of the lifetime of the resistive state was also discussed by Ben-Jacob and coauthors [19] and a useful expression for the retrapping current distribution in terms of the lifetime of the resistive state was given by Barone et al. [20]. The lack of an analytical approach stimulated work on numerical simulations both of single-well escapes [21] and of fluctuation-induced phenomena in Josephson junctions [22].

A particularly important year for the theory of the quantum decay of metastable states was 1981. During that year Wolynes solved the quantum version of the Kramers problem in the regime of strong friction from the high-temperature side [23]; Affleck tracked the crossover from thermal activation to quantum tunneling for the escape rate of thermalized particles [24]; Caldeira and Leggett proposed a general approach to dissipative quantum tunneling [25], which opened a wide field of research in systems with intermediate-to-strong friction. To mention only a few of the immense number of works in this field, we refer here to the articles on dissipative tunneling at finite temperature [26, 27] and to an extension of Affleck's approach to a dissipative case [28] with the use of Miller's technique [10].

Meanwhile, the underdamped Kramers problem remained unsolved. In order to get an interpolation between the already known results, Büttiker, Harris and Landauer [29] have inserted an additional term in the energy-diffusion equation, derived earlier by Kramers, and applied this equation to the calculation of the average energy of escaping particles [30]. From these papers and from the results of very careful numerical studies [31, 32] it was concluded that the behavior of the decay rate for weak friction is nonanalytic, and the amplitude of the first correction to the Kramers result was found with high accuracy [31, 32]. Nearly at that time the research on systems with memory friction resulted in an extension of Kramers' approach to the Brownian motion with long-time memory [33] and BHL's approach to the non-Markovian case with memory damping [34]. A review of activity around the Kramers problem was given by Hänggi [35].

Independently of Iche and Nozières, the present author also proposed the reduction of the Fokker–Planck equation to an integral equation in the energy variable. The first paper [36] contained a short description of this approach, the Gaussian kernel of the integral equation and some mathematics which later turned out to be useful for the solution of the quantum version of the Kramers problem. The most important result here was the conclusion that in the classical limit the kernels of integral equations are universal Gaussian functions, which are specified by a sole parameter for each potential well. An exact solution of the Kramers problem was described in detail both for single- and double-well potentials [37]. The considerations were extended to the study of fluctuation-induced phenomena in a tilted washboard potential. We produced results for the zero-temperature voltage across a biased Josephson junction, the fluctuation-induced voltage–current characteristics, the lifetime of a zero-voltage state and the partial probabilities of the phase jumps by $2\pi n$ (where n is an integer) [38]. These

results were complemented by the calculation of the lifetime of the resistive state and of the retrapping current distribution [39].

Larkin and Ovchinnikov have given a solution of the Kramers problem in the presence of quantum mechanical effects [40]. In contrast to the classical limit, the resulting kernels of integral equations are nonuniversal and depend on the overall shape of a potential. Later this approach was extended to the case of Josephson junction [41]. The general problem in this case is rather complicated due to a large number of parameters involved, and thus explicit results have only been given for the retrapping current distribution [42].

Our main aim is to give a detailed and hopefully pedagogical derivation of a number of physically important results. Explicit reference to previous work will be given only when necessary for the clarity of the exposition. For an exhaustive up to date list of references and a discussion of the various approaches see the review of Hänggi [43].

In the remaining paragraphs we outline the contents of the article. In section 2 we summarize the results for the Brownian motion in deep potential wells with emphasis on the underdamped regime. Discussion of the physical aspects of the problem is given in parallel with the presentation of the results obtained in Kramers' pioneering paper. The main conclusion of these considerations is that the escape rate of a Brownian particle out of a potential well is governed by different mechanisms in the limits of weak and strong friction, and hence a solution of the Kramers problem in these two limits can only be achieved by developing different mathematical approaches. Our general line of reasoning is as follows.

In the underdamped limit the particle's trajectories are only slightly perturbed by dissipation and fluctuations. The energy distribution functions for the particles at different points of a potential well are related through the Green function of the Fokker–Planck equation. When complemented by boundary conditions, these integral relations can be converted into an integral equation for the energy distribution function for particles at the top of the barrier. The resulting one-sided convolution equation with a Gaussian kernel can be easily solved by the Wiener–Hopf method, leading to an explicit expression for the escape rate in the underdamped regime. The shape of the potential well enters the result through a sole parameter: the energy loss during one oscillation for a particle starting at the top of the barrier. The same parameter determines the average energy of the escaping particles. By combining this result with Kramers' expression for the overdamped regime we arrive at an expression for the escape rate valid in the whole range of damping strengths. Also in section 2 we derive the lifetimes of the Brownian particle in particular minima of a double-well potential, which are then used to calculate the rate of relaxation of a nonequilibrium population in the two minima. This section is concluded with a short exposition of theoretical [44] and experimental [45] results on the activated decay of the zero-voltage states in a Josephson junction shunted with a delay line. The experiments have shown that the lifetime of the zero-voltage states oscillates as a function of the delay-line length [45]. These results give the first direct evidence for the oscillation of a particle in a well before escape which is implied by Kramers's energy-diffusion model for the escape from a metastable state.

The approach based on the Wiener–Hopf equations is extended to the quantum case in section 3. The changes due to the energy level quantization near the bottom of the potential well, quantum penetration of the potential barrier and the modification of the kernel of the Wiener–Hopf equation in the presence of quantum fluctuations are naturally incorporated into the scheme developed earlier. A principal difference of the final expression for the escape rate from that in the classical case is the lack of universality, since the escape rate now depends on details of the shape of the potential. (However, for two typical potentials this difference does not exceed a few percent.) In the overdamped case for which the friction coefficient is comparable to the oscillation frequency, the distribution of particles in a

potential well deviates from the Boltzmann distribution and depends on the magnitude of the friction. An explicit expression for the particle distribution in coordinate and momentum is derived for a slowly varying potential. It is shown that the interaction with the high-frequency modes of the thermal bath results in large zero-point fluctuations of the particle momentum. The escape rate in the overdamped regime is calculated with the use of an analytic continuation of the free energy.

Section 4 contains an investigation of a variety of fluctuation-induced phenomena in Josephson junctions. The latter system can be modelled by the Brownian motion of a particle in a tilted washboard potential, the height of which is $\hbar I_c/e$, where I_c is the critical current of the junction. To describe the dissipative properties of the junction we adopt the model of a resistively shunted junction, in which case the coefficient of viscous friction is $1/RC$, where R is the shunting resistance, C is the capacitance. The tilt of the cosine potential is governed by the external current I . As an introductory step we consider the voltage–current characteristics in the absence of thermal noise. For $I < I_0 \equiv I_c/RC\Omega$, where Ω is the Josephson plasma frequency, the junction can only exist in a zero-voltage state, analogous to the small oscillations of the Brownian particle at the bottom of a potential minimum. In the opposite case ($I > I_0$), in addition to the zero-voltage states, the resistive state also becomes possible in analogy with the solution in which the particle is flying above the potential barriers. In the presence of thermal noise the average voltage $\bar{V}(I)$ (calculated in an exponential approximation) remains nearly zero for $I < I_0$; whereas in the $I > I_0$ case $\bar{V}(I)$ is drastically changed compared to the corresponding running-state voltage $V(I)$ due to fluctuation-induced switchings between zero-voltage and running states. At sufficiently large values of I the effects of thermal noise become negligible and $\bar{V}(I) \approx V(I)$ at $I > I_1 \approx 2.63I_0$. In this regime the junction spends most of the time in the running state. By using the same exponential approximation we also calculate the retrapping current distribution, which describes the probability for the junction to switch from the running state into a zero-voltage state in the process of the slow decrease of the external current I . This distribution turns out to be Gaussian with a typical width of order $eT/\hbar RC\Omega$. Below the threshold current I_0 , if fluctuation-induced transitions of the Brownian particle between different potential minima are taken into account, the average voltage $\bar{V}(I)$ becomes nonvanishing, although exponentially small.

The mathematical formulation of this problem is in terms of a system of two integral equations of the Wiener–Hopf type for two distribution functions which describe the particles escaping out of a potential minimum across the left- and right-hand sides of the barriers. (Due to periodicity of the problem, these functions are identical for all the barriers.) With some modifications, this approach can also be exploited in calculating the preexponential factor for the voltage–current characteristics above threshold. Exactly at threshold, $I = I_0$, the function $\bar{V}(I)$ undergoes a finite break in its logarithmic derivative. In a similar way we can also calculate the preexponential factor for the lifetime of the running state.

Also in section 4 we consider the Brownian particle escape rate out of a particular potential minimum. This problem is equivalent to the calculation of the lifetime of a zero-voltage state and its solution requires the investigation of an infinite system of integral equations, due to the presence of nonequivalent minima of the potential. In spite of this problem being rather complicated, a closed solution can still be found, and allows us to calculate the partial jump probabilities between different minima and the probability for the junction to go over from a zero-voltage into the running state. Again, the generalization of these results to the quantum case is straightforward. However, the final expressions are complicated, since they depend on the additional parameter $\hbar\Omega/2\pi T$, a measure of the quantum effects. Accordingly, a detailed expression is only given for the retrapping current distribution which in the ultraquantum regime, $\hbar\Omega \gg T$, has a width of order e/RC . We conclude with a brief summary and discussion of our results in section 5.

2. Lifetime of a classical Brownian particle in deep potential wells

2.1. Formulation of the problem and review of Kramers' results

The starting point in Kramers' model is the Langevin equation

$$m \frac{d^2x}{dt^2} = -m\gamma \frac{dx}{dt} - dU(x)/dx + \eta(t), \quad (2.1)$$

where x is the position of a particle with the mass m , γ is the damping coefficient, $\eta(t)$ is a stationary Gaussian force associated with the coupling to a viscous thermal bath,

$$\langle \eta(t)\eta(t') \rangle = 2m\gamma T\delta(t - t').$$

Equation (2.1) is equivalent to the Fokker-Planck equation,

$$\frac{\partial F}{\partial t} + \frac{p}{m} \frac{\partial F}{\partial x} - \frac{\partial}{\partial p} \left[F \frac{dU}{dx} + \gamma \left(pF + mT \frac{\partial F}{\partial p} \right) \right] = 0, \quad (2.2)$$

for the distribution $F(p, x, t)$ of an ensemble of noninteracting Brownian particles having momentum $p \equiv m dx/dt$ and position x . We begin with the simplest example of the metastable state when a Brownian particle once escaped over the barrier has no chance to return. The corresponding one-well potential $U(x)$ is depicted in fig. 1.

The zero of the potential is chosen to be at the barrier top located at $x = 0$. The boundary condition

$$F(p, x, t) \rightarrow 0 \quad \text{as } x \rightarrow \infty, \quad (2.3)$$

reflects the initial condition that there were no particles at the outer side of the barrier at $t = 0$. From now on it will be assumed that the depth of the well U_0 is large enough compared to T . The ratio T/U_0 is the principal small parameter of the problem. A Brownian particle trapped into a deep potential well resides there for an exponentially long time, exceeding all relaxation times. That is a basic point in a theory of the rate of activated processes, as it allows a sensible definition of the decaying state and

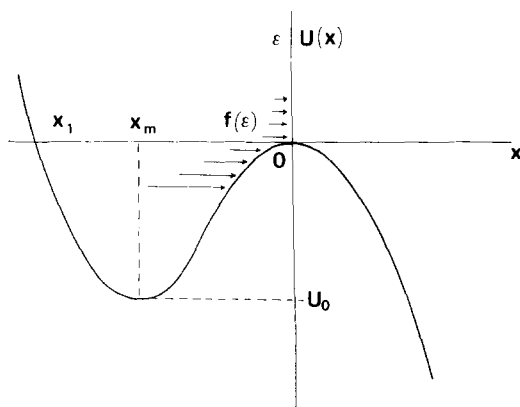


Fig. 1. Schematic representation of a single-well potential. No particles enter the potential well in the underdamped regime.

introduction of the concept of lifetime. The problem of the lifetime of a particle in a potential well can then be formulated rigorously in mathematical terms.

Relaxation of an arbitrary initial distribution of particles interacting with a thermal bath is a two-stage process. After a comparatively short time the distribution of particles inside the potential well and in a vicinity of the barrier approaches its steady-state form. The next stage of relaxation is a purely exponential decay of the distribution as a whole, caused by escapes of the particles over the barrier,

$$F(p, x, t) = F(p, x) \exp(-t/\tau). \quad (2.4)$$

The steady-state distribution $F(p, x)$ can be found from the equation

$$\frac{F}{\tau} - \frac{p}{m} \frac{\partial F}{\partial x} + \frac{\partial}{\partial p} \left[F \frac{dU}{dx} + \gamma \left(pF + mT \frac{\partial F}{\partial p} \right) \right] = 0, \quad (2.5)$$

which is obtained by inserting eq. (2.4) into eq. (2.2). The solution of eq. (2.5) is nonnormalizable, $\int F(p, x) dp dx = \infty$, as $F(p, x)$ diverges at $x \rightarrow \infty$. One encounters the same problem also in the quantum-tunneling theory [50]. The straightforward solution of eq. (2.2) with any reasonable initial condition as in eq. (2.3) is well normalizable and tends to the solution (2.4) for large times. The larger the x , however, the slower is the approach to the solution (2.4). Below we consider $F(p, x)$ only inside the well and in a close vicinity of the barrier.

Near the bottom of the potential well the potential $U(x)$ can be represented by a harmonic oscillator of frequency $\Omega = [U''(x_m)/m]^{1/2}$, x_m being the location of the potential minimum,

$$U(x) \approx -U_0 + \frac{1}{2} m \Omega^2 (x - x_m)^2.$$

The normalized distribution function $F(p, x)$ in this region of variables is only slightly perturbed by escapes over the barrier and retains the Boltzmann form,

$$F(p, x) \approx F_0(p, x) \equiv (\Omega/2\pi T) \exp[-(\varepsilon + U_0)/T], \quad -\varepsilon \gg T, \quad (2.6)$$

$$\varepsilon \equiv p^2/2m + U(x), \quad (2.7)$$

where ε is the total energy, taken with respect to the barrier top, and $F_0(p, x)$ is the equilibrium function. Taking into account rare escapes we can write

$$F(p, x, t) = N(t)F(p, x), \quad N(t) \propto \exp(-t/\tau).$$

$N(t)$ is the number of particles in the well. The main contribution to the normalization condition,

$$\int_{-\infty}^0 dx \int_{-\infty}^{\infty} F(p, x, t) dp = N(t),$$

comes from a narrow region near the bottom of the well, $|x - x_m| \sim \Omega^{-1}(T/m)^{1/2}$. In these terms the escape rate $1/\tau$ can be looked upon as the lowest eigenvalue of eq. (2.5) under given restrictions on the function $F(p, x)$. This approach is developed in full detail in the section devoted to the calculation of

the lifetime of a running state. In order to get a deeper physical insight into the problem we will proceed in the present section in quite a different way, calculating explicitly the flux of escaping particles.

The flux of particles,

$$J = \int_{-\infty}^{\infty} \frac{p}{m} F(p, x, t) dp, \quad (2.8)$$

calculated near the barrier top, does not depend on x as long as $|U(x)| \ll U_0$. The conservation of the number of particles, $dN/dt = -J$, yields the relation between the lifetime τ of a particle and flux J ,

$$1/\tau = J/N. \quad (2.9)$$

Below we shall use this relation to calculate $1/\tau$. The first term in eq. (2.5) is negligibly small. Hence, the steady-state distribution obeys the equation

$$\frac{p}{m} \frac{\partial F}{\partial x} - \frac{\partial}{\partial p} \left[\frac{dU}{dx} F + \gamma \left(pF + mT \frac{\partial F}{\partial p} \right) \right] = 0, \quad (2.10)$$

with the boundary condition (2.6) and asymptotics (2.3). In the general situation the above formulated problem cannot be solved analytically. However, making use of the small parameter T/U_0 , in other words, for sufficiently deep potential wells, one can apply different approaches in the regimes of weak and strong friction and derive in this way an expression for τ , applicable at arbitrary values of γ .

It was shown by Kramers that for large γ eq. (2.10) needs to be solved only near the top of the barrier. In this region the potential $U(x)$ is parabolic,

$$U(x) \approx -\frac{1}{2} m \omega^2 x^2,$$

and eq. (2.10) goes over into an equation with linear coefficients,

$$\frac{p}{m} \frac{\partial F}{\partial x} - \frac{\partial}{\partial p} \left[-m \omega^2 x F + \gamma \left(pF + mT \frac{\partial F}{\partial p} \right) \right] = 0. \quad (2.11)$$

For this equation Kramers has found an exact solution of the above formulated problem,

$$F(p, x) \propto \exp\left(-\frac{p^2}{2mT} + \frac{m\omega^2 x^2}{2T}\right) \int_{x - \lambda p/m\omega^2}^{\infty} \exp\left(-\frac{m\omega^2 \xi^2}{2\gamma\lambda T}\right) d\xi, \quad (2.12)$$

$$\lambda = (\omega^2 + \frac{1}{4}\gamma^2)^{1/2} - \frac{1}{2}\gamma. \quad (2.13)$$

In view of condition (2.3), the upper limit of integration is chosen to be infinity. The integral in eq. (2.12) saturates at

$$\lambda p/m\omega^2 - x \gg (\gamma\lambda T/m\omega^2)^{1/2}. \quad (2.14)$$

In this region of variables the function $F(p, x)$ approaches an equilibrium one. The solution (2.12) has enabled Kramers to find the escape probability per unit time. Namely, one can find the normalization factor in eq. (2.12) by comparison with eq. (2.6) under condition (2.14). Then, using eqs. (2.8) and (2.9) one obtains

$$1/\tau = (\Omega/2\pi)[(1 + \gamma^2/4\omega^2)^{1/2} - \gamma/2\omega] \exp(-U_0/T). \quad (2.15)$$

This expression for the escape rate from a potential well has the following features

(1) It is exponentially small in the parameter U_0/T .

(2) It is proportional to the frequency Ω of small oscillations near the bottom of the well, which enters the problem through the normalization condition for the equilibrium function $F_0(p, x)$.

(3) The factor $(1 + \gamma^2/4\omega^2)^{1/2} - \gamma/2\omega \equiv \lambda/\omega$ is closely related to the increment of the particle motion near the top of the barrier, since the solution of the dynamic equation gives $x(t) \propto \exp(\lambda t)$.

It will be shown below that point 1 holds for all expressions for τ above some temperature, determined by quantum effects. The factor $\Omega/2\pi$ has to be changed in the quantum regime, when the separation of the energy levels $\hbar\Omega$ becomes comparable to T . The factor λ/ω describes explicitly the slowing down of the particle dynamics caused by friction. This factor deviates from unity only at $\gamma \geq \omega$. As γ diminishes it becomes unity, and the corresponding result is frequently referred to as the TST expression [4].

The total flux J in eq. (2.8) is the difference of the fluxes J_R and J_L carried by right- and left-going particles,

$$J = J_R - J_L, \quad J_{R,L} = \int_{-\infty}^{\infty} \theta(\pm p) \frac{|p|}{m} F(p, 0) dp.$$

To the flux J_L contribute the particles which are returning after crossing the barrier. The explicit expression (2.12) for $F(p, x)$ enables us to estimate the relative drop of the flux caused by the recrossings,

$$J_L/J_R = 1 - (1 + \gamma^2/4\omega^2)^{-1/2}. \quad (2.16)$$

This result gives a quantitative confirmation of the intuitive feeling that the origin of the recrossings is the interaction of a particle with the thermal bath. It is worth noting that the ratio J_L/J_R does not depend on the temperature T .

The rigorous condition of validity of the above obtained results will be given below. A simplified condition is that solution (2.12) should approach the equilibrium distribution at such values of x , where the inverted-oscillator approximation for the potential still holds. One should bear in mind that at sufficiently large negative momenta p the function $F(p, x)$ will always deviate from the equilibrium one due to depopulation of this region of the phase space [see eq. (2.12)]. The condition (2.6) should only be satisfied for energies below the barrier top, when

$$p^2/2m - \frac{1}{2}m\omega^2 x^2 < 0.$$

This inequality together with eq. (2.14) yields

$$-x \gg \frac{(\gamma\lambda T/m)^{1/2}}{\omega(\omega - \lambda)}. \quad (2.17)$$

The last condition gives an estimate of a spatial region, where the equilibrium distribution of particles is only negligibly perturbed by their escapes over the barrier. At large and moderate damping $\gamma \geq \omega$ the condition (2.17) taking account of eq. (2.13) yields $-x \gg (T/m\omega^2)^{1/2}$. This means that $F(p, x)$ deviates from the equilibrium function in a rather narrow region of x , where $|U(x)| \sim T \ll U_0$. However, in the underdamped regime, when $\gamma \ll \omega$, condition (2.17) simplifies to

$$-x \gg (\omega/\gamma)^{1/2} (T/m\omega^2)^{1/2},$$

and $F(p, x)$ becomes a nonequilibrium distribution in a much broader region, so that with $\gamma \rightarrow 0$ the inverted-oscillator approximation for $U(x)$ shall inevitably be broken. The width of a typical potential well is of the order of magnitude of about $(U_0/m\omega^2)^{1/2}$. The condition of validity of eq. (2.15) for τ^{-1} in the underdamped regime then becomes

$$\gamma \gg \omega T/U_0. \quad (2.18)$$

To conclude the discussion based on the solution (2.12), we write down the equation for τ^{-1} in the underdamped limit of eq. (2.15),

$$1/\tau = (\Omega/2\pi) \exp(-U_0/T), \quad \omega \gg \gamma \gg \omega T/U_0. \quad (2.19)$$

We have derived the lower limit of validity of the last equation in a rather formal manner. Now it is worthwhile to discuss the physical meaning of these results. It can be seen easily that in the limit of $\gamma \rightarrow 0$ the integral in eq. (2.12) goes over into $\theta(p - m\omega x)$ where $\theta(p)$ is the standard step function. The distribution $F(p, x)$ describes, therefore, the equilibrium flux of particles towards the barrier and the flux of particles reflected from the barrier. It is physically evident, that at $\gamma = 0$ no equilibrium flux towards the barrier is possible, as a nonvanishing interaction with the thermal bath only enables particles to climb up the energy scale from the bottom of the well to the top of the barrier. Equation (2.15) is inapplicable in the limit $\gamma \rightarrow 0$, because it ignores the effects of depopulation below the barrier top.

Considering the motion of a Brownian particle in the extremely underdamped regime as almost conservative with a very slow diffusion on the energy axis, Kramers obtained [5]

$$1/\tau = (\gamma S/T)(\Omega/2\pi) \exp(-U_0/T), \quad \gamma \ll T/S, \quad (2.20)$$

where S is the action per oscillation of a particle at the energy top,

$$S = 2 \int_{x_1}^0 [-2mU(x)]^{1/2} dx,$$

and x_1 is the left-hand side turning point, $U(x_1) = 0$. By order of magnitude $S \sim U_0/\omega$. Note, that in contrast to eq. (2.20), where S depends on the shape of the potential as a whole, eq. (2.15) involves the shape of the potential well only through its curvatures at the initial minimum and at the barrier top. The frequency Ω is inevitably involved in both eq. (2.15) and eq. (2.20), as it is a measure of the phase space available at the initial minimum, and thus of the particle density which has to be depleted by the flux streaming over the barrier. The region of validity of eq. (2.20), $\gamma \ll T/S \sim \omega T/U_0$, does not

overlap with that of eq. (2.15) [see eq. (2.18)]; they are separated by an interval of damping $\gamma \sim \omega T/U_0$.

To calculate τ in this region means to bridge up the whole range of γ , and thus, to obtain a complete solution of the problem. In a series of papers, attempts have been made to solve the outlined problem (see, e.g., ref. [35] and references therein). Unfortunately, these attempts involve unjustified assumptions about the distribution function, and results obtained in such a way represent only ad hoc interpolations between two Kramers' results. In the subsequent sections a rigorous relationship for τ^{-1} valid at arbitrary damping is found. No model assumptions have been used, the only small parameter of the problem being T/U_0 . In what follows we shall systematically neglect algebraically small corrections of order T/U_0 compared to 1, retaining only the leading order term in a low-temperature expansion.

It is convenient to factorize the decay rate into two parts, i.e.,

$$1/\tau = A(\Omega/2\pi) \exp(-U_0/T). \quad (2.21)$$

The dependence on the coupling to the heat bath is absorbed into the factor A , while the second factor describes equilibrium properties of the system and does not require knowledge of the dynamics. It is shown below that in the underdamped regime A depends solely on $\delta \equiv \gamma S$, the loss of energy per oscillation of a particle with energy close to the barrier height. The anticipated expression for A should describe the crossover from the asymptotics

$$A \approx \delta/T \ll 1, \quad (2.22)$$

to the asymptotics

$$A \approx 1, \quad T \ll \delta \ll U_0 \quad (2.23)$$

[see eqs. (2.20) and (2.19)]. On the other hand, at $\delta \sim U_0$ and $\omega \sim \gamma$, the preexponential factor A depends only on the ratio γ/ω ,

$$A = (1 + \gamma^2/4\omega^2)^{1/2} - \gamma/2\omega, \quad (2.24)$$

as can be seen from eq. (2.15). At $\gamma \ll \omega$ the last expression gives $A \approx 1$. This means that eqs. (2.23) and (2.24) have a common region of applicability, $T/\omega U_0 \ll \gamma \ll \omega$, where $A \approx 1$. Hence, one can bridge the ranges of underdamped and overdamped regimes in a very simple manner: once a functional form of A describing the crossover between eq. (2.22) and eq. (2.23) is known, the naive product of it with eq. (2.24) will give the preexponential factor A at arbitrary damping in the low-temperature limit $T/U_0 \ll 1$.

2.2. The Green function for the Fokker–Planck equation

The purpose of this section is to demonstrate that the underdamped Brownian motion in a deep potential well can be adequately described in terms of a Green function of the Fokker–Planck equation. In the next section this approach will be used to derive an integral equation where the kernel is given by a Green function. In accordance with Kramers' results we assume that the flux over the barrier is carried by particles with energies ε in a close vicinity of the barrier top, $|\varepsilon| \leq T$. The scale of the

potential energy exceeds both the thermal energy T and the friction-induced energy loss per oscillation δ , i.e., $U_0 \gg T, \delta$. Therefore, the deterministic dynamics of the escape process is only slightly perturbed by friction and the random force. As the total energy ε [see eq. (2.7)] is the most slowly varying quantity, it is convenient to use it as a new variable instead of the momentum p . The price we pay for this convenience is that now we have to treat the right- and left-going particles separately. To this end we introduce the indices $+$ and $-$. The Fokker–Planck equation (2.10) may then be transformed via substitutions

$$\begin{aligned} p &= \pm \{2m[\varepsilon - U(x)]\}^{1/2} \equiv p_{\pm}(\varepsilon, x), \\ \partial/\partial p|_x &= m^{-1} p_{\pm}(\varepsilon, x) \partial/\partial x|_{\varepsilon}, \quad \partial/\partial x|_p = \partial/\partial x|_{\varepsilon} + (dU/dx) \partial/\partial \varepsilon|_x, \\ f_{R,L}(\varepsilon, x) &\equiv F[p_{\pm}(\varepsilon, x), x]. \end{aligned} \quad (2.25)$$

To proceed further we observe that one can put $\varepsilon = 0$ in the relationship for $p_{\pm}(\varepsilon, x)$. Indeed, our basic trajectory corresponds directly to $\varepsilon = 0$, and the leading contributions to the escape stem from particles in a narrow range of energies $|\varepsilon| \sim T$. Furthermore, the main part of the trajectory lies inside the well, where $|U(x)| \gg |\varepsilon|$. In this approximation the Fokker–Planck equation (2.10) takes on the form

$$\partial f_{R,L}/\partial x = \pm [-2mU(x)]^{1/2} \gamma(\partial/\partial \varepsilon)(f_{R,L} + T \partial f_{R,L}/\partial \varepsilon), \quad (2.26)$$

with coefficients independent of ε . Close to the left-hand turning point, where all the particles are reflected, we have

$$f_R(\varepsilon, x) = f_L(\varepsilon, x), \quad |x - x_1| \ll |x_1|, \quad (2.27)$$

whereas close to the barrier top $f_R = f_L$ only for $\varepsilon < 0$. The function f_L vanishes for $\varepsilon > 0$, as there are no particles going over the barrier into the well.

Equation (2.26) takes on a more convenient form after introduction of the action $s = s(x)$ along the basic trajectory. The relationship between s and x is defined by the differential equation

$$ds/dx = \pm [-2mU(x)]^{1/2}, \quad (2.28)$$

where the sign $+$ or $-$ corresponds to the sign of the particle velocity, so that s is monotonically increasing along the trajectory. We then arrive at the equation

$$\partial f(\varepsilon, s)/\partial s = \gamma(\partial/\partial \varepsilon)[f(\varepsilon, s) + T \partial f(\varepsilon, s)/\partial \varepsilon], \quad (2.29)$$

which describes diffusion and uniform drift in the energy space. Note, that the propagation along the basic trajectory is parameterized not by time or position, as in more familiar cases, but by the action s . Equation (2.29) enables us to find a relationship between the functions f at different values of the action s ,

$$f(\varepsilon, s) = \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon', s - s') f(\varepsilon', s') d\varepsilon', \quad (2.30)$$

where g is the Green function of eq. (2.29) which satisfies an initial condition $g(\varepsilon - \varepsilon', 0) = \delta(\varepsilon - \varepsilon')$. Solution of eq. (2.29) for g gives

$$g(\varepsilon - \varepsilon', s) = (4\pi T\gamma s)^{-1/2} \exp[-(\varepsilon - \varepsilon' + \gamma s)^2/4T\gamma s]. \quad (2.31)$$

Quite naturally, the Gaussian random force induces the Gaussian distribution of variation of the energy. The mean energy loss $\langle \delta\varepsilon \rangle$ ($\delta\varepsilon \equiv \varepsilon - \varepsilon'$) is connected with the mean square value of $\delta\varepsilon - \langle \delta\varepsilon \rangle$ by the fluctuation–dissipation theorem,

$$\langle (\delta\varepsilon - \langle \delta\varepsilon \rangle)^2 \rangle = 2T \langle \delta\varepsilon \rangle = 2T\gamma s,$$

where the brackets $\langle \dots \rangle$ denote averaging over the distribution (2.31). The advantage of the introduction of the action s , instead of using the particle position x , is that s increases steadily along the oscillatory trajectory, whereas the assignment of x is insufficient to specify different cycles of motion.

We have restricted ourselves to the calculation of the Green function of eq. (2.29). A general solution of this equation should be periodic in s for $\varepsilon < 0$. For positive ε this solution is subject to the condition that no particles enter the potential well from outside the barrier. These two solutions must be matched at $\varepsilon = 0$. A detailed investigation of the function $f(\varepsilon, s)$, based on a series expansion combined with numerical calculations, was presented by Risken and coworkers [51].

A word of caution about the indices R and L, which were suppressed in eq. (2.30), is necessary. In the simplest case we insert the function f_L into the integrand of eq. (2.30). In the left-hand side of this equation we then get the function f_L before the left-side turning point x_L , and the function f_R after reflection of the particles. The situation at the barrier top is somewhat more complicated, because only particles with energy $\varepsilon < 0$ are reflected, whereas those with $\varepsilon > 0$ escape over the barrier. An obvious implication of eq. (2.30) is that in the underdamped regime the distribution of the particles throughout the potential well can easily be found from the distribution at a certain x . The transformation from the action s to the position x is defined by eq. (2.28). In view of the oscillatory nature of the motion in a potential well it is rather likely that, if complemented by a proper boundary condition to account for the escape of particles from the well, the integral relationship (2.30) may well be transformed into a closed integral equation.

2.3. Integral equation for the distribution function

In this section we use eq. (2.30) to derive an integral equation for the distribution function, which under certain conditions is equivalent to the original Fokker–Planck equation (2.10). One of these conditions, $\gamma \ll \omega$, already derived above, is necessary for the validity of eq. (2.30). In this regime, eqs. (2.28) and (2.30) determine the distribution function $f(x, \varepsilon)$ within the whole potential well if this function is known at a certain point in the well.

To transform the relationship (2.30) into an integral equation we need additional information about the behavior of the potential $U(x)$ outside the barrier. The simplest situation is that of a single potential well, when, after surmounting the barrier, the particles never return to the well. In this case eq. (2.16) gives an estimate of the probability of recrossing caused by interaction with the heat bath, $J_L/J_R \approx \gamma^2/8\omega^2 \ll 1$. Hence, we shall neglect the recrossings over the barrier. In terms of the functions $f_{R,L}(\varepsilon, x)$ this means that $f_L(\varepsilon, 0) = 0$. Close to the barrier top the flux of the left-going particles arises only due to reflections from the barrier of the right-going particles with $\varepsilon < 0$. This gives the following relationship

between f_R and f_L :

$$f_L[\varepsilon, x(\varepsilon)] = f_R[\varepsilon, x(\varepsilon)], \quad \text{for } \varepsilon < 0; \quad f_L(\varepsilon, 0) = 0, \quad \text{for } \varepsilon > 0, \quad (2.32)$$

where $x(\varepsilon)$ is the root of the equation

$$U(x) = \varepsilon, \quad x_m < x < 0, \quad (2.33)$$

corresponding to the right-side turning point at a given energy ε . It should be emphasized that eq. (2.32) plays the role of a boundary condition, because (1) it interrelates the functions f_R and f_L at negative energies; (2) it specifies the problem by the condition that there should be no left-going particles directly at the barrier top. Particles with different ε are reflected at different values of $x(\varepsilon)$. For $|\varepsilon| \sim T$, however, the range of $x(\varepsilon)$ is very narrow compared to the size of the potential well,

$$|x(-T)| \approx (2T/m\omega^2)^{1/2} \ll |x_1|.$$

We assume, therefore, that all these particles propagate along trajectories very close to the basic trajectory ($\varepsilon \approx 0$) and can be described by the same Green function (2.31).

At a first glance one could doubt whether the motion of all the particles can be described by the function (2.31) as particles with different energies oscillate with different periods. Moreover, the period of oscillation diverges as $\ln(U_0/|\varepsilon|)$ as $\varepsilon \rightarrow 0$. The solution is that we consider a probabilistic problem and are interested, therefore, in the evolution of the distribution function rather than in the dynamics of individual particles. Oscillations with certain periods do not enter the problem, since the probabilistic evolution, governed by eq. (2.31), depends only on the action s along the basic trajectory. In more technical terms, if we introduce the action $S(\varepsilon)$ per oscillation,

$$S(\varepsilon) = \oint \{2m[\varepsilon - U(x)]\}^{1/2} dx, \quad \varepsilon < 0,$$

we obtain for $S(\varepsilon)$ at small energies approximately

$$S(0) - S(\varepsilon) \approx 2\pi|\varepsilon|/\Omega(\varepsilon) \sim (|\varepsilon|/\Omega) \ln(U_0/|\varepsilon|), \quad |\varepsilon| \ll U_0,$$

where $\Omega(\varepsilon)$ is the energy-dependent oscillation frequency. We can safely neglect the difference between $S(0)$ and $S(\varepsilon)$, as it gives small corrections of the order $T/U_0 \ll 1$. In this way we arrive at the basic parameter of the problem,

$$S \equiv S(0) = \oint [-2mU(x)]^{1/2} dx = 2 \int_{x_1}^0 [-2mU(x)]^{1/2} dx,$$

which has already appeared in eq. (2.20) in the discussion of Kramers' results.

The purpose of the considerations was to explain why eq. (2.30) with the action $s = S$, corresponding to $\varepsilon = 0$, could be exploited to describe the evolution of $f(\varepsilon, s)$ with $|\varepsilon| \sim T$. Now we will proceed further with the derivation of an integral equation. The relationship (2.32) will be of crucial importance

in this derivation, as it connects the distribution of the left-going particles f_L at the right-side turning point of the barrier with the distribution of the right-going particles. We begin with the introduction of a new function,

$$f(\varepsilon) = f_R(\varepsilon, 0), \quad \text{for } \varepsilon > 0; \quad f(\varepsilon) = f_R[\varepsilon, x(\varepsilon)], \quad \text{for } \varepsilon < 0, \quad (2.34)$$

where $x(\varepsilon)$ is defined in eq. (2.33). The function $f(\varepsilon)$ describes the rate of escape for $\varepsilon > 0$ and the rate of reflection at the barrier for $\varepsilon < 0$. The reflected particles build up a distribution of left-going particles f_L [see eq. (2.32)]. These particles propagate to the left-hand turning point, where they are reflected again. At this moment the function f_L changes over into the function f_R [see eq. (2.27)]. Propagating across the well these particles should reach the barrier and reproduce the initial distribution $f(\varepsilon)$. That is precisely the condition with which eq. (2.30) has to be complemented, to be transformed into a closed integral equation for the function $f(\varepsilon)$. Evidently, evolution of the particle distribution in vicinity of the closed basic trajectory is governed by Green's function

$$g(\varepsilon - \varepsilon') \equiv g(\varepsilon - \varepsilon', S) = (4\pi\delta T)^{-1/2} \exp[-(\varepsilon - \varepsilon' + \delta)^2/4T\delta], \quad (2.35)$$

where $\delta \equiv \gamma S$ is the energy loss per oscillation. Now we are in the position to write down our fundamental integral equation,

$$f(\varepsilon) = \int_{-\infty}^0 g(\varepsilon - \varepsilon') f(\varepsilon') d\varepsilon', \quad (2.36)$$

where the lower limit of integration is extended to infinity in view of the rapid convergence of the integral. Equations of this type were first introduced by Iche and Nozières [13] and by Leuthäusser [52], but without specifying the expression for $g(\varepsilon - \varepsilon')$.

The boundary condition for $f(\varepsilon)$ deep in the potential well is

$$f(\varepsilon) \approx (\Omega/2\pi T) \exp[-(\varepsilon + U_0)/T], \quad -\varepsilon \gg T. \quad (2.37)$$

With eq. (2.37), the distribution function is normalized to one particle in the well [see eq. (2.6)]. Then, by virtue of eqs. (2.8), (2.9) the decay rate is given by

$$\frac{1}{\tau} = J = \int_0^{\infty} f(\varepsilon) d\varepsilon. \quad (2.38)$$

Here we have used the identity $m^{-1}p dp \equiv d\varepsilon$ and we have taken into account that in the underdamped regime only positive momenta contribute to the integral in eq. (2.8). Equations (2.35)–(2.38) are completely equivalent to the original eqs. (2.2), (2.6) and (2.8)–(2.10) in the region of $\gamma \sim \omega T/U_0$, when $\delta \sim T$. In the extremely underdamped regime, $\delta \ll T$, eq. (2.36) simplifies down to the differential equation

$$\delta(d/d\varepsilon)(f + T df/d\varepsilon) = 0, \quad \delta \ll T, \quad (2.39)$$

subject to the boundary conditions (2.37) and

$$f(0) = 0. \quad (2.40)$$

Integration of eq. (2.36) over positive ε yields in the same approximations

$$1/\tau = -\delta T \left. df/d\varepsilon \right|_{\varepsilon=0}. \quad (2.41)$$

The solution of eq. (2.39) with the boundary condition (2.40) is given by

$$f(\varepsilon) = (\Omega/2\pi T) [\exp(-\varepsilon/T) - 1] \exp(-U_0/T). \quad (2.42)$$

Equations (2.39)–(2.42) were first derived by Kramers, his final result for τ^{-1} being eq. (2.20). Next we proceed by solving eq. (2.36) for arbitrary $\delta \sim T$.

2.4. The Wiener–Hopf method in the Kramers problem

To calculate the escape rate $1/\tau$ one needs to solve eq. (2.36) with the boundary condition (2.37) and then to calculate the integral in eq. (2.38). Equation (2.36) represents a one-sided convolution equation. To solve it by the Wiener–Hopf method [53] we introduce the one-sided Fourier transformations

$$\varphi^\pm(\lambda) = \frac{2\pi}{\Omega} \exp(U_0/T) \int_{-\infty}^{\infty} f(\varepsilon) \theta(\pm\varepsilon) \exp(i\lambda\varepsilon/T) d\varepsilon. \quad (2.43)$$

Comparison of eqs. (2.21), (2.38) and (2.43) yields

$$A = \varphi^+(0). \quad (2.44)$$

The boundary condition (2.37) shows that $\varphi^-(\lambda)$ has a pole at $\lambda = -i$,

$$\varphi^-(\lambda) \approx -i/(\lambda + i), \quad |\lambda + i| \ll 1. \quad (2.45)$$

After Fourier transformation of eq. (2.36) we arrive at a Wiener–Hopf equation

$$\varphi^+(\lambda) + \varphi^-(\lambda) = g(\lambda)\varphi^-(\lambda), \quad (2.46)$$

where

$$g(\lambda) = \exp[-\delta\lambda(\lambda + i)/T] \quad (2.47)$$

is the Fourier-transformed counterpart of eq. (2.35). It is convenient to rewrite eq. (2.46) as

$$\varphi^+(\lambda) + G(\lambda)\varphi^-(\lambda) = 0, \quad (2.48)$$

$$G(\lambda) \equiv 1 - \exp[-\delta\lambda(\lambda + 1)/T]. \quad (2.49)$$

The functions $\varphi^+(\lambda)$ and $\varphi^-(\lambda)$, defined by eq. (2.43), are analytic in the upper and lower complex half-planes of λ , the only exception being the pole (2.45) of $\varphi^-(\lambda)$.

To explain the Wiener–Hopf method in some detail, we rewrite eq. (2.48) as $\ln[-\varphi^+(\lambda)] = \ln \varphi^-(\lambda) + \ln G(\lambda)$, and with the use of the Cauchy theorem decompose $\ln G(\lambda)$ into two terms, $\ln G^+(\lambda)$ and $\ln G^-(\lambda)$, analytic in the corresponding half-planes of λ , arriving at the equation

$$\ln[-\varphi^-(\lambda)] - \ln G^+(\lambda) = \ln \varphi^-(\lambda) + \ln G^-(\lambda), \quad (2.50)$$

$$\ln G^\pm(\lambda) \equiv \pm \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln G(\lambda')}{\lambda' - \lambda \mp i0} d\lambda'. \quad (2.51)$$

The functions $G^\pm(\lambda)$ are entire functions which have no zeros in the half-planes $\text{Im } \lambda > 0$ and $\text{Im } \lambda < 0$ and tend to unity when $\lambda \rightarrow \infty$. Naturally, we have

$$G^+(\lambda)G^-(\lambda) \equiv G(\lambda). \quad (2.52)$$

As the functions on the left- and right-hand sides of eq. (2.50) are analytic in different half-planes of the complex λ , they should be equal to an entire function, which is to be chosen to satisfy eq. (2.45). In this way we arrive at the following solution of eq. (2.48):

$$\varphi^+(\lambda) = \frac{iG^+(\lambda)G^-(i)}{\lambda + i}, \quad \varphi^-(\lambda) = -\frac{iG^-(\lambda)}{G^-(\lambda)(\lambda + i)}. \quad (2.53)$$

Inserting of eq. (2.53) into eq. (2.44) yields

$$A = |G^-(0)|^2. \quad (2.54)$$

Here we have used the fact that $G^+(i)$ is complex-conjugate to $G^-(0)$, which can be verified by displacement of the integration contour in eq. (2.51) to the straight line $\text{Im } \lambda' = -i/2$. Equation (2.54) together with eqs. (2.51) and (2.49) gives an exact solution of the Kramers problem in the underdamped regime.

2.5. Lifetime of the Brownian particle in a single well

Insertion of eq. (2.51) for $G^-(\lambda)$ into eq. (2.54) yields the final result for the preexponential factor A ,

$$A(\delta/T) = \exp\left(\frac{1}{\pi} \int_0^{\infty} \ln\{1 - \exp[-\delta(\lambda^2 + \frac{1}{4})/T]\} \frac{d\lambda}{\lambda^2 + \frac{1}{4}}\right). \quad (2.55)$$

This expression can be represented in several equivalent forms,

$$A(\Delta) = \exp\left(\frac{2}{\pi} \int_0^{\pi/2} \ln[1 - \exp(-\Delta/4 \cos^2 x)] dx\right) \quad (2.56)$$

$$= \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{erfc}[\tfrac{1}{2}(n\Delta)^{1/2}]\right) \quad (2.57)$$

$$= \Delta \exp\left[\left(\frac{\Delta}{\pi}\right)^{1/2} \sum_{n=0}^{\infty} \frac{\zeta(\frac{1}{2}-n)}{n!(2n+1)} \left(\frac{-\Delta}{4}\right)^n\right], \quad (2.58)$$

where $\Delta \equiv \delta/T$, and

$$\operatorname{erfc}(x) \equiv \frac{2}{\pi^{1/2}} \int_x^{\infty} e^{-y^2} dy$$

is the error integral, $\zeta(z)$ is the Riemann zeta function. The series in Δ in eq. (2.58) converges inside the circle $|\Delta| = 8\pi$. The asymptotics of $A(\Delta)$ may be obtained from eq. (2.57) for $\Delta \gg 1$ and from eq. (2.58) for $\Delta \ll 1$,

$$A(\Delta) \approx 1 - 2(\pi\Delta)^{-1/2} \exp(-\Delta/4), \quad \Delta \gg 1, \quad (2.59)$$

$$A(\Delta) \approx \Delta[1 + \zeta(\frac{1}{2})(\Delta/\pi)^{1/2}] \approx \Delta - 0.82\Delta^{3/2}, \quad \Delta \ll 1. \quad (2.60)$$

In ref. [54] instead of $\zeta(\frac{1}{2}) \approx -1.46035$, the value 1.46 was calculated by numerical methods. The dependence of A on Δ is shown in fig. 2.

As one could have expected, $A(\Delta)$ is nonanalytic at $\Delta = 0$ reflecting the fundamental change of the properties of the system with the change of the sign of γ .

The final expression for the lifetime of a Brownian particle in a single potential well can be written down explicitly as a product of eq. (2.55) and eq. (2.15),

$$\frac{1}{\tau} = \frac{\Omega}{2\pi} \left[\left(1 + \frac{\gamma^2}{4\omega^2}\right)^{1/2} - \frac{\gamma}{2\omega} \right] A\left(\frac{\gamma S}{T}\right) \exp(-U_0/T). \quad (2.61)$$

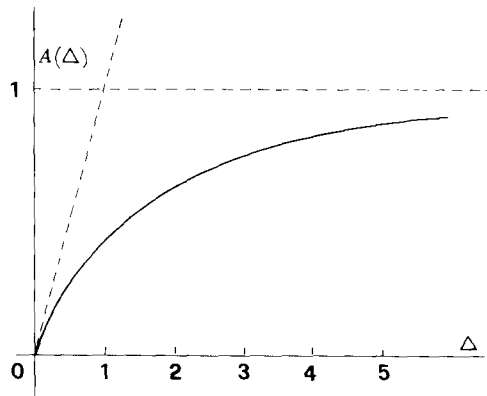


Fig. 2. Dependence of the preexponential factor A in eq. (2.21) on the reduced dissipation $\Delta \equiv \gamma S/T \equiv \delta/T$.

In the intermediate-to-overdamped regime, $\gamma S \gg T$, eq. (2.61) differs from eq. (2.15) only by terms $\sim \exp(-\gamma S/4T)$. In the intermediate-to-underdamped regime, $\gamma \ll \omega$, the factor (2.24) yields in eq. (2.61) a relative error of order γ/ω , which, at $\gamma S \sim T$, is of the order T/U_0 . Hence, the product of eq. (2.55) and eq. (2.15) correctly yields the leading-order term in a low temperature expansion of the particle lifetime at arbitrary damping. Equation (2.61) depends on five quantities U_0 , Ω , ω , S and γ , which completely specify the metastable state of a Brownian particle in a single potential well.

2.6. Energy distribution of escaping particles

The distribution of the escaping particles is given by the inverse Fourier transformation,

$$f(\varepsilon) = \frac{\Omega}{4\pi^2 T} \exp(-U_0/T) \int_{-\infty}^{\infty} \varphi^+(\lambda) \exp(-i\lambda\varepsilon/T) d\lambda, \quad (2.62)$$

where $\varphi^+(\lambda)$ is given by eqs. (2.51) and (2.53). Unfortunately, only the numerical evaluation of the integral in eq. (2.62) is possible, since $\varphi^+(\lambda)$ cannot be represented by a simple analytic form. This can already be seen from the complicated structure of eqs. (2.56)–(2.58) for the preexponential factor $A(\Delta)$, which equals $\varphi^+(\lambda)$ at $\lambda=0$. A comparatively simple expression can be obtained only for the average energy of escaping particles

$$\bar{\varepsilon} \equiv \int_0^{\infty} f(\varepsilon) \varepsilon d\varepsilon / \int_0^{\infty} f(\varepsilon) d\varepsilon.$$

Using eq. (2.43) we find [37]

$$\bar{\varepsilon} = T \left. \frac{d \ln \varphi^+(\lambda)}{d\lambda} \right|_{\lambda=0} = 1 + \frac{2}{\pi} \int_0^{\pi/2} (1 - 2 \cos^2 x) \ln[1 - \exp(-\Delta/4 \cos^2 x)] dx, \quad (2.63)$$

where, as before, $\Delta \equiv \delta/T$. The dependence of $\bar{\varepsilon}/T$ on the ratio δ/T is shown in fig. 3.

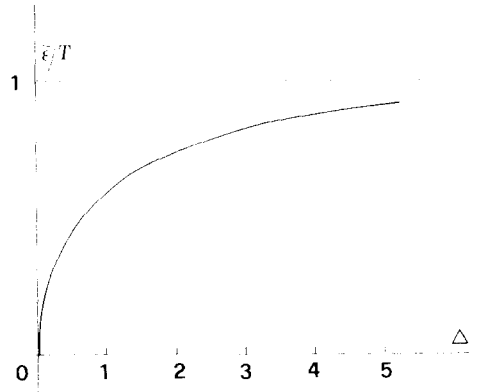


Fig. 3. Dependence of the averaged energy of escaping particles on the reduced dissipation $\Delta \equiv \gamma S/T \equiv \delta/T$.

In the limiting cases one obtains

$$\bar{\varepsilon} \approx 1 - 2(\pi\Delta)^{-1/2} \exp(-\Delta/4), \quad \Delta \gg 1, \quad (2.64)$$

$$\bar{\varepsilon} \approx -\zeta(\frac{1}{2})(\Delta/\pi)^{1/2} \approx 0.82\Delta^{1/2}, \quad \Delta \ll 1. \quad (2.65)$$

For $\delta \gg T$ the distribution $f(\varepsilon)$ is a canonical one and the mean energy $\bar{\varepsilon}$ is given by $\bar{\varepsilon} = T$. For $\delta \ll T$ the average energy of the escaping particles is small and so is $f(\varepsilon)$ at $\varepsilon > 0$,

$$\bar{\varepsilon} \sim (\delta T)^{1/2}, \quad f(0) \sim (\Omega\delta/T^2) \exp(-U_0/T).$$

The last expression gives an estimate of $f(0)$, which has earlier been assumed to be vanishingly small [see eq. (2.40)].

An attempt has been made to solve Kramers' problem at $\delta \sim T$ by inclusion in eq. (2.39) an additional loss term due to escape out of the well [29]. In the extremely underdamped regime this approach yields

$$A_{\text{BHL}}(\Delta) \approx \Delta - \Delta^{3/2}, \quad \bar{\varepsilon}_{\text{BHL}} \approx \Delta^{1/2}, \quad \Delta \ll 1.$$

The parameter α , introduced in the papers by Büttiker, Harris and Landauer [29, 30] has been chosen to be unity, in order to achieve $A = 1$ at $\delta \gg T$. We note, that these results differ from eqs. (2.60) and (2.65) only by a numerical factor $|\zeta(\frac{1}{2})|\pi^{-1/2} \approx 0.82$, which is rather close to unity. However, the distinction between the two approaches is clearly emphasized by their results for $\Delta \gg 1$. The results given by eqs. (2.59) and (2.64) exhibit a sharp dropoff of corrections $\sim \exp(-\Delta/4)$, whereas those of BHL yield relative corrections of the order of Δ^{-1} .

2.7. Double-well potential

Now we return to a more general situation when beyond the barrier there is another well of a finite depth, as shown in fig. 4. In this case there is a finite probability for the particle to return into the initial well 1 after visiting the final well 2. This probability is non-negligible only in the underdamped regime, when the particles having entered well 2 lose energy so slowly, that after several roundtrips fluctuations may still throw them back over the barrier into well 1.

To take account of this process we introduce, in analogy with eq. (2.34), the distributions $f_1(\varepsilon)$ and $f_2(\varepsilon)$ of the particles, moving towards the barrier from the respective wells. In analogy with eq. (2.35) we also introduce the Green functions of the Fokker-Planck equation in the double-well system,

$$g_j(\varepsilon - \varepsilon') = (4\pi\delta_j T)^{-1/2} \exp[-(\varepsilon - \varepsilon' + \delta_j)^2/4\delta_j T], \quad \delta_j \equiv \gamma S_j/T,$$

where S_j is the action per oscillation of the particle with $\varepsilon = 0$ in well j ($j = 1, 2$),

$$S_1 \equiv 2 \int_{x_1}^0 [-2mU(x)]^{1/2} dx, \quad S_2 \equiv 2 \int_0^{x_2} [-2mU(x)]^{1/2} dx.$$

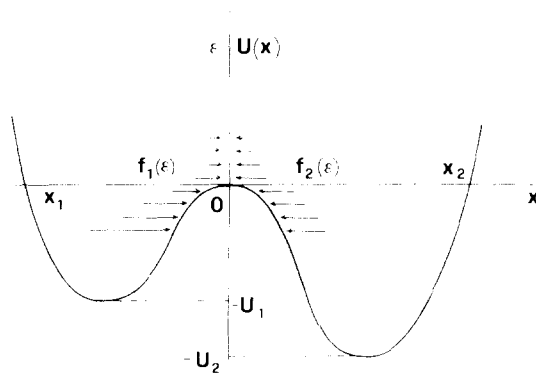


Fig. 4. Schematic representation of a double-well potential.

To write down a system of integral equations for the functions $f_1(\varepsilon)$, $f_2(\varepsilon)$ in analogy with eq. (2.36), one should take into account that there are now two distinct contributions to, say, $f_1(\varepsilon)$. One of them stems from the particles reflected at the barrier one period earlier with the distribution $f_1(\varepsilon)\theta(-\varepsilon)$; the other is due to the particles which have passed over the barrier one period earlier with distribution $f_2(\varepsilon)\theta(\varepsilon)$. The full system of equations is then

$$f_1(\varepsilon) = \int_{-x}^x g_1(\varepsilon - \varepsilon') [f_1(\varepsilon')\theta(-\varepsilon') + f_2(\varepsilon')\theta(\varepsilon')] d\varepsilon',$$

$$f_2(\varepsilon) = \int_{-x}^x g_2(\varepsilon - \varepsilon') [f_2(\varepsilon')\theta(-\varepsilon') + f_1(\varepsilon')\theta(\varepsilon')] d\varepsilon'.$$

The boundary condition (2.37) for $f_1(\varepsilon)$ still holds after substitution of Ω by Ω_1 and U_0 by U_1 , whereas $f_2(\varepsilon)$ has no Boltzmann tail deep in well 2, since initially there were no particles in this well. Following definition (2.43), we introduce the Fourier transforms $\varphi_1^\pm(\lambda)$ and $\varphi_2^\pm(\lambda)$ of $f_1(\varepsilon)$ and $f_2(\varepsilon)$ respectively. These new functions obey the system of equations

$$\varphi_1^+(\lambda) + \varphi_1^-(\lambda) = [1 - G_1(\lambda)][\varphi_1^-(\lambda) + \varphi_2^+(\lambda)], \quad (2.66)$$

$$\varphi_2^-(\lambda) + \varphi_2^+(\lambda) = [1 - G_2(\lambda)][\varphi_2^-(\lambda) + \varphi_1^+(\lambda)],$$

$$G_j(\lambda) \equiv 1 - \exp[-\Delta_j \lambda(\lambda + i)], \quad \Delta_j \equiv \delta_j/T. \quad (2.67)$$

As above, $\varphi_1^+(\lambda)$ is analytic in the upper half-plane, whereas $\varphi_1^-(\lambda)$ is analytic in the lower half-plane but for a pole at $\lambda = -i$ with the residue $-i$. The functions $\varphi_2^+(\lambda)$ and $\varphi_2^-(\lambda)$ are analytic in the upper and in the lower half-plane of λ .

The system (2.66) is reduced into the two independent Wiener-Hopf equations,

$$\varphi_1^-(\lambda) + [G_1(\lambda)G_2(\lambda)/G_{12}(\lambda)]\varphi_1^+(\lambda) = 0, \quad \Psi^+(\lambda) + \Psi^-(\lambda) = 0,$$

where we have introduced the new functions

$$\begin{aligned}\varphi(\lambda) &\equiv \varphi_1(\lambda) - \varphi_2(\lambda), & \Psi(\lambda) &\equiv G_1(\lambda)\varphi_1(\lambda) + G_2(\lambda)\varphi_2(\lambda), \\ G_{12}(\lambda) &\equiv 1 - \exp[-(\Delta_1 + \Delta_2)\lambda(\lambda + i)].\end{aligned}$$

We are only interested in the equation for $\varphi(\lambda)$, since the flux over the barrier is given by

$$\frac{1}{\tau} = \int_0^{\infty} [f_1(\varepsilon) - f_2(\varepsilon)] d\varepsilon. \quad (2.68)$$

We thus arrive at

$$A = \varphi_1^+(0) - \varphi_2^+(0) = \varphi^+(0). \quad (2.69)$$

After factorization of $G_1(\lambda)$, $G_2(\lambda)$ and $G_{12}(\lambda)$ similarly as in eqs. (2.51), (2.52) one obtains

$$\varphi^+(\lambda) = i \frac{G_1^-(-i)G_2^-(-i)G_1^+(\lambda)G_2^+(\lambda)}{(\lambda + i)G_{12}^-(-i)G_{12}^+(\lambda)}, \quad (2.70)$$

and a similar expression for $\varphi^-(\lambda)$. From eqs. (2.69) and (2.70) it follows that the preexponential factor $A(\Delta_1, \Delta_2)$ for a double-well potential in the underdamped regime may be expressed by the function $A(\Delta)$ introduced earlier by eqs. (2.55)–(2.58),

$$A(\Delta_1, \Delta_2) = A(\Delta_1)A(\Delta_2)/A(\Delta_1 + \Delta_2). \quad (2.71)$$

The result takes on such a simple form by virtue of the kernels $G_1(\lambda)$, $G_2(\lambda)$ and $G_{12}(\lambda)$ given in terms of the same function (2.49) with different parameters $\Delta = \delta/T$. In the extremely underdamped limit, $\Delta_1, \Delta_2 \ll 1$, this expression gives [13]

$$A(\Delta_1, \Delta_2) \approx \Delta_1\Delta_2/(\Delta_1 + \Delta_2).$$

For a symmetric double-well potential $\Delta_1 = \Delta_2 = \Delta$, the expansion of eq. (2.71) for $\Delta \ll 1$ takes the form

$$A(\Delta, \Delta) \approx (\Delta/2)[1 + (2 - \sqrt{2})\zeta(\frac{1}{2})(\Delta/\pi)^{1/2}].$$

Numerically $(2 - \sqrt{2})\zeta(\frac{1}{2}) \approx -0.855456$. The value -0.859 for this coefficient was given earlier by Risken and Vollmer [17]. The more accurate value -0.8554 was calculated in refs. [54, 31].

2.8. Population relaxation in a double-well potential

The final relationship for the escape rate in a double-well potential out of the metastable state 1 may be written as the product of eqs. (2.71) and (2.15), yielding

$$\frac{1}{\tau_1} = \frac{\Omega_1}{2\pi} \left[\left(1 + \frac{\gamma^2}{4\omega^2}\right)^{1/2} - \frac{\gamma}{2\omega} \right] \frac{A(\gamma S_1/T)A(\gamma S_2/T)}{A[\gamma(S_1 + S_2)/T]} \exp(-U_1/T). \quad (2.72)$$

This result holds at arbitrary damping γ . The final state of the particle is specified by the parameter S_2 only, which influences the decay rate out of state 1 only in the underdamped regime. Equation (2.72) takes on the form of eq. (2.61) for a single potential well at $S_2 \gg S_1$. For a symmetric double-well potential one finds $S_1 = S_2$. Because the difference of the depths of the initial and final potential wells does not enter the calculations, the friction-dependent part of the preexponential factor is symmetric in the well indices. Equation (2.72) with Ω_1 and U_1 substituted by Ω_2 and U_2 gives the rate of fluctuation-induced transitions τ_2^{-1} from well 2 into well 1. An oscillatory approximation at the bottom of the well is by no means restrictive. In the case of an arbitrary well one only needs to normalize correctly the equilibrium function (2.6). Thus, our results describe completely the activated decay rates out of a one-dimensional metastable state.

The lifetimes τ_1 and τ_2 derived by solving the Kramers problem should be used as elementary rate constants in the phenomenological equations for the well populations,

$$dN_1/dt = -N_1/\tau_1 + N_2/\tau_2, \quad dN_2/dt = -N_2/\tau_2 + N_1/\tau_1.$$

These equations conserve the total population $N_1 + N_2$. Their solution is given by

$$N_1(t) = \frac{N_1(0)\{1 + (\Omega_1/\Omega_2) \exp[(U_2 - U_1)/T - t/\tau]\} + N_2(0)[1 - \exp(-t/\tau)]}{1 + (\Omega_1/\Omega_2) \exp[(U_2 - U_1)/T]},$$

where

$$\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2} = \left[\left(1 + \frac{\gamma^2}{4\omega^2} \right)^{1/2} - \frac{\gamma}{2\omega} \right] \frac{A(\gamma S_1/T)A(\gamma S_2/T)}{A[\gamma(S_1 + S_2)/T]} \times [(\Omega_1/2\pi) \exp(-U_1/T) + (\Omega_2/2\pi) \exp(-U_2/T)], \quad (2.73)$$

is the relaxation rate of a nonequilibrium population of the two wells.

We have derived explicit solutions of the Kramers problem for a single-well and a double-well potential. In the next sections we generalize these results to quantum Brownian motion. Moreover, in section 4 we will derive a rich spectrum of results both for classical and quantum Brownian motion in a washboard potential. These achievements demonstrate the high efficiency as well as the wide range of applicability of the technique developed here. With this experience one would expect that any problem of that kind can be given an explicit solution. However, considering a simple modification of the single-well problem arrives at the conclusion that this is not true. Incidentally, we can easily write down a system of two integral equations for a single-well potential with barriers of finite height on both sides of the well. In the standard way these integral equations can be transformed into a system of equations for four functions $\varphi_{1,2}^{\pm}(\lambda)$. The trouble becomes evident at the next stage when one tries to find a combination of these four functions corresponding to the total flux across two barriers. In contrast to the double-well problem, where the total flux is obtained as the difference of the fluxes from the two wells, in the double-barrier problem the total flux is given by adding two fluxes. This physical circumstance radically changes the analytical structure of the equations and renders the double-barrier problem unsolvable.

2.9. *Experimental verification of Kramers' energy-diffusion model*

In the preceding sections we have presented the theoretical results for the decay rate of metastable states. In most experimental works the nature of the observed activated decay, e.g., chemical

dissociation, is verified by observation of the Arrhenius activation law, in other words, the exponential dependence of the decay rate,

$$1/\tau = (\Omega/2\pi)A \exp(-U_0/T),$$

on temperature T . However, in this case one gets no information on the coupling of the particle to the heat bath, since only the prefactor A depends on this coupling. Because of the predominance of the exponential factor, measurements of the prefactor A are notoriously difficult. To extract unequivocally the friction dependence from the measured decay rate, one must change the damping independently of other relevant parameters.

At the present time, unique experiments appropriate for this purpose are observations of the decay of the zero-voltage state of a Josephson junction [46–48]. In the case of a Josephson junction, the degree of freedom x describes the phase difference across the junction and the mass m corresponds to $C(\hbar/2e)^2$, where C is the capacitance of the junction. Further, the potential is a tilted periodic potential,

$$U(x) = (\hbar I_c/2e)[\cos(x) - (I/I_c)x],$$

where I is the bias current and I_c the critical current of the junction.

The zero-voltage state of the Josephson junction corresponds to the case where $I < I_c$, and the particle is trapped in one of the minima of the potential. The decay of this metastable state can be observed with sizable probability only when the bias current is close to the critical current, $I_c - I \ll I_c$. Then, shifting the origin of x to a local minimum, the relevant part of the potential is very well approximated by a cubic potential

$$U(x) = \frac{1}{2}m\Omega^2x^2(1 - x/x_0),$$

which has a barrier of width x_0 and height $U_0 = \frac{2}{27}m\Omega^2x_0^2$. The potential parameters are related to the parameters of the current-biased Josephson junction via

$$\Omega = (2eI_c/\hbar C)^{1/2}(1 - I/I_c)^{1/4}, \quad U_0 = (2^{3/2}\hbar I_c/3e)(1 - I/I_c)^{3/2}.$$

In the model of a resistively shunted junction, the friction coefficient is given by $1/RC$, where R is the resistance. So long as $RC\Omega < U_0/T$, the decay lies in the region of moderate-to-large damping. In the opposite limit, $RC\Omega \gg U_0/T$, the preexponential factor A is small and depends on the shunting resistance, $A \approx (1/RC\Omega)U_0/T$. Under these conditions, the particle motion in the potential $U(x)$ represents oscillations with a frequency dependent on the particle energy accompanied by a slow energy diffusion. This limit was considered by Kramers and is discussed in section 2.1.

The shunting resistance can be substituted by a more sophisticated shunting circuit. In this way a possibility arises to change in a controllable manner the particle coupling to the heat bath. If a Josephson junction is shunted by a delay line, the friction becomes time-delayed, which is described by a non-Markovian damping kernel [49]. The delay time of the friction depends on the length of the delay line. This allows for the determination of the decay rate of the zero-voltage state of a current-biased Josephson junction as a function of the delay time without affecting other junction parameters. The theory on his effect has been developed by Grabert and Linkwitz [44].

Their starting point is the general Langevin equation

$$m\ddot{x} + \frac{dU(x)}{dx} + \int_{-\infty}^t ds \gamma(t-s)\dot{x}(s) = \eta(t),$$

where $\gamma(t)$ is a non-Markovian damping kernel and $\eta(t)$ a Gaussian noise force with vanishing mean. The noise correlation function $\langle \eta(t)\eta(s) \rangle = mT\gamma(|t-s|)$ is related to the damping kernel by a fluctuation–dissipation theorem.

Consider a Josephson junction shunted by an ideal delay line with a capacitance per length C^* and an inductance per length L^* . The line has a length l and is terminated by a resistance R_t . Then, the applied bias current splits into three time-dependent pieces: a supercurrent $I_c \cos(x)$, a displacement current $C dV/dt$ through the junction capacitance C and a time-dependent current I_r through the shunting delay line. This last current is related to the voltage V across the junction by [49]

$$I_r(t) = \frac{1}{R_c} \left[V(t) + 2 \sum_{n=1}^{\infty} \left(\frac{R_c - R_t}{R_c + R_t} \right)^n V\left(t - \frac{2l}{v} n\right) \right],$$

where $R_c = (C^*/L^*)^{1/2}$ is the characteristic resistance of the delay line and $v = (L^*C^*)^{-1/2}$ the wave-propagation velocity. Translating this into the equivalent model of a damped particle moving in the field of force, one obtains the damping kernel [44]

$$\gamma(t) = 2\gamma \sum_{n=0}^{\infty} a^n \delta(t - n\tau), \quad \gamma = \frac{1}{RC}, \quad \tau = \frac{2l}{v}, \quad a = \frac{R_c - R_t}{R_c + R_t},$$

where a is the reflection coefficient of the line.

In the weak-damping limit the energy-distribution function $f(\varepsilon)$ obeys a Fokker–Planck equation

$$(d/d\varepsilon)\delta(\varepsilon)(1 + T d/d\varepsilon)f(\varepsilon) = 0. \quad (2.74)$$

In contrast to eq. (2.39), $\delta(\varepsilon)$ is now considered energy-dependent,

$$\delta(\varepsilon) = m \oint \oint \gamma(|t-s|) dx(\varepsilon, t) dx(\varepsilon, s),$$

where the integration has to be carried out over the full cycle of motion. The trajectory $x(\varepsilon, t)$ for the motion of a particle in a cubic potential in the absence of friction can be found in terms of elliptic functions. The final answer for $\delta(\varepsilon)$ is [44]

$$\delta(\varepsilon) = (U_0/Q_c)D(\varepsilon/U_0),$$

where Q_c is the quality factor of the delay line,

$$D(r) = 216\pi\mu^5(r)(1-a^2) \sum_{n=1}^{\infty} \frac{n^4}{\sinh^2[\pi n\mu(r)/\mu(1-r)]} \frac{1}{1 - 2a \cos[\pi n\mu(r)d/x_0] + a^2}.$$

The last expression depends on the reduced length of the delay line, d/x_0 , the reflection factor of the line a and the function $\mu(r)$, which describes the dependence of the oscillation frequency on the reduced energy $r = \varepsilon/U_0$, $\Omega(\varepsilon) = \Omega\mu(\varepsilon/U_0)$. Neglecting exponentially small terms, the solution of eq. (2.74) gives the following expression for the preexponential factor:

$$\frac{1}{A} = Q_c \int_0^1 \frac{\exp(-rU_0/T) dr}{D(1-r)}.$$

Numerical calculations for $U_0/T = 10$ and $a = \pm 1/2, 1/3$ have shown that the coefficient A displays damped oscillations of large amplitude as a function of the delay-line length [44]. The modulation of the lifetime of the zero-voltage state by changing the length of a delay line was measured for $A \approx 1/4$ and $U_0/T \approx 10 \pm 3$ [45]. In the limits of the experimental uncertainty of the system parameters, the experimental results agree quite satisfactorily with the theoretical predictions. The amplitude, period, phase and tailing off of the modulation constitute the first direct evidence for the oscillation of a particle in a well before escape, which is implied by Kramers' energy-diffusion model for the escape from a metastable state.

3. Distribution and escape rate of quantum Brownian particles

3.1. Escape rate of thermalized particles

To consider the decay of metastable states in the quantum regime, one has to take into account three different contributions to the final result. First of all, the asymptotics (2.37) for the function $f(\varepsilon)$ deep in the potential well must be modified. To do that one just has to recall that in an oscillatory potential the energy levels equal $(n + 1/2)\hbar\Omega$, where n is an integer. The normalized distribution function is then given by

$$f(\varepsilon) \approx \sinh(\hbar\Omega/2T) (\pi\hbar)^{-1} \exp[-(\varepsilon + U_0)/T], \quad -\varepsilon \gg T. \quad (3.1)$$

This expression will serve as a boundary condition for $f(\varepsilon)$ calculated taking escapes into account.

Similarly to eq. (2.21), we introduce the preexponential factor A as a factor, reflecting effects of dissipation, starting from the expression for the escape rate of thermalized particles, while neglecting friction. Quantum effects become manifest in this process through the quantum penetration probability $[1 + \exp(-2\pi\varepsilon/\hbar\omega)]^{-1}$ for a particle with energy ε . Close to the barrier top the discreteness of the energy levels may be overlooked, since the frequency of oscillations vanishes as $|\varepsilon| \rightarrow 0$. In this way, using eq. (3.1), we obtain

$$\frac{1}{\tau} = \int_{-\infty}^{\infty} \frac{f(\varepsilon) d\varepsilon}{1 + \exp(-2\pi\varepsilon/\hbar\omega)} = \frac{\omega \sinh(\hbar\Omega/2T)}{2\pi \sin(\hbar\omega/2T)} \exp(-U_0/T). \quad (3.2)$$

The lowest-order quantum correction to this preexponential factor was first obtained by Wigner [3].

The expression (3.2) for $1/\tau$ does not contain any dependence on the friction coefficient γ . Therefore, it corresponds to the intermediate friction regime, $T\omega/U_0 \ll \gamma \ll \omega$ (we assume that $\Omega \sim \omega$).

In the underdamped regime, $\gamma \sim \omega T/U_0$, as well as in the overdamped regime, $\gamma \sim \omega$, relation (3.2) must be modified by taking into account the interaction of the Brownian particle with the heat bath. This is the third factor modifying the expression for $1/\tau$. Dissipation will be accounted for if we write $1/\tau$ in the following form:

$$\frac{1}{\tau} = A \frac{\omega \sinh(\hbar\Omega/2T)}{2\pi \sin(\hbar\omega/2T)} \exp(-U_0/T) = \frac{\Omega}{2\pi} A_{\text{tot}} \exp(-U_0/T), \quad (3.3)$$

where the factor A accounts for dissipation effects on the flux of thermalized quantum particles, whereas the factor A_{tot} describes total modification of the result compared to classical TST. In the next sections the preexponential factor A is calculated both in the underdamped and overdamped regimes.

3.2. Derivation of the transition probability

In contrast to the calculation of escape rates in the classical regime when we started from the Langevin equation (2.1) or the Fokker–Planck equation (2.2), in the quantum regime one has to start by specifying the Hamiltonian of the problem. We are particularly interested in the decay rate for systems showing viscous friction in the classical regime. Though this condition is not sufficient to define the system particle + heat bath in a unique way, it is nonetheless sufficiently restrictive to uniquely determine the effective action of the particle, obtained by integration over variables describing the heat bath. This conclusion is very important, as it means that all models of the heat bath are equivalent as far as the results of the escape rate are concerned, provided these models reproduce the same Langevin equation in the classical limit.

In this paper we study two different models of the heat bath. In the underdamped regime, we take into account the interaction of a particle with the heat bath by incorporating a term describing effects of the Johnson–Nyquist noise into the Hamiltonian. In the overdamped regime, we shall use a more physical model assuming that the role of the thermal bath is taken over by a string coupled to the particle and tightened in a direction perpendicular to the direction of motion of the particle.

To consider the quantum problem in the simplest way, we follow as closely as possible the route described in the classical case. There, the first step was the derivation of the kernel $g(\varepsilon - \varepsilon')$. In contrast to the Gaussian function (2.35), which can be written down from almost intuitive considerations, in the quantum case we proceed in a systematic way proposed by Larkin and Ovchinnikov [40]. The starting point is the Hamiltonian

$$\hat{\mathcal{H}}(t) = \hat{p}^2/2m + U(x) + x\hat{\eta}(t),$$

where the last term describes an interaction with the heat bath which is linear in the particle coordinate x . The noise operator $\hat{\eta}(t)$ is written in the Heisenberg representation with respect to the heat-bath degrees of freedom. We assume that $\hat{\eta}(t)$ is Gaussian with the Johnson–Nyquist correlator,

$$\int_{-\infty}^{\infty} \langle \hat{\eta}(t)\hat{\eta}(t') \rangle_T \exp[i\omega(t-t')] dt' = m\gamma\hbar\omega[\coth(\hbar\omega/2T) - 1], \quad (3.4)$$

where the subscript T denotes averaging over heat-bath states. As was already shown in section 2, escaping particles have energies $\varepsilon \sim T$. By virtue of the inequality $T \ll U_0$, we can calculate the

quantum transition probabilities in a semiclassical manner, starting from the classical trajectory $x(t)$ for $\varepsilon = 0$. It is defined by the implicit relation

$$t(x) = \pm \int_{x_1}^x [-2U(x')/m]^{-1/2} dx', \quad (3.5)$$

where x_1 is the left-hand turning point and the signs $+$ and $-$ correspond to positive and negative velocities of the particle respectively. The particle starts from $x = 0$ at $t = -\infty$ and returns to this point for $t \rightarrow \infty$ (see fig. 1).

The amplitude of a quantum transition from the state ε' to the state ε in one cycle of the particle motion in the potential well under the influence of the noise $\hat{\eta}(t)$ is given by

$$A(\varepsilon, \varepsilon') = \langle \varepsilon | \hat{T} \exp\left(-i \int_{-\infty}^{\infty} \hat{\eta}(t) \hat{x}(t) dt / \hbar\right) | \varepsilon' \rangle,$$

where $\langle \varepsilon |$ and $| \varepsilon' \rangle$ are the unperturbed wave functions and \hat{T} denotes time ordering. The probability of transitions from ε' to ε is then given by $g(\varepsilon, \varepsilon') = \langle |A(\varepsilon, \varepsilon')|^2 \rangle_T$. In what follows we calculate $g(\varepsilon, \varepsilon')$ and verify that $g(\varepsilon, \varepsilon') = g(\varepsilon - \varepsilon')$. Applying a perturbation approximation to $A(\varepsilon, \varepsilon')$, we find the first-order contribution to g ,

$$g_1(\varepsilon - \varepsilon') = \delta(\varepsilon - \varepsilon') - \delta(\varepsilon - \varepsilon') \int_{-\infty}^{\infty} w(\varepsilon) d\varepsilon + w(\varepsilon - \varepsilon'), \quad (3.6)$$

where w is the quantum transition probability in perturbation theory,

$$w(\varepsilon - \varepsilon') = 2\pi |\langle \varepsilon | x | \varepsilon' \rangle|^2 m \gamma(\varepsilon - \varepsilon') \{\coth[(\varepsilon - \varepsilon')/2T] - 1\}. \quad (3.7)$$

In the semiclassical approximation the matrix element $\langle \varepsilon | x | \varepsilon' \rangle$ can easily be expressed via the Fourier component of the classical trajectory $x(t)$. To this end we write the normalized semiclassical wave function,

$$|\varepsilon, x\rangle = \left(\frac{1}{2\pi\hbar v(\varepsilon, x)} \right)^{1/2} \exp\left(i \int_0^x p(\varepsilon, x') dx' / \hbar \right),$$

where $v(\varepsilon, x)$ and $p(\varepsilon, x)$ are the velocity and momentum of a particle with the energy ε at a given value of the coordinate x . We assume that $|\varepsilon - \varepsilon'| \ll |U(x)|$ and expand the exponent in the product of the functions $| \varepsilon' \rangle$ and $\langle \varepsilon |$ in $\varepsilon - \varepsilon'$, taking $\varepsilon' = \varepsilon$ elsewhere, which results in the expression

$$\begin{aligned} \langle \varepsilon | x | \varepsilon' \rangle &= \frac{1}{2\pi\hbar} \oint \frac{dx}{v(\varepsilon, x)} x \exp\left(i \frac{(\varepsilon - \varepsilon')}{\hbar} \int_0^x \frac{dx'}{v(\varepsilon, x')} \right) \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} x(t) \exp[i(\varepsilon - \varepsilon')t/\hbar] dt. \end{aligned} \quad (3.8)$$

The integral over ε in eq. (3.6) gives the perturbative contribution of the first vacuum loop. The Gaussian nature of the Johnson–Nyquist noise (3.4) leads to the situation that summing up over all the vacuum loops results in a general factor $\exp(-\int_{-\infty}^{\infty} w(\varepsilon) d\varepsilon)$ in the expression for $g(\varepsilon - \varepsilon')$. The perturbation series is therefore of the form

$$g(\varepsilon - \varepsilon') = \exp\left(-\int_{-\infty}^{\infty} w(\varepsilon) d\varepsilon\right) \left(w(\varepsilon - \varepsilon') + \frac{1}{2} \int_{-\infty}^{\infty} w(\varepsilon - \varepsilon'') w(\varepsilon'' - \varepsilon') d\varepsilon'' + \frac{1}{6} \int_{-\infty}^{\infty} w(\varepsilon - \varepsilon'') w(\varepsilon'' - \varepsilon''') w(\varepsilon''' - \varepsilon') d\varepsilon'' d\varepsilon''' + \dots \right). \quad (3.9)$$

Fourier-transforming this expression,

$$h(\lambda) \equiv \int_{-\infty}^{\infty} h(\varepsilon) \exp(i\lambda\varepsilon/T) d\varepsilon, \quad (3.10)$$

and performing the summation on its right-hand side yield [40]

$$g(\lambda) = \exp[w(\lambda) - w(0)]. \quad (3.11)$$

Though both $w(0)$ and $w(\lambda)$ are divergent, their difference is finite, which justifies the above formal manipulations.

It is convenient to separate the factor δ/T from $w(\lambda)$ and to represent $g(\lambda)$ in the following form:

$$g(\lambda) = \exp[-(\delta/T)Z(\lambda + i/2, \hbar\omega/2\pi T)],$$

where ω is a typical frequency of the motion. At $\lambda = i/2$ the function $Z(\lambda, y)$ vanishes in accordance with the fact that the interaction with the heat bath tends to relax the particle distribution function to $f(\varepsilon) \propto \exp(-\varepsilon/T)$. At $\lambda = -i/2$ the function $Z(\lambda, y)$ vanishes due to particle conservation. The function $Z(\lambda, y)$ depends on the shape of $U(x)$ through the Fourier transformation (3.8) of the basic trajectories $x(t)$ [see eq. (3.5)] and is calculated in the next section for two typical potentials.

3.3. Integral equation and its solution

In order to write down an integral equation similar to eq. (2.36), one has to recall that in the quantum situation the penetration of a potential barrier becomes a probabilistic process specified by the penetration coefficient. The energies of escaping particles are distributed in a narrow range, $|\varepsilon| \sim T \ll U_0$, so that the potential can be approximated by a parabola. A particle with energy ε' is reflected from the parabolic potential barrier $U(x) \approx -m\omega^2 x^2/2$ with probability $[1 + \exp(-2\pi\varepsilon'/\hbar\omega)]^{-1}$, whereas its penetration coefficient equals $[1 + \exp(2\pi\varepsilon'/\hbar\omega)]^{-1}$ [55]. The reflected particles after a cycle of motion in the potential well will reproduce the distribution function $f(\varepsilon)$.

Using the transition probability (3.9), we can state the equation for $f(\varepsilon)$,

$$f(\varepsilon) = \int_{-\infty}^{\infty} \frac{g(\varepsilon - \varepsilon')}{1 + \exp(2\pi\varepsilon'/\hbar\omega)} f(\varepsilon') d\varepsilon'.$$

Contributions to the escape rate come only from particles penetrating the barrier, so that the lifetime τ is given by the equation

$$\frac{1}{\tau} = \int_{-\infty}^{\infty} \frac{f(\varepsilon) d\varepsilon}{1 + \exp(-2\pi\varepsilon/\hbar\omega)}. \quad (3.12)$$

After the substitution

$$\varphi(\varepsilon) = f(\varepsilon)[1 + \exp(2\pi\varepsilon/\hbar\omega)], \quad (3.13)$$

the Fourier transformation (3.10) gives the finite-difference equation

$$\varphi(\lambda - 2\pi iT/\hbar\omega) = -G(\lambda)\varphi(\lambda), \quad (3.14)$$

where $\varphi(\lambda)$ is the Fourier transform of the function $f(\varepsilon)$. Equation (3.14) has to be solved with the boundary condition

$$\varphi(\lambda) \approx -iT \sinh(\hbar\Omega/2T) \exp(-U_0/T)/\pi\hbar(\lambda + i), \quad |\lambda + i| \ll 1, \quad (3.15)$$

which follows from the asymptotics (3.1). The function $G(\lambda)$ grows monotonically both in the upper and in the lower half-planes of λ , so that both direct and inverse iterations of eq. (3.14) result in divergent infinite products.

In order to solve this equation we factorize $G(\lambda)$ into a product of the functions $G^+(\lambda)$ and $G^-(\lambda)$ according to eqs. (2.51), (2.52), respectively. These functions decrease in the upper and the lower half-planes of λ and can be used, respectively, by inverse and direct iterations of eq. (3.14). One can check directly that the auxiliary function obtained in this way,

$$\psi(\lambda) = \frac{\Gamma}{G^-(\lambda)} \prod_{n=1}^{\infty} \frac{G^+(\lambda + 2\pi nT/\hbar\omega)}{G^-(\lambda - 2\pi nT/\hbar\omega)}, \quad (3.16)$$

satisfies the following relation:

$$\psi(\lambda - 2\pi iT/\hbar\omega) = \psi(\lambda)G^+(\lambda)G^-(\lambda) = \psi(\lambda)G(\lambda). \quad (3.17)$$

Substituting eq. (2.51) into eq. (3.16) and performing the infinite summation yield

$$\psi(\lambda) = \exp\left(\frac{\hbar\omega}{4\pi iT} \int_{-\infty}^{\infty} d\lambda' \frac{\ln G(\lambda)}{\tanh[\hbar\omega(\lambda - \lambda')/2T]}\right), \quad \text{Im } \lambda < 0.$$

Comparison of eq. (3.14) with eq. (3.17) and eq. (3.15) shows that $\varphi(\lambda)$ differs from $\psi(\lambda)$ by a function which changes its sign upon shifting λ by $2\pi iT/\hbar\omega$ and has a pole at $\lambda = -i$. It is obvious that this function is simply given by $1/\sinh[\hbar\omega(\lambda + i)/2T]$. Thus, the solution of eq. (3.14) with the boundary condition (3.15) is given by

$$\varphi(\lambda) = -\frac{i \sinh(\hbar\Omega/2T) \omega \psi(\lambda) \exp(-U_0/T)}{2\pi \sinh[\hbar\omega(\lambda + i)/2T] \psi(-i)}. \quad (3.18)$$

3.4. Escape rate in the underdamped quantum regime

The lifetime τ is related to $\varphi(\lambda)$ by the equation

$$1/\tau = \varphi(-2\pi i T/\hbar\omega), \quad (3.19)$$

which follows from eq. (3.12) using the substitution (3.13) and the Fourier transformation (3.10). For the preexponential factor $A(\delta/T, \hbar\omega/2\pi T)$, defined by eq. (3.3), eqs. (3.18) and (3.19) finally yield the expression

$$A(\Delta, y) = \exp\left(\int_{-\infty}^{\infty} \frac{y \sin \pi y \ln\{1 - \exp[-\Delta Z(\lambda, y)]\} d\lambda}{\cosh 2\pi\lambda y - \cos \pi y}\right), \quad (3.20)$$

which is written in a form slightly different from the original result [40]. In the extremely underdamped regime, $\Delta \ll 1$, the inner exponent in eq. (3.20) can be expanded up to the term linear in Δ , which gives

$$A(\Delta, \hbar\omega/2\pi T) \approx (\Delta)^{1-\hbar\omega/2\pi T} a(\hbar\omega/2\pi T), \quad \Delta \ll 1, \quad (3.21)$$

$$a(y) \equiv \exp\left(y \sin \pi y \int_{-\infty}^{\infty} \frac{\ln Z(\lambda, y) d\lambda}{\cosh 2\pi\lambda y - \cos \pi y}\right).$$

These expressions show quite clearly that, with decreasing temperature T , contributions of quantum tunneling predominate over the effects of depletion of the distribution function. Therefore, the escape rate extrapolated to $T = T_0 = \hbar\omega/2\pi$ becomes independent of dissipation. At large temperature, $T \gg \hbar\omega$, eq. (3.21) gives the classical result, $A \approx \Delta$.

Further progress is only possible for explicit potentials $U(x)$. We shall consider both a cubic potential and a cosine potential. For the cubic potential,

$$U(x) = -\frac{1}{2}m\omega^2 x^2(1 - x/x_1),$$

eq. (3.5) gives $x(t) = x_1/\cosh^2(\omega t/2)$,

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} x(t) \exp(i\epsilon t/\hbar) dt = \frac{2\epsilon x_1}{(\hbar\omega)^2 \sinh(\pi\epsilon/\hbar\omega)},$$

and taking account of eq. (3.7) we obtain

$$Z(\lambda, y) = \frac{15}{2\pi^4 y^5} \int_{-\infty}^{\infty} \frac{(\cosh x - \cos 2\lambda x)x^3 dx}{\sinh x \sinh^2(x/y)}. \quad (3.22)$$

In a similar way, for the periodic potential

$$U(x) = (m\omega^2 x_1^2/2\pi^2) \cos^2(\pi x/x_1)$$

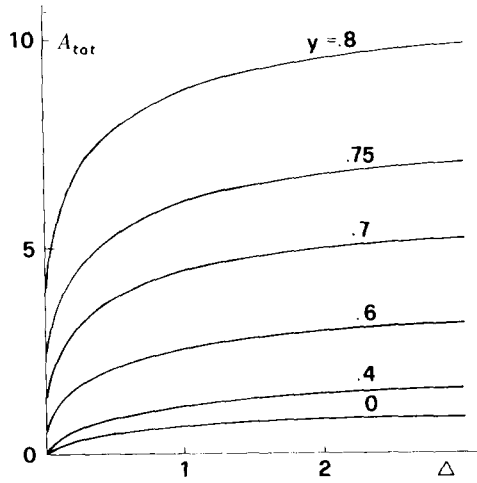


Fig. 5. Dependence of the preexponential factor A_{tot} in eq. (3.2) on the reduced dissipation $\Delta \equiv \gamma S/T \equiv \delta/T$ at several values of the quantum parameter $y \equiv \hbar\Omega/2\pi T$.

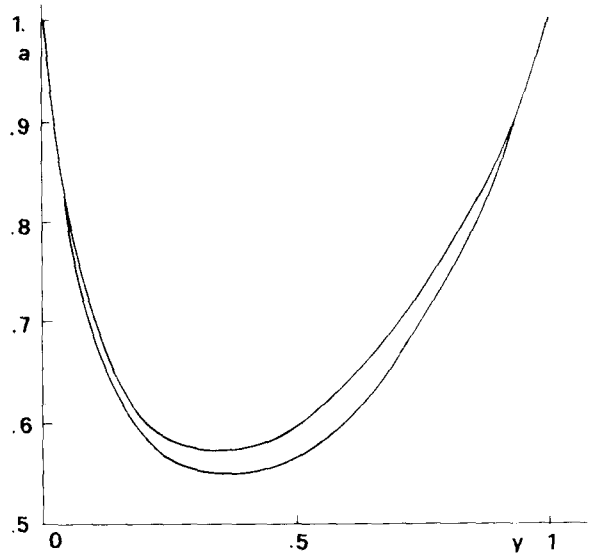


Fig. 6. Function $a(y)$ entering eq. (3.21) for the cases of a cosine potential (upper curve) and cubic potential (lower curve).

we obtain $x(t) = (2x_1/\pi) \arctan(\exp \omega t)$,

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} x(t) \exp(i\epsilon t/\hbar) dt = \frac{ix_1}{2\epsilon \cosh(\pi\epsilon/2\hbar\omega)}, \quad Z(\lambda, y) = \frac{1}{8y} \int_{-\infty}^{\infty} \frac{(\cosh x - \cos 2\lambda x) dx}{x \sinh x \cosh^2(x/2y)}. \quad (3.23)$$

It can easily be checked that in the classical limit, $\hbar\omega/2\pi T = y \rightarrow 0$, both expressions for $Z(\lambda, y)$ go over to $Z_{\text{cl}} = \lambda^2 + 1/4$ [see eq. (2.47) for $g(\lambda)$]. Results of numerical calculations of

$$A_{\text{tot}}(\Delta, y) \equiv \frac{\sinh(\pi y)}{\sin(\pi y)} A(\Delta, y) \quad (3.24)$$

are presented in fig. 5 for the case of a cosine potential, when $\Omega \equiv \omega$. The corresponding results for a cubic potential differ only by 3–4%.

The results for $a(y)$ are presented in fig. 6 for both types of the potentials. Note that the procedure of solution of eq. (3.14) has been developed in refs. [36, 56, 57]. Unfortunately, in these articles the noise was assumed to be classical. Hence, the results obtained there are only applicable in the limit of a very wide potential well.

3.5. Distribution function in the overdamped quantum regime

In the classical overdamped regime the escape rate is governed by the viscous dynamics of the Brownian particle near the top of the barrier. To examine the quantum situation, we shall introduce the heat bath explicitly in the form of an infinite string, attached to the particle and tightened perpendicular

to the axis of particle motion [58]. The dynamics of this system is governed by the equations

$$m \, d^2x/dt^2 = -dU(x)/dx + \rho s^2 \, \partial x(z, t)/\partial z|_{z=0}, \quad (3.25)$$

$$\partial^2 x(z, t)/\partial t^2 - s^2 \, \partial^2 x(z, t)/\partial z^2 = 0, \quad (3.26)$$

where $x(z, t)$ is the displacement of the string, $x(t) \equiv x(0, t)$. Neglecting thermal noise, the motion of the string is completely specified by the particle motion, since for waves emanating from the particle we have $x(z, t) = x(z - st)$. From this relation one obtains

$$\partial x(z, t)/\partial z|_{z=0} = -(1/s) \, dx/dt.$$

Substitution of this result into eq. (3.25) shows that, in the classical limit, the action of the string on the particle is equivalent to the effect of a viscous medium with the coefficient of friction $\gamma = \rho s/m$, where ρ is the linear density of the string and s is the velocity of the waves traveling along the string. In the quantum case only this combination of ρ and s enters the final results. Due to introduction of the string, the number of degrees of freedom becomes infinite but this inconvenience is compensated by the fact that we obtain a conservative dynamical system which can be quantized in the usual manner. The approach presented here is applicable to any medium with linear response.

We consider the quantum regime using the action for the system particle + string,

$$S = \int_0^{\hbar/T} \left\{ \frac{m}{2} \left(\frac{dx(0, t)}{dt} \right)^2 + U[x(0, t)] + \frac{1}{2} \int_0^\infty \left[\rho \left(\frac{\partial x(z, t)}{\partial t} \right)^2 + \rho s^2 \left(\frac{\partial x(z, t)}{\partial z} \right)^2 \right] dz \right\} dt. \quad (3.27)$$

This is equivalent to the system (3.25), (3.26), with the only difference that t now denotes imaginary time. Eliminating the string variables by solution of the dynamic equations for a given trajectory $x(t) \equiv x(0, t)$ of the particle, we arrive at the effective action [26],

$$S = \int_0^{\hbar/T} \left(\frac{m}{2} \left(\frac{dx(t)}{dt} \right)^2 + U[x(t)] + \frac{\pi\gamma}{4} \int_0^{\hbar/T} \frac{[x(t) - x(t')]^2}{\sin^2[\pi\hbar(t-t')/T]} dt' \right) dt, \quad (3.28)$$

First we derive the partition function $Z(T)$ of quantum Brownian particles in a potential $U(x)$ at temperature T . The partition function can be expressed in terms of the path integral

$$Z(T) = \int \exp\{-S[x(t)]/\hbar\} D[x(t)], \quad (3.29)$$

over trajectories $x(t)$. If the temperature is small compared to the typical drop in the potential, in calculating the integral in eq. (3.29) we can assume that the potential $U(x)$ is quadratic near the point $x = x_m$,

$$U(x) \approx -U_0 + \frac{1}{2} m \Omega^2 (x - x_m)^2.$$

The coordinate of the particle is represented by a Fourier series with respect to the Matsubara

frequencies,

$$x(t) = x_m + a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t), \quad \omega_n \equiv 2\pi n T / \hbar. \quad (3.30)$$

In practical calculations it is sometimes convenient to return to eq. (3.27), and to substitute into it the string trajectory,

$$x(z, t) = x_m + a_0 + \sum_{n=1}^{\infty} [a_n \cos \omega_n t + b_n \sin \omega_n t] \exp(-z\omega_n t/s).$$

For the action S we obtain

$$S = -\frac{\hbar U_0}{T} + \frac{\hbar m}{2T} \Omega^2 a_0^2 + \frac{\hbar m}{4T} \sum_{n=1}^{\infty} (\omega_n^2 + \gamma \omega_n + \Omega^2)(a_n^2 + b_n^2).$$

The path integral in eq. (3.29) is thus reduced to Gaussian integrals over the a_n and b_n and can easily be calculated,

$$Z(T) = \Omega^{-1} \Gamma(1 - \hbar \Lambda^+ / 2\pi T) \Gamma(1 - \hbar \Lambda^- / 2\pi T) \exp(U_0/T), \quad (3.31)$$

$$\Lambda^{\pm} = -\frac{1}{2}\gamma \pm (\frac{1}{4}\gamma^2 - \Omega^2)^{1/2},$$

where we have omitted a factor independent of γ and Ω . Note that Λ^{\pm} is complex for $\gamma < 2\Omega$ and negative for $\gamma > 2\Omega$, which makes $Z(T)$ finite at any real value of the parameters γ and Ω and for $T > 0$.

The normalized distribution of particles in the coordinate x is given by the expression

$$N(x) = Z^{-1}(T) \int \exp\{-S[x(t)]/\hbar\} D[x(t)], \quad (3.32)$$

where integration goes over all trajectories subject to the boundary conditions $x(0) = x(\hbar/T) = x$. At sufficiently small T only a close neighbourhood of the point x contributes into the integral in eq. (3.32), and the potential $U[x(t)]$ can be expanded up to second order near this point,

$$U[x(t)] \approx U(x) + U'(x)[x(t) - x] + \frac{1}{2}U''(x)[x(t) - x]^2.$$

The Fourier series for $x(t)$ now takes the form

$$x(t) = x + a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t),$$

where the Fourier coefficients a_n must satisfy the condition

$$\sum_{n=0}^{\infty} a_n = 0. \quad (3.33)$$

Calculating the action S as the function of the coefficients a_n and b_n and integrating over these

coefficients taking account of the condition (3.33) gives the particle distribution [59],

$$N(x) = Z^{-1}(T) \sigma^{1/2}(x) I [1 - \hbar A^+(x)/\pi T] I [1 - \hbar A^-(x)/\pi T] \quad (3.34)$$

$$\times \exp\left(-\frac{U(x)}{T} + [1 - \sigma(x)] \frac{[U'(x)]^2}{2TU''(x)}\right),$$

$$\sigma(x) \equiv \left(\frac{U''(x)}{m} \sum_{n=-\infty}^{\infty} \frac{1}{\omega_n^2 + \gamma|\omega_n| + U''(x)/m} \right)^{-1}, \quad (3.35)$$

where A^+ and A^- are the roots of the equation

$$A^2 + \gamma A + U''(x)/m = 0. \quad (3.36)$$

Expression (3.28) is applicable in the regions of x where $U''(x)$ exceeds some (negative) boundary value U''_b , corresponding to the rightmost pole of σ as a function of $U''(x)$. As the region where $U'' \leq U''_b$ is approached, the particle density $N(x)$ is determined by trajectories that are increasingly more distant from x , so that the quadratic approximation finally becomes inapplicable. At $\gamma = 0$, we have

$$\sigma(x) = [2T/\hbar\Omega(x)] \tanh[\hbar\Omega(x)/2T], \quad \Omega(x) \equiv [U''(x)/m]^{1/2}, \quad U''_b = -m(\pi T/\hbar)^2.$$

The equilibrium distribution function $f(p, x)$ is related to the density matrix $\rho(x, x')$ by the Wigner transformation

$$f(p, x) = \int_{-\infty}^{\infty} \rho(x + y/2, x - y/2) \exp(ip y/\hbar) dy. \quad (3.37)$$

The density matrix $\rho(x, x')$ in its turn is given by the path integral

$$\rho(x, x') = \int \exp\{-S[x(t)]/\hbar\} D[x(t)], \quad (3.38)$$

and the trajectories $x(z, t)$ correspond to the motion of the particle from a point x at $t = 0$ to the point x' at $t = \hbar/T$,

$$x(0) = x, \quad x(\hbar/T) = x'. \quad (3.39)$$

In calculating the integral in eq. (3.37) we again assume that the potential $U(x)$ is quadratic near x . In the case of a harmonic oscillator, the expression obtained below for $f(p, x)$ is exact. To account explicitly for the particle motion from the point x to the point x' , the trajectory $x(t)$ must be written as

$$x(t) = x + (x' - x)tT/\hbar + a_0 + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t),$$

while the motion of the string is periodic,

$$x(z, t) = (x' + x)/2 + a_0 + \sum_{n=1}^{\infty} \{a_n \cos \omega_n t + [b_n - (x - x')/\pi n] \sin \omega_n t\} \exp(-z\omega_n/s).$$

After substitution of these series into the expression for the action S one has to calculate integrals in the a_n and b_n . Integration in the a_n , subject to condition (3.33), reproduces eq. (3.34) for $N(x)$. Integration in the b_n gives a Gaussian function of $x - x'$. After the transformation (3.37) one finally obtains

$$f(p, x) = N(x)[2\pi mT\xi(x)]^{-1/2} \exp[-p^2/2mT\xi(x)], \quad (3.40)$$

$$\xi(x) = \sum_{n=-\infty}^{\infty} \frac{\gamma|\omega_n| + U''(x)}{\omega_n^2 + \gamma|\omega_n| + U''(x)/m}. \quad (3.41)$$

It is evident that the sum for $\xi(x)$ is logarithmically divergent for $n \rightarrow \infty$. To obtain a finite result one should, e.g., take the internal structure of the string into account. Assuming that the maximum energy of the string phonons is $\varepsilon_g \gg T$, one obtains $\xi(x) \sim \gamma \ln(\varepsilon_g/T)$. The divergence of the expression for $p^2 \propto mT\xi(x)$ reflects the large amplitude of the quantum fluctuations of the particle momentum p under the influence of the interaction with the heat bath. A similar phenomenon has already been observed in the case of a large particle interacting with a Fermi gas [60].

3.6. Escape rate in the overdamped quantum regime

The aim of the present section is to generalize eq. (2.24) to the quantum regime. This problem was first solved by Wolynes [23]. Later, his result was reproduced by a different method [59]. Here, we shall follow a standard procedure of calculation of $1/\tau$, developed by Langer [8, 9] (see also refs. [61, 26]). The starting point is the relation between the decay rate $1/\tau(E)$ of a quantum eigenstate with energy E and the imaginary part of the energy eigenvalue, $1/\tau = (2 \operatorname{Im} E)/\hbar$. After averaging over the equilibrium distribution one obtains

$$1/\tau = (2 \operatorname{Im} F)/\hbar, \quad (3.42)$$

where F is the free energy, $F \equiv T \ln Z$. For a stable potential the partition function Z is real and $1/\tau = 0$. Analytic continuation in the parameters of the potential produces an exponentially small imaginary contribution to Z , related to the finite height of the potential barrier. The expression for $1/\tau$ given above is written down assuming that the main contribution to the escape rate comes from decay of the quantized levels, since only in this case the concept of the complex eigenvalue makes sense.

Earlier we showed that at $T \sim \hbar\omega$, the contribution of the subbarrier penetration is of the same order of magnitude as the contribution of the overbarrier escapes. The latter process is not taken into account in the above expression for $1/\tau$, hence, this expression is only of limited validity. At sufficiently high temperature, when the overbarrier escapes become important, it has to be modified. This question has been discussed in detail by Affleck [24]. Expression (3.2) for the escape rate becomes singular at a temperature $T_0 = \hbar\omega/2\pi$, since at lower temperatures the decrease of the penetration coefficient, $D \sim \exp(2\pi\varepsilon/\hbar\omega)$, at large negative energies yields in competition with the increase of the particle population, $f(\varepsilon) \sim \exp(-\varepsilon/T)$, and the integral for the particle flux diverges at large negative energies. It is almost obvious that in the dissipative case the role of ω is played by the parameter $\lambda^+ = (\omega^2 + \frac{1}{4}\gamma^2)^{1/2} - \gamma/2\omega$, which describes the particle motion near the barrier top [9]. The range of applicability of the anticipated result for $1/\tau$ is then given by the inequality $T > \hbar\lambda^+/2\pi$. Therefore, one can conclude that, in accord with Langer [9], in the high-temperature range the expression for the

escape rate is

$$1/\tau = (\hbar\lambda^+/\pi)Z_1(T)/Z(T), \quad T > \hbar\lambda^+/2\pi. \quad (3.43)$$

where $Z(T)$ is given by eq. (3.31). The correctness of eq. (3.43) is also supported by the argument that all subsequent calculations are carried through in a quadratic approximation, when in the eigenmode representation the system particle + string becomes a set of harmonic oscillators. The eigenvalue λ^+ is just the imaginary frequency of the single unstable mode. Only this mode is responsible for escapes over the barrier, whereas all the other modes determine the particle density at the barrier.

To calculate $Z_1(T)$ we expand the action S quadratically near the barrier top. In terms of the coefficients of the Fourier series,

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)],$$

the action takes the form

$$S = -\frac{\hbar m}{2T} \omega^2 a_0^2 + \frac{\hbar m}{4T} \sum_{n=1}^{\infty} [\omega_n^2 + \gamma\omega_n - \omega^2](a_n^2 + b_n^2), \quad (3.44)$$

where again $\omega^2 \equiv -U''(0)/m$. The integrals in the a_n and the b_n , with $n > 0$, can be done in the standard way. On the other hand, the integral in a_0 over the real axis is divergent. To render it finite, one has to shift the contour of integration into the complex plane [8],

$$iZ_1 = \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} da_n db_n \int_0^{i\infty} da_0 \exp(-S/\hbar). \quad (3.45)$$

The final result for $Z_1(T)$ can easily be obtained by comparing eq. (3.45) with eq. (3.31). These expressions differ only in two points: (1) a factor 1/2 appears in Z_1 , since the integration in a_0 goes over a semiaxis only; (2) the parameter Ω should be replaced by $-i\omega$. Substitution of the resulting expression for Z_1 into eq. (3.43) gives

$$\frac{1}{\tau} = \frac{\lambda^+ \Omega \Gamma(1 - \hbar\lambda^+/2\pi T) \Gamma(1 - \hbar\lambda^-/2\pi T)}{2\pi\omega \Gamma(1 - \hbar\Lambda^+/2\pi T) \Gamma(1 - \hbar\Lambda^-/2\pi T)} \exp(-U_0/T), \quad (3.46)$$

$$\lambda^{\pm} = -\frac{1}{2}\gamma \pm (\frac{1}{4}\gamma^2 + \omega^2)^{1/2}, \quad \Lambda^{\pm} = -\frac{1}{2}\gamma \pm (\frac{1}{4}\gamma^2 - \Omega^2)^{1/2}.$$

When the temperature decreases, the singularity in τ^{-1} arises only when $\Gamma(1 - \hbar\lambda^+/2\pi T)$ becomes infinite, so that the region of applicability of eq. (3.46), $T > T_0 = \hbar\lambda^+/2\pi T$, is wider than that of eqs. (3.34) and (3.40). To calculate the flux at temperatures close to T_0 and below, the deviation of the barrier from parabolic shape must be taken into account. The crossover from eq. (3.43) to eq. (3.42) in a narrow region, $|1 - \hbar\lambda^+/2\pi T| \sim (T/U_0)^{1/2}$, has been treated by Affleck in a conservative case [24] and by Larkin and Ovchinnikov in the presence of dissipation [26]. At large temperatures, the expression obtained for τ^{-1} becomes the Kramers expression (2.15), while in the limit $\gamma \ll \omega, \Omega$, using the relation $\Gamma(1+z)\Gamma(1-z) = \pi z/\sin \pi z$, eq. (3.46) can be reduced to eq. (3.2).

It should be noted that the result for $1/\tau$ is valid in a wider range of parameters than the result for the particle density $N(x)$. Indeed, calculation of $\sigma(x)$ at $x = 0$ gives

$$\sigma = \left\{ -1 + \left(\frac{\hbar\omega}{2\pi T} \right)^2 \left(1 + \frac{\gamma^2}{4\omega^2} \right)^{-1} \left[\psi \left(-\frac{\hbar\lambda^-}{2\pi T} \right) - \psi \left(-\frac{\hbar\lambda^+}{2\pi T} \right) \right] \right\}^{-1},$$

where $\psi(z)$ is the digamma function. It follows from this expression that the function $\psi(-\hbar\lambda^+/2\pi T)$ has a pole at $T = \hbar\lambda^+/2\pi$. Hence, at a certain value of $T > \hbar\lambda^+/2\pi$, the parameter σ entering the particle distribution $N(x)$ goes to infinity. The physical reason of the difference between the validity regions of eqs. (3.34) and (3.46) is quite clear: while all particles penetrating the barrier up to $x = 0$ do contribute to the particle density, only those particles which have penetrated the whole barrier contribute to the escape rate. An intuitive guess that the latter quantity should behave in a less singular manner is quantitatively supported by the above considerations.

4. The Brownian particle in a tilted periodical potential; fluctuation-induced phenomena in Josephson junctions

4.1. Basic equations for a resistively shunted junction

Initially, the treatment of decay of a metastable state of a Brownian particle was formulated by Kramers [5] for the problem of dissociation of a molecule. As such Kramers did use several simplifying assumptions. The point is that the Fokker–Planck equation or its equivalent Langevin equation driven by Gaussian white noise, is valid only for the description of a slowly vibrating molecule which interacts with a gas of light particles. This condition is difficult to satisfy in real situations. Another simplification of the model presented in section 2 is the assumption of a one-dimensional potential. Molecules, in addition to vibrational degrees of freedom, possess also rotational ones, and this circumstance complicates substantially the process of dissociation. Consequently, the Kramers model can only qualitatively describe chemical reactions.

However, there exist physical objects of quite a different nature which under certain conditions manifest accurately the features prescribed by the Kramers model. As was indicated by Josephson [11], the physical state of a contact of two superconductors separated by a thin layer of a normal metal is specified by a parameter φ that gives the difference of the order-parameter phases of the two superconductors in contact. The supercurrent through the junction is given by the relation

$$I = I_c \cos \varphi, \quad (4.1)$$

where I_c is the critical current of the junction. The voltage V across the junction is connected with the time derivative of φ by the Josephson relation,

$$V = (\hbar/2e) d\varphi/dt. \quad (4.2)$$

These relations show that at $I < I_c$ the junction will remain in the superconducting state, when $\varphi = \text{const.}$ and $V = 0$. At larger currents, $I > I_c$, the superconducting state is no longer stable. To describe the junction state in this regime, one needs to take into account, apart from the critical current I_c , some other parameters of the junction.

In a relatively simple manner this can be achieved with the model of a resistively shunted Josephson junction [12] which assumes that the nonlinear element, described by eqs. (4.1) and (4.2), is shunted by a resistance R and a capacitance C . The total current through the junction is then given by the sum of the supercurrent (4.1) and the normal current $V/R + C dV/dt$. At finite temperature T the resistor R generates a fluctuating current $I_T(t)$. Taking account all these factors, the parameter $\varphi(t)$ is governed by the equation

$$C \frac{d^2\varphi}{dt^2} + \frac{1}{R} \frac{d\varphi}{dt} + \frac{2e}{\hbar} [I_c \sin \varphi + I + I_T(t)] = 0, \quad (4.3)$$

where I is the external current controlled by the experimenter, $I_T(t)$ is a Gaussian fluctuating current satisfying

$$\langle I_T(t) I_T(t') \rangle = (2T/R) \delta(t - t'), \quad (4.4)$$

and the angular brackets denote the thermal average. From eq. (4.3) it follows that in the absence of dissipation, i.e. $R \rightarrow \infty$, the frequency of small oscillations with $I = 0$ is given by $\Omega = (2eI_c/\hbar C)^{1/2}$. This quantity is known as the Josephson plasma frequency. If the current I is assumed to be given, the problem of the junction static current–voltage characteristics reduces to solving eq. (4.3) and subsequently averaging the Josephson relation (4.2), i.e.,

$$\bar{V}(I) = (\hbar/2e) \langle d\varphi/dt \rangle. \quad (4.5)$$

The Langevin equation (4.3) with the Gaussian noise (4.4) is equivalent to the Fokker–Planck equation for the distribution function $F(\varphi, \dot{\varphi})$,

$$\dot{\varphi} \frac{\partial F}{\partial \varphi} - \frac{4e^2}{C} \frac{\partial U(\varphi)}{\partial \varphi} \frac{\partial F}{\partial \dot{\varphi}} = \frac{1}{RC} \frac{\partial}{\partial \dot{\varphi}} \left(\frac{4e^2}{C} T \frac{\partial F}{\partial \dot{\varphi}} + \dot{\varphi} F \right), \quad (4.6)$$

where $U(\varphi)$ is the potential energy of a particle in the tilted washboard potential,

$$U(\varphi) = (\hbar I_c/2e)(\cos \varphi - 1 - I\varphi/I_c). \quad (4.7)$$

This potential is depicted in fig. 7.

The time derivative in eq. (4.6) has been omitted, for only the steady state is of interest for the following. The function $F(\varphi, \dot{\varphi})$ must be periodic in φ , and normalized, i.e.,

$$F(\varphi + 2\pi, \dot{\varphi}) = F(\varphi, \dot{\varphi}), \quad \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} \dot{\varphi} F(\varphi, \dot{\varphi}) d\dot{\varphi} = 1. \quad (4.8)$$

The average junction voltage is given by the integral

$$\bar{V}(I) = \frac{\hbar}{2e} \int_{-\infty}^{\infty} \dot{\varphi} d\dot{\varphi} \int_0^{2\pi} F(\varphi, \dot{\varphi}) d\varphi = \frac{\hbar \pi}{e} \int_{-\infty}^{\infty} \dot{\varphi} F(\varphi, \dot{\varphi}) d\dot{\varphi}, \quad (4.9)$$

where account is taken of the fact that under stationary conditions the flux is independent of the phase.

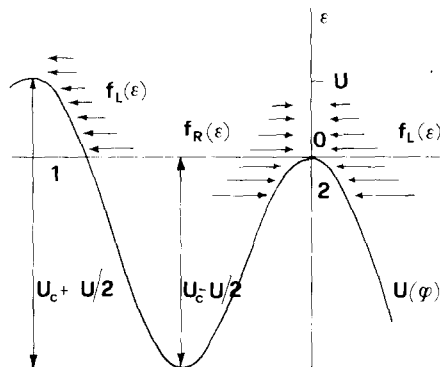


Fig. 7. Effective potential for the Brownina particle. f_R and f_L are the energy densities for the right- and left-going particles. $U_c = \hbar I_c/e$ and $U = \pi \hbar I/e$.

It is obvious that for $I < I_c$ the tilted periodic potential has a series of minima. Similarly to section 2, one can consider fluctuation-induced escapes of a Brownian particle placed initially into one of the minima. However, the periodic potential is qualitatively different from a single-well or a double-well potential and provides a richer picture of fluctuation-induced phenomena. To make our presentation more transparent, we consider in the next section the voltage–current characteristics in an underdamped Josephson junction in the absence of noise. Then a series of fluctuation-induced effects will be considered. The results given below are nearly exhaustive in what concerns the classical regime. In the quantum regime we have restricted ourselves to the calculation of the retrapping current distribution.

4.2. Dissipative phase dynamics in Josephson junctions

At sufficiently low temperatures fluctuation-induced effects are small and the noise current $I_T(t)$ in eq. (4.3) can be neglected. On the other hand, in assumption of high- Q junction, i.e. $RC\Omega \gg 1$, effects of dissipation and external current I can be treated in a perturbative manner. Therefore, our starting point is the unperturbed classical trajectory. With zero friction ($R \rightarrow \infty$) and without noise [$I_T(t) = 0$] and zero tilt ($I = 0$) one finds $d^2\varphi/dt^2 = -\Omega^2 \sin \varphi$. The first integral of this equation is

$$\varepsilon = (\hbar I_c/e) \left[(1/4\Omega^2) (d\varphi/dt)^2 + \frac{1}{2} (\cos \varphi - 1) \right], \quad (4.10)$$

where ε specifies the energy of a particle moving in a washboard potential. For $\varepsilon < 0$ the particle orbits in a finite region and the junction is in a zero-voltage state. For $\varepsilon > 0$ the particle flies over the potential barriers and the junction is in the running state. At finite R the energy ε of the Josephson junction is dissipated due to the Ohmic losses,

$$d\varepsilon/dt = -V^2/R = -(\hbar^2/4e^2R) (d\varphi/dt)^2. \quad (4.11)$$

The unperturbed solution for $\varphi(t)$ given by eq. (4.10) has the property $\varphi(t + 2\pi/\omega) = 2\pi + \varphi(t)$. The frequency ω depends on ε and is given by the equation $\omega = \pi^{-1} \Omega s(e\varepsilon/\hbar I_c)$, where

$$s(y) \equiv \pi^2 \left(\int_0^{\pi/2} \frac{d\varphi}{\sqrt{y + \sin^2 \varphi}} \right)^{-1}, \quad y = \frac{e\varepsilon}{\hbar I_c}. \quad (4.12)$$

For its asymptotic behaviour we find

$$s(y) \approx 2\pi^2/\ln(4/y), \quad y \ll 1. \quad (4.13)$$

$$s(y) \approx 2\pi y^{1/2}(1 + 1/4y + 7/64y^2), \quad y \gg 1. \quad (4.14)$$

When shifted by 2π in the tilted potential (4.7) ($I \neq 0$), the particle gains the energy $\pi\hbar I/e$. At the same time the loss of energy caused by the Ohmic dissipation can be obtained by integration of eq. (4.11) over the time interval $2\pi/\omega$. With the use of eq. (4.10) we obtain the following expression for the loss of energy

$$\delta(\varepsilon) = \frac{\hbar^2}{4e^2 R} \int_0^{2\pi/\omega} \left(\frac{d\varphi}{dt} \right)^2 dt = \delta r(y), \quad y \equiv e\varepsilon/\hbar I_c, \quad (4.15)$$

where δ is the loss of energy per period for a particle moving precisely at the level of the potential maximum ($\varepsilon = 0$),

$$\delta = \frac{1}{R} \left(\frac{2\hbar}{e} \right)^{3/2} \left(\frac{I_c}{C} \right)^{1/2} = \frac{\pi\hbar I_0}{e} = \frac{2\hbar^2 \Omega}{e^2 R} = \frac{4\hbar I_c}{eRC\Omega},$$

and the function $r(y)$ is determined by the equation

$$r(y) \equiv \int_0^{\pi/2} (y + \sin^2 \varphi)^{1/2} d\varphi. \quad (4.16)$$

Asymptotically, we have

$$r(y) \approx 1 + \frac{1}{4}y \ln(4/y), \quad y \ll 1, \quad (4.17)$$

$$r(y) \approx \frac{1}{2}\pi y^{1/2}(1 + 1/4y - 3/64y^2), \quad y \gg 1. \quad (4.18)$$

The energy $\varepsilon(I)$ of a particle in a running state is determined by the balance between the energy loss $\delta(\varepsilon)$ due to friction and the energy gain $\pi\hbar I/e$ due to the bias. This results in

$$r[e\varepsilon(I)/\hbar I_c] = I/I_0, \quad I_0 = e\delta/\pi\hbar = 4I_c/\pi RC\Omega. \quad (4.19)$$

From the inequality $r(y) > 1$ it follows that the running states only exist under the condition $I > I_0$. The averaged potential $V(I)$ can be determined from eqs. (4.2) and (4.5) with the use of eq. (4.10), i.e.,

$$V(I) = \frac{\hbar}{2e} \left\langle \frac{d\varphi}{dt} \right\rangle = \frac{\hbar\omega}{2e} = \frac{\hbar\Omega}{2\pi e} s\left(\frac{e\varepsilon(I)}{\hbar I_c} \right). \quad (4.20)$$

From eqs. (4.20) and (4.19) we obtain the current–voltage characteristics,

$$V(I) = 0, \quad I < I_0, \quad V(I) = RI_0 v(I/I_0), \quad I > I_0, \quad (4.21)$$

where the function $v(\eta)$ is determined via the implicit relations

$$v(\eta) = s(y)/4, \quad \eta = r(y). \quad (4.22)$$

For currents I close to I_0 one obtains from eq. (4.13) and eq. (4.17) [38] (see also ref. [62])

$$I(V) \approx I_0 + I_0(\pi^2 RI_0/2V) \exp(-\pi^2 RI_0/2V), \quad V \ll RI_0.$$

To calculate $V(I)$ at large currents, $I \gg I_0$, we substitute into eq. (4.22) the asymptotic result for $s(y)$ and $r(y)$ given by (4.14) and (4.18). This gives

$$V(I) \approx RI[1 - \frac{1}{512} \pi^4 (I_0/I)^4], \quad I \gg I_0. \quad (4.23)$$

A plot of the current–voltage characteristics $V(I)$ is shown in fig. 8.

From this, as well as from eq. (4.21) and the asymptotics (4.23), it follows that the voltage $V(I)$ is a nonanalytic function which vanishes for $I < I_0$, grows nearly jump-like when I exceeds I_0 (dashed line in fig. 8) and approaches the Ohmic law quite rapidly for $I > I_0$. We shall distinguish hereafter between values of the current I below-threshold, $I < I_0$, and above-threshold, $I > I_0$. It is clear that for $I < I_0$ a zero-voltage state is the only option for the junction. For $I > I_0$ the junction becomes bistable and thermal fluctuations can switch it between the zero-voltage states and the running state. The notation $V(I)$ will be retained for the voltage given by eq. (4.21), the average voltage calculated taking account of thermal fluctuations will be denoted as $\bar{V}(I)$. Below we shall show that for $I < I_0$ the voltage $\bar{V}(I)$ is nonvanishing, though exponentially small. The strong nonanalyticity of $V(I)$ is smoothed out in such a way that the derivative of $\bar{V}(I)$ in I performs a finite jump of an exponentially small magnitude. For $I > I_0$ the junction would spend long periods of time either in a zero-voltage or the running states. At I close to I_0 the voltage $\bar{V}(I)$ is exponentially small, since the junction spends most of the time in a zero-voltage state. With increasing I the time spent in the running state will grow, and for sufficiently large currents the effects of fluctuations can be neglected, since the random switchings into a zero-voltage state become extremely rare. It will be shown that $\bar{V}(I)$ nearly coincides with $V(I)$ for currents $I > I_1$, where $I_1 \approx 2.63I_0$ is a new threshold value of the current I .

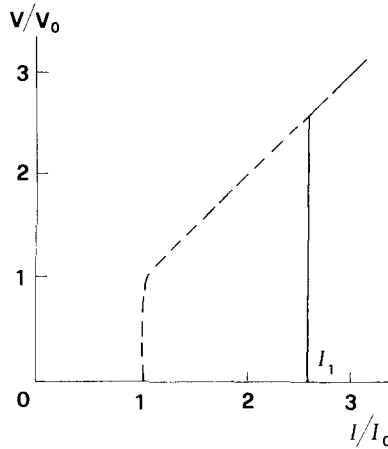


Fig. 8. Current–voltage characteristics for the running state.

4.3. Fluctuation-induced current–voltage characteristics; an exponential approximation

Thermal fluctuations affect the current–voltage characteristics of a Josephson junction in two ways. For $I < I_0$, a finite but exponentially small voltage $\bar{V}(I)$ has its origin in random jumps of the phase φ between minima of the potential energy $U(\varphi)$. The calculation of $\bar{V}(I)$ under these conditions is given in the next section. For $I > I_0$ the thermal fluctuations induce transitions between the junction states with $V=0$ and $V=V(I)$ (see dashed line in fig. 8). The physical picture is in this case the following: let initially $V=0$, which corresponds to finding the particle at the bottom of a potential well. After a time $\tau_a \sim \Omega^{-1} \exp(U_0/T)$ the particle is expected to be ejected from the minimum and be set in motion at an average energy $\varepsilon(I)$ determined by eq. (4.19). We note that the activation time τ_a can be regarded as weakly dependent on the bias current I . In contrast, the time τ_t of the trapping of a particle from the running state depends strongly on the energy $\varepsilon(I)$, hence also on the current I , so that

$$\tau_t \sim \Omega^{-1} \exp[a(I/I_0)U_0/T], \quad (4.24)$$

where $a(\eta)$ is a function to be defined below. Thus, the device that averages $V(t)$ over the time intervals $\tau_a, \tau_t \gg t \gg \Omega^{-1}$, will read either $V=0$ or $V=V(I)$. The times of transition from $V=0$ to $V=V(I)$ form a Poisson random process with parameter τ_a , while the reverse transition time is on average τ_t . Averaging over a time $t \gg \tau_a, \tau_t$ we get

$$\bar{V}(I) \approx V(I)\tau_t/(\tau_a + \tau_t).$$

Clearly, for $\tau_a \ll \tau_t$ the fluctuations change the junction voltage only slightly. On the contrary, for $\tau_a \gg \tau_t$ we get in an exponential approximation

$$\ln \bar{V}(I) \approx (U_0/T)[a(I/I_0) - 1].$$

It thus follows that, with thermal fluctuations taken into account, a new characteristic value I_1 of the current I appears, defined by the condition $a(I_1/I_0) = 1$. For $I < I_1$ the voltage $\bar{V}(I)$ depends exponentially on I , whereas for $I > I_1$, when $\tau_t \gg \tau_a$, the fluctuations have little effect and $\bar{V}(I) \approx V(I)$. To find the function $a(\eta)$ and to determine the current I_1 we must solve the kinetic problem.

In the presence of thermal noise, the distribution $f(\varepsilon)$ of the particles in energy is concentrated in two regions: around the energy $\varepsilon(I)$ and near the bottom of the well. We determine $f(\varepsilon)$ from the following considerations. When a particle is displaced by 2π in the potential (4.7) it loses an amount of energy $\delta(\varepsilon)$ [see eq. (4.15)] due to friction and gains an amount of energy $U \equiv \pi\hbar/e$ due to the tilt of the potential. In addition, thermal fluctuations broaden the distribution function by an amount $[\delta(\varepsilon)T]^{1/2}$. In close similarity to eq. (2.36), the periodicity condition for the stationary function $f(\varepsilon)$ takes the form of the integral equation

$$f(\varepsilon) = \frac{1}{[4\pi\delta(\varepsilon)T]^{1/2}} \int_{-\infty}^{\infty} f(\varepsilon') \exp\{-[\varepsilon - \varepsilon' - U + \delta(\varepsilon)]^2/4\delta(\varepsilon)T\} d\varepsilon'. \quad (4.25)$$

The variation scale of the function $\delta(\varepsilon)$ is $\varepsilon \sim U_c \gg |\varepsilon - \varepsilon'| \sim [\delta(\varepsilon)T]^{1/2}$. This enables one to search for

a solution of eq. (4.25) in the form

$$f(\varepsilon) = \exp\left(-\int \frac{\lambda(\varepsilon')}{T} d\varepsilon'\right),$$

with $\lambda(\varepsilon)$ varying on the scale $\varepsilon \sim U_c$. Substituting this expression into eq. (4.25) and calculating the integral under the assumption of a slow variation of $\lambda(\varepsilon)$, we obtain a quadratic equation for λ ,

$$\lambda[\lambda + U/\delta(\varepsilon) - 1] = 0. \quad (4.26)$$

The root $\lambda = 0$ corresponds to $f(\varepsilon) = \text{const}$. This solution does not satisfy the boundary condition $f \rightarrow 0$ as $\varepsilon \rightarrow \infty$. Thus we must discard it.

The second root $\lambda = 1 - U/\delta(\varepsilon)$ gives the physical solution,

$$f(\varepsilon) \approx C_1 \exp\left[-\int_0^\varepsilon \left(1 - \frac{U}{\delta(\varepsilon')}\right) \frac{d\varepsilon'}{T}\right]. \quad (4.27)$$

This expression was first obtained by Iche and Nozières [14] and by Risken and Vollmer [17]. The maximum of $f(\varepsilon)$ is reached at an energy $\varepsilon(I)$ determined by the relation (4.19). The constant C_1 is determined from the condition $f(0) \sim \exp(-U_c/T)$. For the particle density at the energy $\varepsilon(I)$ we obtain the following relation:

$$\ln f[\varepsilon(I)] \approx -\frac{U_c}{T} + \int_0^{\varepsilon(I)} \left(\frac{U}{\delta(\varepsilon)} - 1\right) \frac{d\varepsilon}{T}. \quad (4.28)$$

It follows from eq. (4.9) that the average voltage is proportional to the flux of particles, which in an exponential approximation coincides with $f[\varepsilon(I)]$, the maximal density of the flux. Taking eqs. (4.19) and (4.28) into account, we obtain for the average junction voltage in the fluctuation-governed regime the implicit relations

$$\frac{T}{U_c} \ln \bar{V}(I) = a(I/I_0) - 1, \quad a(\eta) = \int_0^\eta \left(\frac{r(y)}{r(x)} - 1\right) dx, \quad r(y) = \eta. \quad (4.29)$$

The same function $a(\eta)$ determines the life time $\tau_r(I)$ of the running state [see eq. (4.24)].

The current I_1 and the voltage V_1 corresponding to the departure of $\bar{V}(I)$ from the fluctuation-governed regime are determined from relations (4.22) and (4.21) by substituting the value of $\eta = I/I_0$ which yields a vanishing value for eq. (4.29), i.e., $I_1 = 2.63I_0$, $V_1 = RI_1 = 1.68\hbar/e$. It follows from these results that $\bar{V}(I)$ is exponentially small compared to $V(I)$ so long as $I < I_1$. When applied to fig. 8, this means that $\bar{V}(I)$ is essentially zero so long as $I < I_1$, and increases in a step-like manner up to $V_1 = V(I_1)$, after which point it follows the plot shown in fig. 8. The dependence $\ln \bar{V}(I)$ in the region $I_0 < I < I_1$ is depicted in fig. 9.

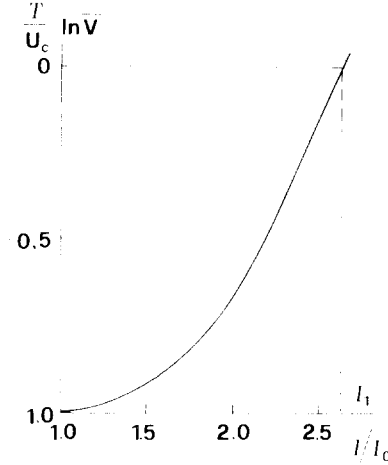


Fig. 9. Fluctuation-induced current-voltage characteristics in an exponential approximation.

4.4. Retrapping current distribution

The temperature T is taken to be small compared to the height of the potential barriers $\hbar I_c / e$. Then we can make use of the large magnitude of the parameter $\hbar I_c / eT$ to find the distribution of the retrapping current for the case that the current I is decreasing with time linearly and very slowly, $dI/dt = -s$, $s \ll \Omega I_c$. The normalized probability $P(I)$ of the retrapping at a current of magnitude I is given by the relation [20]

$$P(I) = \frac{1}{s\tau(I)} \exp\left(-\frac{1}{s} \int_I^{\infty} \frac{dI'}{\tau(I')}\right).$$

For the sake of convenience this can be written in the form

$$P(I) = \exp[-D(I, s)], \quad D(I, s) \equiv \int_I^{\infty} \frac{dI'}{s\tau(I')} + \ln[s\tau(I)],$$

and $\tau(I)$ is the lifetime of the running state $\tau_t(I)$ introduced in the previous section. For the current I_m corresponding to the maximum of the probability $P(I)$ we get then the equation (the prime denotes the partial derivative in I_m)

$$D'(I_m, s) = -1/s\tau(I_m) + \tau'(I_m)/\tau(I_m) = 0, \quad (4.30)$$

wherefrom we get

$$\tau'(I_m) = 1/s. \quad (4.31)$$

This relation enables one to find I_m at a given s if $\tau(I)$ is known. In a more general case of arbitrary dependence $I(t)$ the moment of time t corresponding to the current I_m can be found from the equation

$d\tau[I(t)]/dt = 1$. The complete solution for $\tau(I)$, including the preexponential factor, is given below [see eq. (4.70)]. For our present purposes we need only know τ in an exponential approximation,

$$\ln[\Omega\tau(I)] \approx (\hbar I_c/eT)a(I/I_0). \quad (4.32)$$

Expanding $D(I_m, s)$ near $I = I_m$ up to the quadratic term we get

$$D(I, s) \approx D(I_m, s) + \frac{1}{2}D''(I_m, s)(I - I_m)^2.$$

For $\hbar I_c/eT \gg 1$ the main contribution to D'' comes from differentiating of the first term in eq. (4.30), which gives

$$D''(I_m, s) = \frac{\tau'(I_m)}{s\tau^2(I_m)} = \left(\frac{\tau'(I_m)}{\tau(I_m)}\right)^2 = \left(\frac{\hbar I_c}{eT I_0}\right)^2 \left[a'\left(\frac{I_m}{I_0}\right)\right]^2. \quad (4.33)$$

Going over to the variable $\eta \equiv I/I_0$, one finally gets [39]

$$P(\eta) = \frac{\hbar I_c}{(2\pi)^{1/2} eT \Sigma(\eta_m)} \exp\left[-\left(\frac{\hbar I_c}{eT}\right)^2 \frac{(\eta - \eta_m)^2}{2\Sigma^2(\eta_m)}\right], \quad (4.34)$$

$$\Sigma(\eta_m) = \left(\int_0^x \frac{dy}{r(y)}\right)^{-1}, \quad (4.35)$$

and the upper limit of integration is defined by the equation $r(x) = \eta_m$. The dependence $\Sigma(\eta_m)$ is shown in fig. 17. In the limit of large currents, $\eta_m \gg 1$, the asymptotics (4.18) gives

$$\Sigma(\eta_m) \approx \pi^2/8\eta_m, \quad (4.36)$$

and a typical width of the retrapping current distribution is $|I - I_m| \sim eT/\hbar RC\Omega$. The parameter $I_m \equiv \eta_m I_0$ gives the most probable value of the switching current I . It is related to the rate s of the current change, $s = -dI/dt$, through eq. (4.31). Equations (4.31) and (4.36) imply that with the slowing down of the switching-off process, $s \rightarrow 0$, the peak in the distribution of the retrapping currents becomes more pronounced.

It is worth pointing out that in eq. (4.34) the half-width Σ is introduced in a different way to our original paper [39]. To conclude this section, we note that even in the simplest case of the classical limit the result for $\Sigma(\eta_m)$ can only be written down as a set of implicit equations. In the quantum regime, where a new parameter $\hbar\Omega/2\pi T$ appears, the situation becomes even more complicated.

4.5. Fluctuation-induced current–voltage characteristics below threshold

Thermal noise can activate a Brownian particle to an energy sufficient for escaping over the potential barrier. This means that even for $I < I_0$, when, neglecting thermal noise, a Josephson junction remains forever in a zero-voltage state, fluctuation-induced jumps of the phase φ between neighboring minima will produce a finite voltage $V(I)$. To calculate it, we have to find the average flux of the Brownian particles at a given value of the tilt.

A particle initially placed at a minimum of the tilted periodic potential can escape from it either through the right or through the left barrier. Accordingly, in close similarity to the problem of a double-well potential, we introduce the functions $f_R(\varepsilon)$ and $f_L(\varepsilon)$, which give the number of particles that move over the barrier with velocities directed to the right and to the left, respectively. In the stationary case these functions are identical for all barriers if the energy ε is measured with respect to the barrier tops (see fig. 7). The function $f_R(\varepsilon)$ at barrier 2 is formed both of particles that have already passed over barrier 1 and of particles reflected from the same barrier. These particles are described by the functions $f_R(\varepsilon')\theta(\varepsilon')$ and $f_L(\varepsilon')\theta(-\varepsilon')$, and the shift of the points of reference of ε' and ε is equal to U . The periodicity conditions for $f_R(\varepsilon)$ and $f_L(\varepsilon)$ take the form of integral equations

$$\begin{aligned} f_R(\varepsilon) &= \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon' - U)[f_R(\varepsilon')\theta(\varepsilon') + f_L(\varepsilon')\theta(-\varepsilon')] d\varepsilon', \\ f_L(\varepsilon) &= \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon' + U)[f_R(\varepsilon')\theta(-\varepsilon') + f_L(\varepsilon')\theta(\varepsilon')] d\varepsilon', \end{aligned} \quad (4.37)$$

where the shift of the argument of the function g by $\pm U$ is due to the different points of energy reference at the different barriers. The normalization condition (4.8) corresponds to the presence of one particle at each potential minimum. At $-\varepsilon \gg T$, the function $f_R(\varepsilon)$ should be the Boltzmann distribution, so that we get the boundary condition

$$f_{R,L}(\varepsilon) \approx (\Omega/2\pi T) \exp[-(1/T)(U_c + \varepsilon \mp \frac{1}{2}U)], \quad -\varepsilon \gg T. \quad (4.38)$$

Solving the system (4.37) with the boundary condition (4.38) allows us to express the voltage across the junction in the form

$$\bar{V}(I) = \frac{\pi\hbar}{e} \int_0^{\infty} [f_R(\varepsilon) - f_L(\varepsilon)] d\varepsilon. \quad (4.39)$$

The integral-equation system (4.37) is solved by the Wiener-Hopf method. The unilateral Fourier transformation

$$\varphi_{R,L}^{\pm}(\lambda) \equiv \int_{-\infty}^{\infty} f_{R,L}(\varepsilon)\theta(\pm\varepsilon) \exp(i\lambda\varepsilon/T) d\varepsilon \quad (4.40)$$

transforms it into

$$\varphi_R^+(\lambda) + \varphi_R^-(\lambda) = g^+(\lambda)[\varphi_R^+(\lambda) + \varphi_L(\lambda)], \quad \varphi_L^+(\lambda) + \varphi_L^-(\lambda) = g^-(\lambda)[\varphi_L^+(\lambda) + \varphi_R^-(\lambda)], \quad (4.41)$$

$$g^{\pm}(\lambda) \equiv \exp\{[\delta\lambda^2 + i\lambda(\delta \pm U)/T]\}.$$

The junction voltage is connected with $\varphi_{R,L}(\varepsilon)$ by the relation $\bar{V} = (\pi\hbar/e)[\varphi_R^+(0) - \varphi_L^+(0)]$, which is

obtained from eq. (4.39) when eq. (4.40) is taken into account. Thus, for our purposes, it is sufficient to find the difference $\varphi_R(\lambda) - \varphi_L(\lambda) \equiv \varphi(\lambda)$.

Solving eqs. (4.41) for $\varphi_R^+(\lambda)$ and $\varphi_L^+(\lambda)$ and taking the difference $\varphi_R^+ - \varphi_L^+$, we obtain an equation for $\varphi(\lambda)$,

$$\varphi^+(\lambda) = -G(\lambda)\varphi^-(\lambda), \quad (4.42)$$

$$G(\lambda) \equiv \frac{1 - g^+(\lambda)g^-(\lambda)}{[1 - g^+(\lambda)][1 - g^-(\lambda)]}. \quad (4.43)$$

Condition (4.38) means that $\varphi(\lambda)$ has a pole of the form

$$\varphi^-(\lambda) \approx -\frac{i\Omega}{\pi} \frac{\sinh(U/2T) \exp(-U_c/T)}{\lambda + i}, \quad |\lambda + i| \ll 1. \quad (4.44)$$

To solve eq. (4.42), the kernel $G(\lambda)$ is expressed as

$$G(\lambda) \equiv G^+(\lambda)G^-(\lambda), \quad (4.45)$$

where $G^+(\lambda)$ and $G^-(\lambda)$, determined earlier by relation (2.51), are analytic in the upper and lower halves of the λ -plane, respectively, and their analyticity regions overlap in a certain band. The singular points of $G(\lambda)$ that are closest to the real λ axis are located at $\lambda = 0$ and $\lambda = -i(1 - U/\delta)$. This determines the common region of analyticity of $G^+(\lambda)$ and $G^-(\lambda)$. Note that factorization in (4.45) can be achieved in various ways which differ by the contour of integration in eq. (2.51). The factorization used here corresponds to an integration along the real axis of λ' .

The solution of eq. (4.42) follows from the factorization condition written together with the boundary condition (4.44),

$$\frac{\varphi^+(\lambda)}{G^+(\lambda)} = -G^-(\lambda)\varphi^-(\lambda) = \frac{i\omega}{\pi} \frac{\sinh(U/2T)G^-(-i) \exp(-U_c/T)}{\lambda + i}. \quad (4.46)$$

The voltage $\bar{V}(I)$ is equal to $(\pi\hbar/e)\varphi^+(0)$, so that

$$\bar{V}(I) = (\hbar\Omega/e)G^+(0)G^-(-i) \sinh(U/2T) \exp(-U_c/T). \quad (4.47)$$

Using the multiplicative structure of $G(\lambda)$ [see eq. (4.43)] we can write the expressions for $G^\pm(\lambda)$ as follows:

$$G^+(\lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)}{\Phi(\alpha - 2i\lambda, \alpha)\Phi(\beta - 2i\lambda, \beta)}, \quad \text{Im } \lambda > -\frac{1}{2}\beta, \quad (4.48)$$

$$G^-(\lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)}{\Phi(\alpha - 2i\lambda, \alpha)\Phi(\beta - 2i\lambda, \beta)}, \quad \text{Im } \lambda < -\frac{1}{2}\alpha, \quad (4.49)$$

$$\alpha \equiv 1 + U/\delta = 1 + I/I_0, \quad \beta \equiv 1 - U/\delta = 1 - I/I_0.$$

The function Φ is defined by the relation

$$\ln \Phi(\mu, \nu) = \int_0^{\pi/2} \frac{dx}{\pi} \ln \{1 - \exp[-(\delta/4T)(\mu^2 \tan^2 x + \nu^2)]\}, \quad (4.50)$$

and $\Phi(2\delta, \mu, \nu)$ is given by the same expression, but with δ replaced by 2δ . It should be pointed out that in the following the arguments μ and ν of the functions $\Phi(\mu, \nu)$ are positive, so that we need not consider the analytical properties of $\Phi(\mu, \nu)$. Substitution of eqs. (4.48) and (4.49) into eq. (4.47) yields

$$\bar{V}(I) = \frac{\hbar\Omega}{e} \frac{\sinh(U/2T) A(2\delta/T) \exp(-U_c/T)}{\Phi(\alpha, \alpha)\Phi(\beta, \beta)\Phi(\beta, \alpha)\Phi(\alpha, \beta)}, \quad 1 - \frac{U}{\delta} \gg \left(\frac{T}{U_c}\right)^{1/2}, \quad (4.51)$$

where $A(\Delta)$ is the factor preceding the exponential in the problem of the decay of a metastable state of a Brownian particle in a single potential well [see section 2, eq. (2.55)],

$$\ln A(\delta/T) = \frac{2}{\pi} \int_0^{\pi/2} \ln[1 - \exp(-\delta/4T \cos^2 x)] dx \equiv 2 \ln \Phi(1, 1).$$

The condition indicated in eq. (4.51) will be explained below. For $U \ll \delta$, which is equivalent to $I \ll I_0$, the junction has Ohmic conductance,

$$\bar{V}(I) \approx \frac{\Omega I}{e^2 T} \frac{\exp(-I_c/eT)}{\sigma(\delta/T)}, \quad \sigma(\delta/T) = \frac{2A^2(\delta/T)}{\pi A(2\delta/T)}.$$

In the limiting cases we have the expressions

$$\sigma(\delta/T) \approx \delta/\pi T, \quad \delta \ll T, \quad \sigma(\delta/T) \approx 2/\pi, \quad \delta \gg T.$$

At low dissipation we find

$$\Phi(\mu, \nu) \approx (\delta/4T)^{1/2}(\mu + \nu), \quad \delta \ll T,$$

so that we get from eq. (4.51)

$$\bar{V}(I) \approx \frac{\pi}{2} \frac{IR \exp(-\hbar I_c/eT)}{1 - (I/I_c)^2}, \quad \delta \ll T.$$

Here, the dependence on the junction parameters is explicitly indicated. We note that in the low-dissipation model the Ohmic conductance of the junction does not depend on its capacitance.

In the opposite limiting case we have ($I > 0$)

$$\bar{V}(I) \approx \frac{\hbar\Omega}{e} \frac{\sinh(\pi\hbar I/2eT) \exp(-\hbar I_c/eT)}{A^{1/2}[\delta(1 - I/I_0)^2/T]}, \quad \delta \gg T.$$

This implies that the current–voltage characteristics of the junction depend on δ only near threshold, where $I_0 - I \sim I_0(T/\delta)^{1/2}$. As $I \rightarrow I_0$, and under assumption that δ does not depend on energy, the dependence $\bar{V}(I)$ has a singularity of the form $(1 - I/I_0)^{-1}$. Figure 10 shows plots of $\bar{V}(I)$ versus the reduced current I/I_0 for various values of δ/T .

We now consider how the current–voltage characteristics can be extended to the threshold current. The appearance of a singularity of the $(1 - I/I_0)^{-1} \equiv \beta^{-1}$ type in $\bar{V}(I)$ is due to the integration of the function $f_R(\varepsilon)$ which, neglecting the ε -dependence of δ , is proportional to $\exp(-\varepsilon\beta/T)$. In eq. (4.51) this singularity follows from the asymptotic relation

$$\Phi(\beta, \beta) \approx \beta(\delta/T)^{1/2} = (1 - I/I_0)(\delta/T)^{1/2}. \quad (4.52)$$

To remove this singularity, one has to allow for ε -dependence of δ . Making use of eq. (4.27), we introduce an auxiliary function

$$V_{\text{exp}}(U) \equiv \int_0^\infty \frac{d\varepsilon}{T} \exp\left[-\frac{U_c}{T} - \int_0^\varepsilon \left(1 - \frac{U}{\delta(\varepsilon')}\right) \frac{d\varepsilon'}{T}\right],$$

which determines the contribution of $f_R(\varepsilon)$ to the junction voltage at values of U close to δ .

This expression is valid up to the threshold $U = \delta$, where

$$V_{\text{exp}}(\delta) \approx 4\pi^{1/2} [U_c/T \ln(U_c/T)]^{1/2} \exp(-U_c/T),$$

which is independent of δ . For $1 \gg (1 - U/\delta) \gg (T/U_c)^{1/2}$ we have

$$V_{\text{exp}}(U) \approx (1 - U/\delta)^{-1} \exp(-U_c/T).$$

Comparison with eqs. (4.51) and (4.52) shows that, in order to continue eq. (4.51) to the near-threshold region, we must replace $\Phi^{-1}(\beta, \beta) \exp(-U_c/T)$ by $(T/\delta)^{1/2} V_{\text{exp}}(U)$, and substitute $U = \delta$,

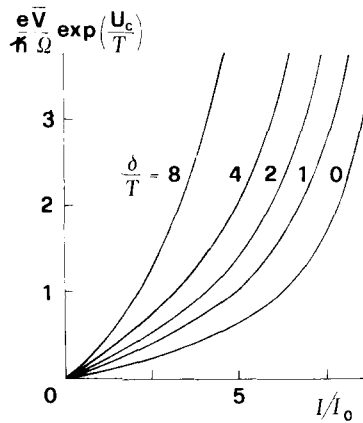


Fig. 10. Fluctuation-induced $\bar{V}(I)$ below threshold.

$\alpha = 2$ and $\beta = 0$ in the remaining functions. The result is

$$\bar{V} = \frac{e}{\Omega} \left(\frac{T}{\delta} \right)^{1/2} \frac{\sinh(\delta/2T) A(2\delta/T) V_{\text{exp}}(U)}{\Phi(2, 2)\Phi(2, 0)\Phi(0, 2)}. \quad (4.53)$$

This expression together with eq. (4.51) determines $\bar{V}(I)$ in the whole region below threshold.

4.6. Fluctuation-induced current–voltage characteristics above threshold

In this subsection, we examine how the solution of eq. (4.42) has to be modified in the region $U > \delta$. Neglecting the dependence of δ on ε , the equation

$$f_{\text{R}}(\varepsilon) = \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon' - U) f_{\text{R}}(\varepsilon') d\varepsilon',$$

for sufficiently large $\varepsilon \gg U$, δ and T , has the two solutions $f_{\text{R}} \propto \text{const.}$ and $f_{\text{R}} \propto \exp[-\varepsilon(1 - U/\delta)/T]$, which correspond to the zeros of $1 - g^-(\lambda)$ at $\lambda = 0$ and $\lambda = \lambda_0 \equiv i(U/\delta - 1)$. It was shown above that an exponential solution for $f_{\text{R}}(\varepsilon)$ taking the energy dependence of $\delta(\varepsilon)$ into account, introduces a factor $V_{\text{exp}}(U)$ in the expression for the voltage. At the same time, the solution $f = \text{const.}$ corresponds to a nonnormalizable distribution and must be discarded. When applied to the function $\varphi(\lambda)$ this means that $\varphi^+(\lambda)$ should have a pole at λ_0 and be finite at $\lambda = 0$. The factorization (4.46) does not satisfy either condition.

The point is that the inequalities $\beta < 0$ and $\alpha > 2$ hold for $U > \delta$, so that expressions (4.48) and (4.49) are insufficient to determine $G^+(0)$ and $G^-(0)$. To continue $G^+(\lambda)$ to the significant regions of λ , it is necessary to return to the original definition (2.51). This results in

$$G^-(\lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)\Phi(\beta - 2i\lambda, \beta)}{\Phi(\alpha - 2i\lambda, \alpha)[1 - g^-(\lambda)]}, \quad -\frac{1}{2}\beta > \text{Im } \lambda > -\frac{1}{2}, \quad (4.54)$$

$$G^-(\lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)\Phi(\alpha - 2i\lambda, \alpha)}{\Phi(\beta - 2i\lambda, \beta)[1 - g^-(\lambda)]}, \quad -\frac{1}{2} > \text{Im } \lambda > -\frac{1}{2}\alpha. \quad (4.55)$$

Assuming that $\varphi^+(\lambda) \propto G^+(\lambda)/(\lambda + i)$, as follows from eq. (4.45), the function $\varphi^-(\lambda)$ will be finite at $\lambda = \lambda_0$. On the other hand, in the vicinity of $\lambda = 0$ the factor $G^+(\lambda)$ is given by eq. (4.54) and has a pole singularity. Therefore, it is necessary to choose a different factorization of eq. (4.42) instead of eq. (4.46), such that $\varphi^+(\lambda)$ obtains a pole at $\lambda = \lambda_0$ and the pole of $G^+(\lambda)$ is eliminated. These requirements are satisfied if we write, using the boundary condition (4.44),

$$\frac{\varphi^+(\lambda)}{G^+(\lambda)} = -\varphi^-(\lambda)G^-(\lambda) = \frac{iU}{\delta} \frac{\Omega \sinh(U/2T)\lambda G^-(-i) \exp(-U_c/T)}{\pi(\lambda + i)(\lambda - \lambda_0)}.$$

Thus, we have found the solution of eq. (4.42) for the region $U > \delta$. It is clear from the preceding arguments that, to calculate the voltage, we have to match this solution with the function $f_{\text{R}}(\varepsilon)$ for

$\varepsilon \gg U, \delta, T$. To this end it suffices to write

$$f_R(\varepsilon) = \frac{C_1}{T} \exp\left[-\frac{U_c}{T} + \int_0^\varepsilon \left(\frac{U}{\delta(\varepsilon')} - 1\right) \frac{d\varepsilon'}{T}\right], \quad (4.56)$$

and to note that if the ε -dependence of δ is neglected we obtain a pole of the form

$$\varphi^+(\lambda) \approx iC_1 \exp(-U_c/T) / (\lambda - \lambda_0), \quad |\lambda - \lambda_0| \ll 1.$$

The coefficient C_1 is determined by the residue of $\varphi^+(\lambda)$ at the point λ_0 , after which integration of eq. (4.56) over ε yields the voltage $\bar{V} = \pi\hbar C_1 V_{\text{exp}}(U)/e$. Since $\beta < 0$, we have $\text{Im } \lambda_0 = -\beta > -\beta/2$, and the residue must be determined by using eq. (4.48) for $G^+(\lambda)$ and eq. (4.55) for $G^-(\lambda)$. The result is

$$\bar{V} = (\hbar\Omega/e)B(\delta, U)V_{\text{exp}}(U), \quad (4.57)$$

$$B(\delta, U) = \frac{1}{2} \exp(U/2T) \frac{|\beta| \Phi(2\delta, 1, 1) \Phi(2\delta, 2\alpha - 3, 1) \Phi(\beta, \alpha)}{\Phi(3\alpha - 4, \alpha) \Phi(\alpha, \beta) \Phi(\beta, \beta)}. \quad (4.58)$$

In the limiting cases we have near threshold

$$B \approx \frac{1}{2}, \quad \delta \ll T, \quad B \approx \frac{|\beta| \exp(U/2T)}{2A^{1/2}(\delta\beta^2/T)}, \quad \delta \gg T.$$

Expression (4.57) is matched to eq. (4.53) at $U = \delta$. The relation (4.57) was obtained, with exponential accuracy, by Vollmer and Risken [18], but with an incorrect prefactor.

The transition, near $I = I_1$, of the expression for the current–voltage characteristics from $\bar{V}(I)$ to $V(I)$ can be easily understood if one takes account of the contribution to the normalization of the function $f(\varepsilon)$ from the positive-energy particles.

4.7. *Cusp in the current–voltage characteristics at threshold*

The fact that the expressions for $\bar{V}(I)$ are different at $I < I_0$ and at $I > I_0$ suggests that the current–voltage characteristics of a Josephson junction has a singularity at $I = I_0$. In order to calculate the magnitude of this singularity given by a jump of the logarithmic derivative, we note that, also at $I < I_0$ ($U < \delta$), the principal part of $V(I)$, near threshold, is given by an expression such as (4.57), while the function $B(\delta, U)$ is given by the residue of $\varphi^+(\lambda)$ at $\lambda = -i\beta$. Since below threshold we have $\beta > 0$, the residue has to be determined using expression (4.54) for $G^+(\lambda)$ and expression (4.49) for $G^-(\lambda)$. We then obtain

$$B(\delta, U) = (T/U) \sinh(U/2T) \frac{\Phi(2\delta, 1, 1) \Phi(2\delta, 2\alpha - 3, 1) \Phi(\beta, \beta)}{\beta \Phi(\alpha, \beta) \Phi(\beta, \alpha)}, \quad U < \delta. \quad (4.59)$$

Since $V_{\text{exp}}(U)$ has no singularity at $U = \delta$, we differentiate eqs. (4.58) and (4.59) to find the jump in its derivative. It is important here that β reverses sign at $U = \delta$. The calculations show that only the

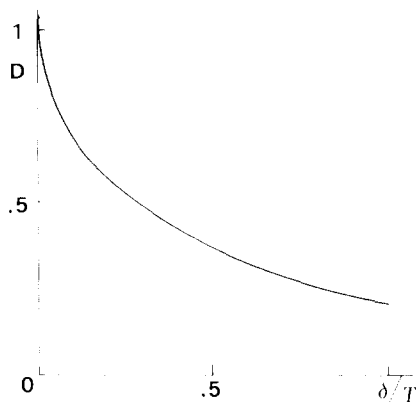


Fig. 11. Discontinuity of the logarithmic derivative of $\bar{V}(I)$ at threshold $I = I_0$.

function $\Phi(3\alpha - 4, \alpha)$ contributes to this jump, so that

$$(d \ln \bar{V} / d \ln I)_+ - (d \ln \bar{V} / d \ln I)_- = -D(\delta/T), \quad (4.60)$$

$$D(\Delta) = \frac{\partial \ln \Phi(3\alpha - 4, \alpha)}{\partial \alpha} \Big|_{\alpha=2} = \Delta \int_0^{\pi/2} \frac{dx}{\pi} \frac{3 \tan^2 x + 1}{\exp(\Delta / \cos^2 x) - 1}.$$

In the limiting cases we have

$$D(\Delta) \approx 1 + \frac{3}{2} \zeta(\frac{1}{2})(\Delta/\pi)^{1/2}, \quad \Delta \ll 1, \quad D(\Delta) \approx \frac{1}{2}(\pi/\Delta)^{1/2} \exp(-\Delta), \quad \Delta \gg 1.$$

A plot of $D(\Delta)$ is shown in fig. 11. It follows from these results that the logarithmic derivative of $\bar{V}(I)$, of which the order of magnitude is $(U_0/T)^{1/2} \gg 1$, has a negative jump of order unity at $I = I_0$.

4.8. Lifetime of the running state

The aim of this section is to derive the complete expression for the lifetime $\tau(I)$ of the running state, which had been calculated above in an exponential approximation [see eq. (4.32)]. We assume that all potential wells are empty, so that the distribution function decays slowly due to the fluctuation-induced retrapping of particles running high above the potential barriers, $F(\varphi, \dot{\varphi}, t) = F(\varphi, \dot{\varphi}) \exp(-t/\tau)$. It is convenient to change from the variables $\varphi, \dot{\varphi}$ to φ and ε , according to eq. (4.10). The stationary distribution function is governed by the equation

$$\frac{F(\varphi, \dot{\varphi})}{\tau \dot{\varphi}(\varphi, \varepsilon)} + \frac{\partial F}{\dot{\varphi}(\varphi, \varepsilon) \partial \varphi} = \frac{1}{RC\Omega} \frac{\partial}{\partial \varphi} \dot{\varphi}(\varphi, \varepsilon) \left(T \frac{\partial F}{\partial \varepsilon} + F \right) + \frac{\hbar I}{2e} \frac{\partial F}{\partial \varepsilon}. \quad (4.61)$$

Neglecting its first term, this equation can be transformed either into the integral equation (4.25) for energies $\varepsilon \gg T$ or into the system of integral equations (4.37) at $|\varepsilon| \sim T$, where reflections from the potential barriers become important. Solutions of these equations have already been described in detail and will be used later.

Now we consider the vicinity of the energy $\tilde{\varepsilon} \equiv \varepsilon(I)$, which gives the major contribution to the flux of particles. Equation (4.26) demonstrates that, in this region of energy ε , the function F varies slowly, since both roots $\lambda_{1,2}$ of the equation are close to zero. This fact helps us to derive a simplified equation for F . We begin by integrating eq. (4.61) over φ in the interval $(0, \pi)$. Integrating the term $\partial F/\partial \varphi$ gives a vanishing contribution due to periodicity of the function $F(\varphi, \varepsilon)$. In the other terms one can neglect the φ -dependence of F . The result is

$$f(\varepsilon)/\mu = (d/d\varepsilon)\{\delta(\varepsilon)T df(\varepsilon)/d\varepsilon + [\delta(\varepsilon) - U]f(\varepsilon)\}, \quad (4.62)$$

where $f(\varepsilon)$ is a function of the single argument ε , introduced instead of $F(\varphi, \varepsilon)$. Furthermore, $\mu \equiv (\Omega\tau/2\pi^2)s(e\tilde{\varepsilon}/\hbar I_c)$ is a small parameter of the problem and $s(y)$ is defined by eq. (4.12).

Neglecting the term f/μ , we obtain

$$f(\varepsilon) = \exp\left[-\int_0^\varepsilon \frac{d\varepsilon'}{T} \left(1 - \frac{U}{\delta(\varepsilon')}\right)\right] + \frac{C_1}{\delta(\varepsilon) - U}, \quad (4.63)$$

where C_1 is a constant.

It should be emphasized that the first term in this expression coincides with the solution of the integral equation (4.25), while the second term replaces the solution $f(\varepsilon) = \text{constant}$. An estimate of corrections to this solution shows that it is valid in a broad region of ε with the exception of a narrow vicinity of $\varepsilon = \tilde{\varepsilon}$, where this term is singular. The applicability of the first term of eq. (4.63) is justified by the solution of the integral equation (4.25). The value of C_1 can be found by solving eq. (4.63) with the boundary condition $f(\varepsilon) \rightarrow 0$, $\varepsilon \rightarrow \infty$. In a small neighborhood of the energy $\tilde{\varepsilon} \equiv \varepsilon(I)$ which satisfies eq. (4.19) we can use the expansion

$$\delta(\varepsilon) - U \approx \Delta(\varepsilon - \tilde{\varepsilon}), \quad \Delta \equiv \delta'(\tilde{\varepsilon}) = \frac{2\pi^2}{RC\Omega s(\tilde{\varepsilon}e/\hbar I_c)},$$

where the function $s(y)$ is defined by eq. (4.12), and we replace $\delta(\varepsilon)$ by $\delta(\tilde{\varepsilon}) = U$. Substituting $f = \psi \exp[-\Delta(\varepsilon - \tilde{\varepsilon})^2/4UT]$ and switching to the variable $x \equiv (\varepsilon - \tilde{\varepsilon})(\Delta/UT)^{1/2}$ leads to the equation

$$\psi'' + \left(\frac{1}{2} + \mu/\Delta - \frac{1}{4}x^2\right)\psi = 0.$$

This is the Schrödinger equation for an oscillator whose energy exceeds the ground-state energy by $\mu/\Delta \ll 1$. The function $f(\varepsilon)$ is normalizable if $\psi(x)$ decays as $x \rightarrow \infty$. A nonzero value of μ/Δ then means that $\psi(x)$ contains an exponentially growing term at large negative x ,

$$\psi = \exp(-\frac{1}{4}x^2) + [(2\pi)^{1/2}\mu/\Delta x] \exp(\frac{1}{4}x^2), \quad x < 0, \quad |x| \gg 1.$$

Comparison of this expression with eq. (4.63) yields the constant C_1 . In the region $\tilde{\varepsilon} \gg \varepsilon \gg T$ we can now set $\delta(\varepsilon) \approx \delta(0) \equiv \delta$, but need not yet allow for the change in $f(\varepsilon)$ due to reflections of the particles from the potential barriers. Here, we find

$$f(\varepsilon) = \exp(-\beta\varepsilon/T) + \frac{\mu}{\beta I_0} \left(\frac{2\pi UT}{\delta\Delta}\right)^{1/2} \exp\left[-\int_0^{\tilde{\varepsilon}} \frac{d\varepsilon}{T} \left(1 - \frac{U}{\delta(\varepsilon)}\right)\right], \quad (4.64)$$

with $\beta \equiv 1 - U/\delta$. We will use this expression below as a boundary condition for the function describing the distribution of particles at energies $\varepsilon \sim T$ near the top of the barrier. Here we have $\beta < 0$, so the first term in eq. (4.64) increases exponentially with the energy ε . Expression (4.64) contains an unknown parameter μ , which can be calculated by matching this solution to the solution of the integral equations (4.37). These equations take reflections of particles from the potential barriers into account and are valid for $\varepsilon \ll \hbar I_0/e$. As already shown, these equations are equivalent to the Wiener–Hopf equation

$$\varphi^-(\lambda) = -G(\lambda)\varphi^+(\lambda), \quad (4.65)$$

$$G(\lambda) \equiv \frac{1 - g^+(\lambda)g^-(\lambda)}{[1 - g^+(\lambda)][1 - g^-(\lambda)]}, \quad g^\pm(\lambda) \equiv \exp\{-[\delta\lambda^2 + i\lambda(\delta \pm U)]/T\}. \quad (4.66)$$

Under which conditions should eq. (4.65) now be solved? The presence of a certain number of particles with the Boltzmann distribution, $f(\varepsilon) \propto \exp(-\varepsilon/T)$, would correspond to a pole of $\varphi^-(\lambda)$ at point $\lambda = -i$. In the absence of such particles, $\varphi^-(\lambda)$ is finite, and by virtue of the condition $G(-i) = 0$ the condition

$$\varphi^+(-i) = 0 \quad (4.67)$$

holds. In its turn, expression (4.64) shows that $\varphi^+(\lambda)$ has poles at the points $\lambda = -i\beta$ and $\lambda = 0$. If we define $\varphi^+(\lambda)$ in such a way that the residue at $\lambda = -i\beta$ corresponds to the amplitude of the first term in eq. (4.64),

$$\varphi^+(\lambda) \approx iT/(\lambda + i\beta), \quad |\lambda + i| \ll 1, \quad (4.68)$$

then the residue of $\varphi^+(\lambda)$ at $\lambda = 0$ gives the unknown parameter μ . With the boundary conditions (4.67) and (4.68), we obtain the following solution of eq. (4.65):

$$\varphi^+(\lambda)/G^-(\lambda) = G^-(\lambda)\varphi^-(\lambda) = iT/(\lambda + i\beta)G^+(-i\beta), \quad (4.69)$$

where the $G^\pm(\lambda)$ are defined by eq. (2.50). Condition (4.67) then indeed holds by virtue of the equality $G^+(-i) = 0$, and the validity of the condition (4.68) is obvious from eq. (4.69).

Formally, the problem has now been solved. Taking the inequality $\beta < 0$ into account, we use expression (4.54) for $G^+(\lambda)$,

$$G^+(\lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)\Phi(\beta - 2i\lambda, \beta)}{\Phi(\alpha - 2i\lambda, \alpha)[1 - g^-(\lambda)]}.$$

Note that $G^+(\lambda)$ has a pole at $\lambda = 0$. To calculate the function $G^+(-i\beta)$, we use the expression

$$G^+(-i\beta) = \frac{\Phi(2\delta, 1 - 2\beta, 1)}{\Phi(\alpha - 2\beta, \alpha)\Phi(\beta, \beta)}.$$

Substituting it into eq. (4.69) and taking the limit $\lambda \rightarrow 0$, we find

$$\frac{i\lambda\varphi^+(\lambda)}{T} = \frac{T}{\beta^2\delta} \frac{\Phi(2\delta, 1 - 2\beta, \alpha)\Phi^2(\beta, \beta)\Phi(\alpha - 2\beta, \alpha)}{\Phi(2\delta, 1, 1)\Phi(\alpha, \alpha)}.$$

The right-hand side of this expression should coincide with the second term in eq. (4.64) if we replace β by $|\beta|$. Making use of the relationship between μ and τ , we write the final result for τ as a function of the reduced current, $\eta \equiv I/I_0$, in the form of the parametric relations

$$\begin{aligned} \tau(\eta) &= (2\pi/\Omega)(4\pi\hbar I_c/eT)^{1/2} b(\eta) \exp[(\hbar I_c/eT)a(\eta)], \\ a(\eta) &\equiv \int_0^x [r(x)/r(y) - 1] dy, \quad r(x) = \eta, \\ b(\eta) &= \frac{\eta^{1/2}(\eta-1)\Phi(2\delta, 2\eta-1, 1)\Phi(1+\eta, 1+\eta)}{s^{1/2}(x)\Phi(2\delta, 1, 1)\Phi^2(1-\eta, 1-\eta)\Phi(3\eta-1, 1+\eta)}. \end{aligned} \quad (4.70)$$

The functions $r(y)$ and $s(y)$ are defined by eqs. (4.16) and (4.12), the functions $\Phi(\mu, \nu)$ and $\Phi(2\delta, \mu, \nu)$ are defined by eq. (4.50).

4.9. Lifetime of a zero-voltage state

So long as the current I through the junction is less than the critical current I_c , the superconducting state of the junction corresponds to a certain minimum of its potential energy. For the junction to switch into the resistive state or into one of the neighboring potential wells it is necessary to surmount a potential barrier. Accordingly, the lifetime τ of the superconducting state and the probability w_n of a transition with a phase flip by $2\pi n$ are determined by activation processes.

Assume that initially the junction state corresponds to a phase distribution near one of the minima of the potential $U(\varphi)$. The activation processes flip the phase over to the neighboring minima and destroy thereby the superconductivity. In this situation one of the potential minima is singled out, so that the translational symmetry used for the problem in the preceding subsection is violated. Instead of the equations for the functions $f_{R,L}(\varepsilon)$ we must write therefore the following infinite system of equations:

$$\begin{aligned} f_{Rn}(\varepsilon) &= \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon' - U)[f_{Rn-1}(\varepsilon')\theta(\varepsilon') + f_{Ln}(\varepsilon')\theta(-\varepsilon')] d\varepsilon', \\ f_{Ln}(\varepsilon) &= \int_{-\infty}^{\infty} g(\varepsilon - \varepsilon' + U)[f_{Rn}(\varepsilon')\theta(-\varepsilon') + f_{Ln+1}(\varepsilon')\theta(\varepsilon')] d\varepsilon', \end{aligned} \quad (4.71)$$

where n denotes the site of a minimum. The solution of these equations must satisfy the boundary conditions

$$f_{R,L0}(\varepsilon) \approx (\Omega/2\pi T) \exp[-(1/T)(U_c + \varepsilon \mp \frac{1}{2}U)], \quad f_{R,Ln}(\varepsilon) \approx \text{const.}, \quad -\varepsilon \gg T,$$

assuming that initially the particle was placed into the minimum $n=0$.

Carrying out the transformation (4.40), we obtain from eqs. (4.71) the system

$$\begin{aligned} \varphi_{Rn}^+(\lambda) + \varphi_{Rn}^-(\lambda) &= g^-(\lambda)[\varphi_{Rn-1}^+(\lambda) + \varphi_{Ln}^-(\lambda)] \\ \varphi_{Ln}^+(\lambda) + \varphi_{Ln}^-(\lambda) &= g^+(\lambda)[\varphi_{Ln+1}^+(\lambda) + \varphi_{Rn}^-(\lambda)]. \end{aligned}$$

Solving these equations for φ_{Rn}^- and $\varphi_{L,n}^-$ (it is just these functions that contain no shift with respect to n), and taking into account the expression for the decay rate,

$$\tau^{-1} = \int_0^{\infty} [f_{R0}(\varepsilon) - f_{L,1}(\varepsilon) + f_{L,0}(\varepsilon) - f_{R,-1}(\varepsilon)] d\varepsilon,$$

we consider the difference $\varphi_n = \varphi_{Rn}^- - \varphi_{L,n+1}^-$. The new functions satisfy the system of equations

$$(1 - g^+ g^-) \varphi_n^- = g^- \varphi_{n-1}^- - (1 + g^+ g^-) \varphi_n^+ + g^+ \varphi_{n+1}^+, \quad 1/\tau = \varphi_0^+(0) - \varphi_{-1}^-(0). \quad (4.72)$$

The boundary conditions for $\varphi_n(\lambda)$ are

$$\begin{aligned} \varphi_0^-(\lambda) &\approx -\frac{i\Omega}{2\pi} \frac{\exp[-(U_c - U/2)/T]}{\lambda + i}, \quad |\lambda + i| \ll 1, \\ \varphi_{-1}^+(\lambda) &\approx \frac{i\Omega}{2\pi} \frac{\exp[-(U_c + U/2)/T]}{\lambda + i}, \quad |\lambda + i| \ll 1, \end{aligned} \quad (4.73)$$

while the remaining $\varphi_n(\lambda)$ have no poles at $\lambda = -i$.

To solve the system (4.72), we introduce the functions

$$\varphi^{\pm}(k, \lambda) \equiv \sum_{n=-\infty}^{\infty} \varphi_n^{\pm} \exp[-ik(n + \frac{1}{2})],$$

which satisfy the equation

$$\varphi^+(k, \lambda) = -H(k, \lambda) \varphi^-(k, \lambda), \quad (4.74)$$

$$H(k, \lambda) \equiv (1 - h^+ h^-) / (1 - h^+)(1 - h^-),$$

$$h^{\pm}(k, \lambda) \equiv g^{\pm}(\lambda) \exp(\pm ik\lambda) = \exp\{-(\delta/T)[\lambda^2 + i\lambda(1 \pm U/\delta)] \pm ik\lambda\}.$$

From the conditions (4.73) we obtain for $\varphi^-(k, \lambda)$ the boundary condition

$$\varphi^-(k, \lambda) \approx -\frac{\Omega}{\pi} \frac{\sin(\frac{1}{2}k + iU/2T) \exp(-U_c/T)}{\lambda + i}, \quad |\lambda + i| \ll 1,$$

where τ is connected with $\varphi(k, \lambda)$ by the relation

$$\frac{1}{\tau} = i \int_{-\pi}^{\pi} \frac{dk}{\pi} \sin(\frac{1}{2}k) \varphi^+(k, 0).$$

Expressing the kernel of eq. (4.74) as a product of $H^+(k, \lambda)$ and $H^-(k, \lambda)$ [in full analogy with expressions (4.42) and (4.45)], we obtain the solution for $\varphi^+(k, \lambda)$ in the form

$$\varphi^+(k, \lambda) = \frac{\Omega \sin(\frac{1}{2}k + iU/2T) H^+(k, \lambda) H^-(k, -i) \exp(-U_c/T)}{\pi(\lambda + i)}.$$

It is convenient to write the decay rate in the Arrhenius form,

$$1/\tau = (\Omega/2\pi)\mathcal{A}(\delta, U) \exp[-(1/T)(U_c - \frac{1}{2}U)], \quad (4.75)$$

where the exponent contains the height of the lowest barrier $U_c - \frac{1}{2}U$. The preexponential factor \mathcal{A} is then given by the expression

$$\mathcal{A}(\delta, U) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} w(k, \delta, U), \quad (4.76)$$

$$w(k, \delta, U) = 4 \sin(\frac{1}{2}k) \sin(\frac{1}{2}k + iU/2T) H^+(k, 0) H^-(k, -i) \exp(-U/2T). \quad (4.77)$$

Just like the current-voltage characteristics $\bar{V}(I)$, the function w is given by different expressions for $U < \delta$ and for $U > \delta$. To find the corresponding expressions we write down the factors $H^{\pm}(k, \lambda)$ in different regions for λ

$$H^+(k, \lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)}{\Phi_1(k, \alpha - 2i\lambda, \alpha)\Phi_1(-k, \beta - 2i\lambda, \beta)}, \quad \text{Im } \lambda > -\frac{1}{2}\beta, \quad (4.78)$$

$$H^+(k, \lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)\Phi_1(-k, \beta - 2i\lambda, \beta)}{\Phi_1(k, \alpha - 2i\lambda, \alpha)[1 - h^-(k, \lambda)]}, \quad -\frac{1}{2}\beta > \text{Im } \lambda > -\frac{1}{2}, \quad (4.79)$$

$$H^-(k, \lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)}{\Phi_1(k, \alpha - 2i\lambda, \alpha)\Phi_1(-k, \beta - 2i\lambda, \beta)}, \quad \text{Im } \lambda < -\frac{1}{2}\alpha, \quad (4.80)$$

$$H^-(k, \lambda) = \frac{\Phi(2\delta, 1 - 2i\lambda, 1)\Phi_1(k, \alpha - 2i\lambda, \alpha)}{\Phi_1(-k, \beta - 2i\lambda, \beta)[1 - h^+(k, \lambda)]}, \quad -\frac{1}{2} > \text{Im } \lambda > -\frac{1}{2}\alpha, \quad (4.81)$$

$$\alpha \equiv 1 + U/\delta \equiv 1 + I/I_0, \quad \beta \equiv 1 - U/\delta \equiv 1 - I/I_0,$$

and the new function

$$\Phi_1(k, \mu, \nu) = \exp\left(\int_0^{\pi/2} \frac{dx}{\pi} \ln\{1 - \exp[-(\delta/4T)(\mu^2 \tan^2 x + \nu^2) + ik]\}\right)$$

has been introduced. In accordance with eq. (4.77), we obtain [38]

$$w(k, \delta, U) = \frac{4A(2\delta/T) \sin(\frac{1}{2}k) \sin(\frac{1}{2}k + iU/2T) \exp(-U/2T)}{\Phi_1(k, \alpha, \alpha)\Phi_1(-k, \beta, \beta)\Phi_1(k, \beta, \alpha)\Phi_1(-k, \alpha, \beta)}, \quad U < \delta, \quad (4.82)$$

$$w(k, \delta, U) = A(2\delta/T) \frac{\Phi_1(k, \beta, \alpha)\Phi_1(-k, \beta, \beta)}{\Phi_1(k, \alpha, \alpha)\Phi_1(-k, \alpha, \beta)}, \quad U > \delta, \quad (4.83)$$

where expressions (4.78) and (4.80) have been used for $U < \delta$ and (4.79) and (4.81) for $U > \delta$.

The function $w(k, \delta, U)$ vanishes at $k = 0$ if $U < \delta$ and differs from zero if $U > \delta$. The reason is that for $U > \delta$ a particle that leaves a potential well will be trapped in one of the wells after some time. For

$U < \delta$, on the contrary, there is a finite probability for the particle to go into a state of accelerated motion, since we have neglected the energy dependence of δ in the scheme developed here.

In certain cases it may be more convenient to use the expression

$$w(k, \delta, U) = 4A(2\delta/T) \sin(\frac{1}{2}k) \sin(\frac{1}{2}k + iU/2T) \exp[\Sigma(k, \delta, U) - U/2T], \quad (4.84)$$

$$\begin{aligned} \Sigma(k, \delta, U) = & \sum_{n=1}^{\infty} \frac{1}{n} \cos[n(k + iU/2T)] \\ & \times [\operatorname{erfc}(\frac{1}{2}\alpha\sqrt{\delta n/T}) \exp(nU/2T) + \operatorname{erfc}(\frac{1}{2}\beta\sqrt{\delta n/T}) \exp(-nU/2T)], \end{aligned} \quad (4.85)$$

being valid for both $U < \delta$ and $U > \delta$, where $\operatorname{erfc}(x)$ is the complementary error function,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} \exp(-t^2) dt.$$

The existence of such an expression shows that, in contrast to the function $\bar{V}(I)$, the activation probabilities have no singularities at threshold, $U = \delta$. With the use of the function $w(k, \delta, U)$ and eq. (4.76) one can calculate the preexponential factor \mathcal{A} , which enters the expression for the lifetime of a zero-voltage state (4.75). Moreover, the same function $w(k, \delta, U)$ determines the partial probabilities of the phase jumps by an integer number of 2π and the relative probability for the Josephson junction to switch into the running state at such events.

4.10. Partial probabilities of the phase jumps

The probability of the transition of the particle to the n th minimum, i.e., the probability of a phase flip by $2\pi n$, is given by

$$w_n(\delta, U) = - \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikn} w(k, \delta, U) / \int_{-\pi}^{\pi} \frac{dk}{2\pi} w(k, \delta, U). \quad (4.86)$$

The sum of w_n over n is equal to unity for $U < \delta$, since in this case $w(0, \delta, U) = 0$. It is less than unity for $U > \delta$, when a probability $P(\delta, U)$ appears that the particle will not be trapped in any of the potential wells,

$$P(\delta, U) = 1 - \sum_{n \neq 0} w_n(\delta, U) = w(0, \delta, U) \left(\int_{-\pi}^{\pi} \frac{dk}{2\pi} w(k, \delta, U) \right)^{-1}. \quad (4.87)$$

The dependence of w_1 , w_2 and P on the reduced current I/I_0 for different δ/T is shown in figs. 12–14. In limiting cases it is possible to obtain simpler expressions for these quantities.

Most interesting is the case $\delta \ll T$. To find $w(k, \delta, U)$ in this case we use the asymptotic relations

$$\Phi_1(k, \mu, \nu) = (\delta/4T)^{1/2} [\mu + (\nu^2 - 4ikT/\delta)^{1/2}], \quad \delta/T, k \ll 1, \quad (4.88)$$

$$\Phi_1(k, \mu, \nu) = (1 - e^{ik})^{1/2} \left[1 + \sum_{n=1}^{\infty} \left(\frac{\mu^2 \delta}{4\pi n T} \right)^{1/2} e^{ink} \right], \quad \delta/T \ll k. \quad (4.89)$$

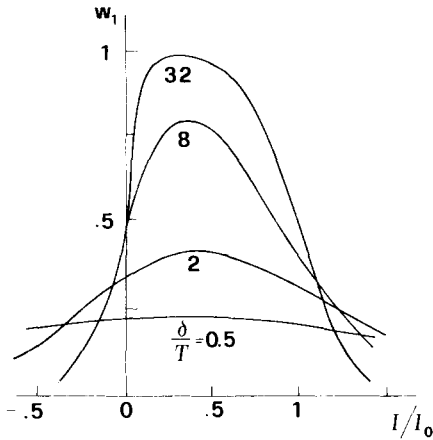


Fig. 12. Partial probability of a 2π phase flip.

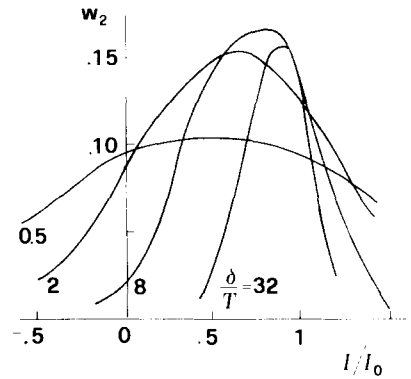


Fig. 13. Partial probability of a 4π phase flip.

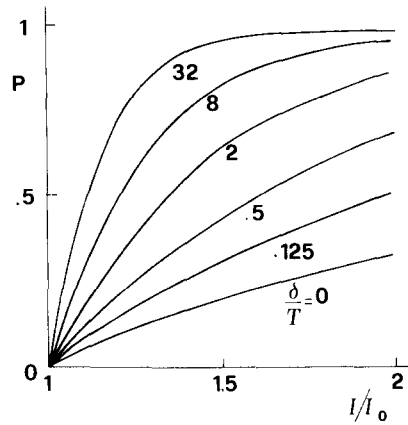


Fig. 14. Partial probability of activation of a junction into the resistive state.

Calculating the denominators of eqs. (4.86) and (4.87) we can assume that $w = 2\delta$. Taking into account eq. (4.88), we obtain

$$P = \frac{U - \delta}{U + \delta} = \frac{I - I_0}{I + I_0}, \quad \frac{\delta}{T} \ll 1, \quad I > I_0.$$

Calculating $w_n(\delta, U)$ for $n \sim 1$, the entire interval $(-\pi, \pi)$ contributes to the integral (4.86), so that the use of the asymptotic form (4.89) yields

$$w_n = (\delta/\pi n T)^{1/2} + (\delta/\pi T) \ln(n\delta/\pi T), \quad T/\delta \gg n \geq 1, \quad (4.90)$$

where the second term takes into account the properties of the function \mathcal{P} defined below. It can be seen that, in this limit, w_n is independent of the current through the junction.

If $n \gg 1$, the main contribution to the integral (4.86) comes from $k \ll 1$. The asymptotic form (4.88) must therefore be used and the integral must be extended over the entire k axis. Shifting the integration

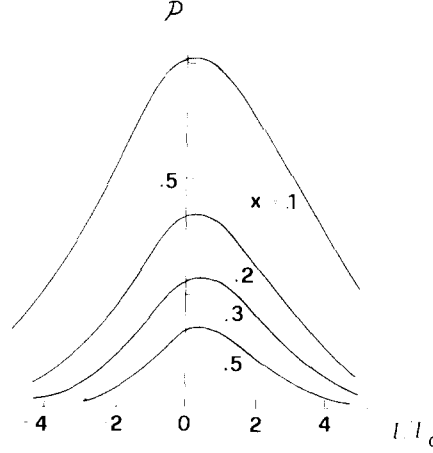


Fig. 15. Function \mathcal{P} entering eq. (4.91) for the relative probability of the phase jumps in extremely underdamped regime.

contour into the upper half-plane of complex k , in such a way that it passes along the imaginary axis and encloses the branch point $k = i\beta^2\delta/4T$, we obtain

$$w_n(\delta, U) = (\delta/T)\mathcal{P}(n\delta/T, U), \quad \delta/T \ll 1, \quad (4.91)$$

$$\mathcal{P}(x, U) = \exp(-\frac{1}{4}x\beta^2) \int_0^\infty \frac{dz}{\pi} \frac{z^2 \exp(-xz^2/4)}{[\alpha + (\alpha^2 + \beta^2 + z^2)^{1/2}][\beta + (\alpha^2 + \beta^2 + z^2)^{1/2}]}. \quad (4.92)$$

This expression is valid for both $U < \delta$ ($\beta > 0$) and $U > \delta$ ($\beta < 0$). To obtain $w_n(\delta, U)$ for $n < 0$ we use the identity $w_{-n}(\delta, U) = w_n(\delta, -U)$. We note that reversal of the sign of U is equivalent to the interchange $\alpha \leftrightarrow \beta$ in eq. (4.92). For small $n\delta/T$, eq. (4.91) yields the asymptotic form (4.90), so that eq. (4.91) is valid for all n . The dependence of \mathcal{P} on I/I_0 is shown in fig. 15 for $x = 0.1, 0.2, 0.3$ and 0.5 .

The relatively weak dependence of \mathcal{P} on I/I_0 at $I \sim I_0$ agrees with the fact that the asymptotic form (4.90) does not depend on I at all. For $x \geq 1$ the function \mathcal{P} depends on I exponentially, as can be seen from the factor preceding the exponential. In the high-friction limit, $\delta \gg T$, below threshold, when $\delta - U \gg (\delta T)^{1/2}$, only transitions to the nearest minima are possible with probabilities $w_{\pm} = [1 + \exp(\mp U/T)]^{-1}$. Exactly at threshold, transitions with any $n > 0$ are possible, since

$$w(k, \delta, U) = (1 - e^{-ik})^{1/2}, \quad w_n = \frac{\Gamma(n - \frac{1}{2})}{2\Gamma(\frac{1}{2})\Gamma(n + 1)}.$$

For the first three values of n we have $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{8}$, $w_3 = \frac{1}{16}$. The probability of the junction becoming resistive for $\delta \gg T$ is $P(\delta, U) = A^{1/2}[(U - \delta)^2/\delta T]$, whence it follows that for $U - \delta \gg (\delta T)^{1/2}$ each activation event makes the junction resistive, and the probability of finite phase flips is negligibly small.

4.11. Retrapping current distribution in the quantum regime

As a starting point for calculations in the quantum regime we write down the Hamiltonian

$$\mathcal{H}(\varphi, d\varphi/dt) = \frac{\hbar I_c}{e} \left[\frac{1}{4\Omega^2} \left(\frac{d\varphi}{dt} \right)^2 + \frac{1}{2}(\cos \varphi - 1) - \frac{I\varphi}{2I_c} + \frac{I_T(t)\varphi}{2I_c} \right].$$

The last term in this expression describes the interaction with the heat bath, which gives rise to the Ohmic dissipation and to fluctuation-induced transitions. The noise current $I_T(t)$ is Gaussian and obeys the Johnson–Nyquist correlation law,

$$\int_{-\infty}^{\infty} \langle I_T(0)I_T(t) \rangle \exp(i\nu t) dt = \frac{\hbar\nu}{R} [\coth(\hbar\nu/2T) - 1].$$

In analogy with the classical limit, we assume that an interaction with the heat bath as well as the tilt of the potential caused by the external current I can be treated perturbatively. Under this condition the stationary quantum states can be specified by the energy ε [see eq. (4.10)]. We consider transitions between different running states in a semiclassical approximation, when the tilt of the potential causes the energy ε to increase by $\pi\hbar I/e$ and φ increases by 2π , while under the action of the fluctuating current the junction undergoes transitions between running states with different ε .

We will proceed in the following way. First we derive an expression for the function $g(\varepsilon - \varepsilon')$, which describes the quantum probability for the particle with energy ε' to go over into the state with energy ε during the time $2\pi/\omega$ in which the phase φ is increased by 2π . The effect of the tilt will be accounted for, by using the function $f(\varepsilon)$ at the point φ and the function $f(\varepsilon' - \pi\hbar I/e)$ at the point $\varphi + 2\pi$, since $\pi\hbar I/e$ is just the energy gain per period of the potential. As the points φ and $\varphi + 2\pi$ are equivalent, they correspond to identical distribution functions. This will enable us to write down an equation for $f(\varepsilon)$. It should be noted that typically $|\varepsilon - \varepsilon'| \sim \delta(\varepsilon)$ (for simplicity we assume that $\delta \sim T \sim \hbar\Omega$) and $\delta(\varepsilon)$ depends on the absolute magnitude of the energy ε . We stress therefore, that the function g depends on the energy ε in two different respects: g is concentrated in an interval $\varepsilon - \varepsilon' \sim \delta$, but also depends on ε on a much larger scale, $\varepsilon \sim \hbar I_c/e$. In this respect the situation is similar to that described by eq. (4.27) in the classical limit, when the function $f(\varepsilon)$ is locally of the canonical form, but with the temperature $T/\lambda(\varepsilon)$ depending on the energy ε .

The derivation of the kernel $g(\varepsilon - \varepsilon')$ follows the same steps, as for quantum Brownian particles escaping out of a potential well. The only difference is that the basic trajectory $\varphi(t)$ is now a sum of the Fourier harmonics with frequencies $n\omega$. Accordingly, we must calculate probabilities $g(n\hbar\omega)$ for a particle to lose or gain the energy $n\hbar\omega$. The amplitude of the quantum transition from a state ε' to a state ε during the time t in the presence of the noise current $I_T(t)$ is given by

$$A(\varepsilon, \varepsilon', t) = \langle \varepsilon | \hat{T} \exp\left(-\frac{i\hbar}{2e} \int_0^t \hat{I}(t)\hat{\varphi}(t) dt\right) | \varepsilon' \rangle,$$

where the symbol \hat{T} means time ordering. In the first order of perturbation theory one gets for the transition probability per period $2\pi/\omega$ the following expression:

$$w(\varepsilon - \varepsilon') = \frac{2\pi}{\omega} \frac{\hbar^2}{4e^2} |\langle \varepsilon | \varphi | \varepsilon' \rangle|^2 \int_{-\infty}^{\infty} \langle \hat{I}_T(0)\hat{I}_T(t) \rangle \exp[i(\varepsilon - \varepsilon')t/\hbar] dt,$$

$$\langle \varepsilon | \varphi | \varepsilon' \rangle = \frac{2\pi}{\omega} \int_0^{\omega/2\pi} \varphi(t) \exp[i(\varepsilon - \varepsilon')t/\hbar] dt. \quad (4.93)$$

Solution of eq. (4.10) for $\varphi(t)$ yields $\varphi(t) = 2 \operatorname{am}(\Omega t/k)$, where $\operatorname{am}(z)$ is the Jacobian elliptic amplitude with the modulus $k \equiv (1 + e\varepsilon/\hbar I_c)^{-1/2}$. Expansion of $\varphi(t)$ into a trigonometric series gives

$$\varphi(t) = \omega t + 2 \sum_{n=1}^{\infty} \frac{\sin(\omega n t)}{n \cosh[np(e\varepsilon/\hbar I_c)]}, \quad (4.94)$$

where ω is given by eq. (4.12). The relative magnitudes of the partial harmonics in eq. (4.94) are determined by the function

$$p(x) \equiv \pi \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1+x \sin^2 \varphi}} \bigg/ \int_0^{\pi/2} \frac{d\varphi}{\sqrt{x + \sin^2 \varphi}}, \quad (4.95)$$

with the following asymptotics:

$$p(x) \approx \pi^2/\ln(4/x), \quad x \ll 1, \quad p(x) \approx \ln(16x), \quad x \gg 1.$$

Substituting of eq. (4.94) into eq. (4.15) yields the identity

$$\frac{\omega}{\Omega} \sum_{n=-\infty}^{\infty} \frac{1}{\cosh^2[np(e\varepsilon/\hbar I_c)]} = \frac{4}{\pi} r(e\varepsilon/\hbar I_c), \quad (4.96)$$

where the function $r(x)$ is determined by eq. (4.16). The n th term of the sum in eq. (4.96) is proportional to the transfer of energy by the quanta $n\hbar\omega$, as will be shown later. It should be noted that the energy transfer for $n=0$ is finite. From eqs. (4.94) and (4.93) we get

$$\varphi_n = \frac{i\omega}{2\pi} \int_0^t \varphi(t) \sin(n\omega t) dt = \frac{i}{n \cosh[np(e\varepsilon/\hbar I_c)]}.$$

The corresponding transition probability is then given by

$$w(\varepsilon - \varepsilon') = (\pi\delta\omega/4\Omega) \{ \coth[(\varepsilon - \varepsilon')/2T] - 1 \} \\ \times \left(\delta'(\varepsilon - \varepsilon') + \sum_{n=-\infty}^{\infty} \frac{\delta(\varepsilon - \varepsilon' - n\hbar\omega)}{n\hbar\omega \cosh^2[np(e\varepsilon/\hbar I_c)]} \right), \quad n \neq 0. \quad (4.97)$$

The term $\delta'(\varepsilon - \varepsilon')$ has its origin in the first term in the right-hand side of eq. (4.94). For the mean loss of energy from eqs. (4.97) and (4.96) we obtain

$$\int_{-\infty}^{\infty} (\varepsilon - \varepsilon') w(\varepsilon - \varepsilon') d\varepsilon' = \delta r(e\varepsilon/\hbar I_c).$$

The function $w(\varepsilon - \varepsilon')$ gives the first-order contribution to the transition probability $g(\varepsilon - \varepsilon')$.

Summation of the whole perturbation series can be done as before [see eqs. (3.9)–(3.11)], with the only modification that the integration over the energy ε must be substituted by a summation over the

quantum states. For simplicity, instead of the Fourier transformation we will use the transformation

$$g(\lambda) = \int_{-\infty}^{\infty} g(\varepsilon) \exp(\varepsilon\lambda/T) d\varepsilon.$$

The final result for $g(\lambda)$ reads

$$g(\lambda) = \exp[-(\delta/T)Z(\lambda, \hbar\Omega/2\pi T, e\varepsilon/\hbar I_c)], \quad (4.98)$$

$$Z(\lambda, y, x) = \frac{1}{8y} \sum_{n=-\infty}^{\infty} \frac{\cosh[(2\lambda + 1)ns(x)y] - \cosh[ns(x)y]}{n \sinh[ns(x)y] \cosh^2[np(x)]}. \quad (4.99)$$

The term with $n = 0$ in this expression is obtained as the limiting value of the summand at $n \rightarrow 0$.

The finite-difference equation for the distribution function,

$$f(\varepsilon) = \sum_{n=-\infty}^{\infty} g(n\hbar\omega) f(\varepsilon - n\hbar\omega - \pi\hbar I/e), \quad (4.100)$$

is equivalent to the condition of periodicity of the distribution function with respect to the shift of φ by 2π . The function $g(n\hbar\omega)$ is connected with $g(\lambda)$ via the relation

$$g(\lambda) = \sum_{n=-\infty}^{\infty} g(n\hbar\omega) \exp(-\lambda n\hbar\omega/T).$$

It is worthwhile to notice that in a tilted potential the energy spectrum is continuous, and periodicity of the unperturbed potential manifests itself only in the quantization of the energy changes $\varepsilon - \varepsilon' = n\hbar\omega$. Equation (4.100) differs from eq. (3.10) in ref. [41] in the following points:

(1) we have neglected here the effects of classical reflection at the barrier top for energies $\varepsilon < 0$ and quantum over-barrier reflection at $\varepsilon > 0$, assuming that the actual range of energies, $\varepsilon \sim \hbar I_c/e$, exceeds both $\hbar\Omega$ and T by orders of magnitude;

(2) in contrast to the integral equations in ref. [41], valid for continuous changes of the energy ε , eq. (4.100) takes into account explicit quantization of the energy transfers in a periodic potential;

(3) only one function $f(\varepsilon)$ (corresponding to f^R in ref. [41]) enters eq. (4.100), since far above the barrier tops the number of particles going upwards the potential tilt is exponentially small.

Similarly to the classical limit, an existence of the two energy scales, $\hbar\Omega \sim T$ and $\hbar I_c/e \gg \hbar\Omega$, enables us to use the ansatz

$$f(\varepsilon) = \exp\left(-\int_0^{\varepsilon} \lambda(\varepsilon') d\varepsilon'/T\right),$$

where the function $\lambda(\varepsilon)$ varies on the scale $\varepsilon \sim \hbar I_c/e$. When substituting this expression into eq. (4.100) we assume that $\lambda(\varepsilon')$ on the right-hand side is a constant, specified by the energy ε on the left-hand side of the equation. Implicitly, the parameter ε enters the kernel of eq. (4.100) through the functions $s(x)$

and $p(x)$ [see eqs. (4.12) and (4.95)]. After summation over n one gets

$$g(\lambda) \exp(-\lambda\pi\hbar I/eT) = 1.$$

Taking account of eq. (4.98) we obtain

$$\lambda\eta = Z(\lambda, y, x), \quad (4.101)$$

which determines λ as an implicit function of parameters η , y and x , $\lambda = \lambda(\eta, y, x)$. The parameters η and y are specified by the external current I , $\eta \equiv I/I_0$, and the temperature T , $y \equiv \hbar\Omega/2\pi T$. The parameter x is related to the energy ε , $x \equiv e\varepsilon/\hbar I_c$. The range of variation of x at given η and y is determined by the range of variation of the energy ε . The maximal energy ε_m corresponds to the extremal value of $f(\varepsilon)$, when $\lambda = 0$. Expanding $Z(\lambda, y, x)$ for small λ and taking into account eq. (4.96) we get $Z(\lambda, y, z) \approx \lambda r(x_m)$, $\lambda \ll 1$. From eq. (4.101) we then obtain $r(x_m) = \eta$.

The minimal energy $\varepsilon = 0$ corresponds to $x = 0$. In the same way as in section 2, we obtain

$$\ln[\Omega\tau(I)] = \frac{\hbar I_c}{eT} \int_0^{x_m} \lambda(\eta, y, x) dx. \quad (4.102)$$

Combining this equation with eq. (4.33), which expresses the parameter of the Gaussian distribution of the retrapping current by the logarithmic derivative of the retrapping time $\tau(I)$, we see that the final result is expressed in terms of the derivative $\partial\lambda(\eta, y, x)/\partial\eta$. From eq. (4.101) follows

$$\partial\lambda/\partial\eta = \lambda/[Z'_\lambda(\lambda, y, x) - \eta].$$

Hence, the half-width of the retrapping current distribution entering expression (4.34) is determined by the equations

$$\Sigma(\eta_m, y) = \left(\int_0^{x_m} \frac{\lambda(\eta_m, y, x) dx}{Z'_\lambda(\lambda, y, x) - \eta_m} \right)^{-1}, \quad (4.103)$$

$$\lambda\eta_m = \frac{1}{4}s(x)\lambda(\lambda + 1) + \frac{1}{4y} \sum_{n=1}^{\infty} \frac{\cosh[(2\lambda + 1)ns(x)y] - \cosh[ns(x)y]}{n \sinh[ns(x)y] \cosh^2[np(x)]}, \quad (4.104)$$

$$Z'_\lambda(\lambda, y, x) = \frac{1}{4}s(x) \left(2\lambda + 1 + 2 \sum_{n=1}^{\infty} \frac{\sinh[(2\lambda + 1)ns(x)y]}{\sinh[ns(x)y] \cosh^2[np(x)]} \right). \quad (4.105)$$

The results of the classical consideration follow from these equations in the limit $x \rightarrow 0$. Retaining in the right-hand side of eq. (4.104) only terms finite at $y = 0$ and taking account of eq. (4.96), we obtain the equation $\lambda\eta_m = r(x)\lambda(\lambda + 1)$. It then follows, that in the classical limit $\partial\lambda/\partial\eta_m$ does not depend on η_m .

$$\partial\lambda/\partial\eta_m = 1/r(x), \quad (4.106)$$

and we thus reproduce the result (4.35).

In the quantum regime, when $y \geq 1$, only numerical results can be obtained. In the limit of very small x the sums in eqs. (4.104) and (4.105) should be transformed into integrals by use of the relation $p(x)/s(x) \rightarrow 1/2$ as $x \rightarrow 0$. Hence, at $x = 0$ the values of λ and Z'_λ must be calculated from the equations

$$\lambda \eta_m = \frac{1}{4y} \int_0^\infty \frac{\{\cosh[(2\lambda + 1)vy] - \cosh(vy)\} dv}{v \sinh(vy) \cosh^2(v/2)}, \quad Z'_\lambda = \frac{1}{2} \int_0^\infty \frac{\sinh[(2\lambda + 1)vy] dv}{\sinh(vy) \cosh^2(v/2)}. \quad (4.107)$$

In the limit of large y and η_m eq. (4.107) reduces to $\ln(1 - 2\lambda y) = -\lambda y \eta_m$. In a first approximation we can take the argument of the logarithm to be vanishing, getting $\lambda \approx 1/2y$. The next iteration then gives $\lambda \approx (1/2y)[1 - \exp(-\frac{1}{2}\eta_m)]$. From this equation we obtain $\partial\lambda/\partial\eta_m \approx \exp(-\frac{1}{2}\eta_m)/4y$. Comparing this asymptotics with eq. (4.106), one concludes that in the quantum regime ($y \sim 1$) for large currents $\eta_m \gg 1$ the main contribution to the integral (4.103) comes from a region near its upper limit. The results of numerical calculations are presented in fig. 16 for several values of $y = \hbar\Omega/2\pi T$.

Now we consider the ultraquantum regime, when $y \gg 1$. For $\eta_m - 1 \geq 1$, in the main region of $x \geq 1$ the inequality $s(x)y \gg 1$ enables us to neglect the term $\exp[-s(x)y]$ compared to $\exp[s(x)y]$ in eqs. (4.104) and (4.105). In this way we obtain expressions, where λ enters the arguments of the summands only in the combination λy . The rescaling $\lambda y \rightarrow \lambda$ yields then the following result for the retrapping current distribution:

$$P(\eta) = \frac{\Gamma}{(2\pi)^{1/2} \sigma(\eta_m)} \exp[-\Gamma^2(\eta - \eta_m)^2/2\sigma^2(\eta_m)], \quad \Gamma \equiv \frac{\hbar I_c}{eyT} \equiv \frac{2\pi I_c}{e\Omega}, \quad (4.108)$$

$$\sigma(\eta_m) = \left(\int_0^{x_m} \frac{\lambda(\eta_m, x) dx}{Z'_\lambda(\lambda, x) - \eta_m} \right)^{-1}. \quad (4.109)$$

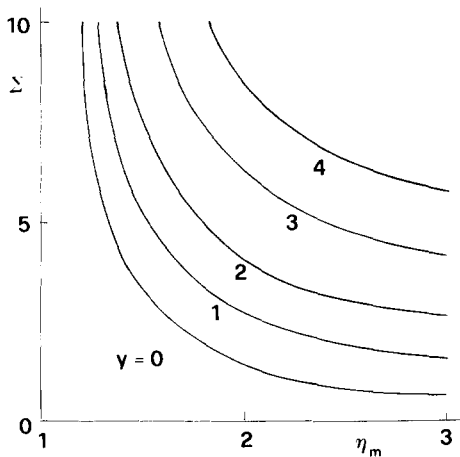


Fig. 16. Dependence of the reduced half-width $\Sigma(\eta_m, x)$ on the reduced current $\eta_m \equiv \pi I_m RC\Omega/4I_c$ at different values of the quantum parameter $y = \hbar\Omega/2\pi T$.

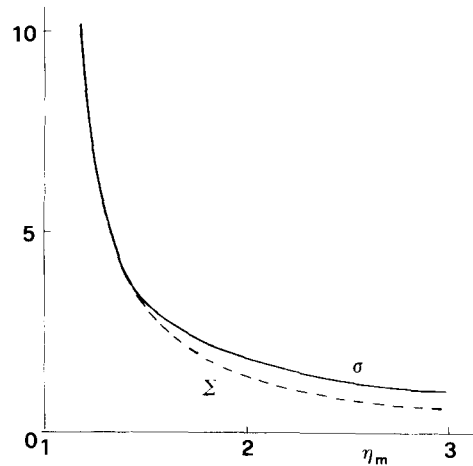


Fig. 17. Dependence of the half-width of the retrapping current distribution on the reduced current η_m in the classical (Σ) and ultraquantum (σ) regimes.

The function $\lambda(\eta_m, x)$ obeys the equation

$$\lambda\eta_m = \frac{1}{4}\lambda s(x) + \frac{1}{4} \sum_{n=1}^{\infty} \frac{\exp[2\lambda ns(x)] - 1}{n \cosh^2[np(x)]}, \quad (4.110)$$

$$Z'_\lambda(\lambda, x) = \frac{1}{4}s(x) \left(1 + 2 \sum_{n=1}^{\infty} \frac{\exp[2\lambda ns(x)]}{\cosh^2[np(x)]} \right). \quad (4.111)$$

Corrections to eq. (4.109) are due to the following two sources. For $x \sim 1$, the terms neglected in the derivation of eq. (4.110) are relatively small, i.e., of the order of $\exp[-2s(x)y]$. On the other hand, for $x \ll 1$ the inequality $s(x)y \gg 1$ does not hold. Asymptotically we find $s(x) \approx 2\pi^2/\ln(1/x)$, $x \ll 1$. It then follows that for large y the inequality $s(x)y \ll 1$ holds only in a very narrow region, $x \leq \exp(-2\pi^2y)$, and the corresponding correction to the function $\sigma(\eta_m)$ is again exponentially small for $y \gg 1$. The analytic structure of eqs. (4.109), (4.110) and (4.111) is very complicated even in the limit of $\eta_m \rightarrow \infty$. Therefore, we have no definite answer for the asymptotic behavior of $\sigma(\eta_m)$.

Comparison of eqs. (4.34) and (4.108) shows that for a given location of the peak $I = I_m$ of the retrapping current distribution, the half-width of the distribution diminishes as the temperature is lowered, reaching a constant value $\sim e/RC$ at $T=0$. The reduced half-widths $\Sigma(\eta_m)$ for large temperatures $T \gg \hbar\Omega$ ($y=0$) and $\sigma(\eta_m)$ at low temperatures $T \ll \hbar\Omega$ ($y \rightarrow \infty$) are depicted in fig. 17.

5. Summary and final remarks

In this paper we have presented consistent derivations of a number of results relevant to Kramers' problem. It is worthwhile to start with the list of principal final expressions. Considering the escape rate out of a single-well potential, we have succeeded in bridging the region between the extremely underdamped regime and the regime of moderate friction, deriving expression (2.55) for the pre-exponential factor $A(\Delta)$ shown in fig. 2 as function of the reduced dissipation $\Delta \equiv \delta/T$. This result has enabled us to write down eq. (2.55) for the escape rate valid for arbitrary damping. For the average energy of escaping particles we have obtained eq. (2.63), which is plotted in fig. 3. The same approach when applied to a double-well potential results in eq. (2.71) for the preexponential factor of the Brownian particle's lifetime in a single-well potential and in eq. (2.73) for the relaxation rate of nonequilibrium population in the two wells.

The rate of escape of a quantum Brownian particle is treated in the framework of well known methods. For the underdamped regime we have presented the general expression (3.20) for the preexponential factor $A(\Delta, y)$, which depends on the reduced dissipation $\Delta \equiv \delta/T$ and on the quantum parameter $y \equiv \hbar\omega/2\pi T$. For several values of y the total preexponential factor $A_{\text{tot}}(\Delta, y)$ [see eq. (3.24)] is plotted in fig. 5. In the extremely underdamped regime we have obtained eq. (3.21), which shows that for $\Delta \ll 1$ the function A depends on Δ in a universal manner, $A(\Delta, y) \approx \Delta^{1-y}a(y)$. The function $a(y)$ is depicted in fig. 6 for both cubic and cosine potentials; the two results differ by at most a few per cent. In contrast to the classical case, in the quantum overdamped case the distribution function becomes explicitly dependent on the friction strength. We have calculated the partition function of a Brownian particle in a semiclassical potential well [see eq. (3.31)], the distribution $N(x)$ of the density of Brownian particles [see eq. (3.34)] and the distribution of particles in the coordinate and momentum $f(p, x)$ [see eq. (3.40)]. From these results the conclusion follows that the interaction with the

high-frequency modes of the heat bath causes large zero-point fluctuations of the particle momentum. For the escape rate in the overdamped regime we derived eq. (3.46) which is applicable at a sufficiently high temperature.

A large part of this paper is devoted to the investigation of fluctuation-induced phenomena in Josephson junctions. We begin with calculation of the voltage–current characteristics in the absence of noise [see eq. (2.32) and fig. 8]. The effects of noise substantially change the voltage–current characteristics in a certain range of the external current, since under the effects of noise a Josephson junction switches between the zero-voltage and running states. The result is a fluctuation-induced voltage across the junction, given (in an exponential approximation) by eq. (4.29) and depicted in fig. 9. The random events of the retrapping into zero-voltage states under a very slow decrease of the external current are described by the retrapping current distribution. We have shown that typically this distribution is Gaussian and derived eq. (4.35) (see fig. 17) for the dependence of the reduced half-width of the distribution on the location of its maximum. The onset of the retrapping state is related to a threshold value of the external current. The fluctuation-induced voltage–current characteristics are calculated in a slightly different way below threshold [see eq. (4.51) and fig. 10] and above threshold [see eq. (4.57)]. Exactly at threshold it has a jump of finite magnitude in its first derivative, given by eq. (4.60) and plotted in fig. 11.

A complete solution of the resistive-state lifetime is presented, eq. (4.70). Much more complicated is the problem of the lifetime of a zero-voltage state. Its solution is given by eqs. (4.75), (4.76), (4.84) and (4.85). The mathematical approach developed in the process of the solution of this problem has also provided us with expression (4.86) for the partial probabilities of the phase jumps by $2\pi n$ (n is an integer) (numerical results for $n = 1$ and $n = 2$ are plotted in figs. 12 and 13) and with expression (4.87) for the probability of switching from a zero-voltage state into the running state, plotted in fig. 14. In the extremely underdamped case the probabilities w_n of the phase jumps are given by eq. (4.91) which is illustrated in fig. 15.

Application of the general method to the investigation of the fluctuation-induced phenomena in a Josephson junction in the quantum regime is hindered due to a large number of the parameters involved. Therefore, we have restricted ourselves to calculation of the half-width of the retrapping current distribution which is determined by eqs. (4.103)–(4.105) in the general case and by eqs. (4.108)–(4.111) in the ultraquantum regime. The results of numerical calculations of these quantities are presented in figs. 16–17.

All these results were derived upon condition that the ratio T/U_0 should be small. It should be noted, however, that this parameter enters the problem in a twofold manner. On the one hand, under condition $T \ll U_0$ the escape rate is exponentially slow, $\tau^{-1} \propto \exp(-U_0/T)$. In this respect the values $U_0/T \sim 6-7$ seem to be sufficiently large, since during its lifetime in the potential well the Brownian particle could undergo $\exp(U_0/T) \sim 10^2-10^3$ oscillations. On the other hand, the ratio U_0/T should not be too large, since then the decay events become extremely rare and are nearly beyond observation. The results of this paper are only correct in the leading approximation in T/U_0 , and corrections from the next terms of the expansion may be quite substantial. Therefore, the calculation of the preexponential factor for finite values of T/U_0 represents an absolutely necessary task in solving Kramers' problem completely [63].

Another very important approximation which has drastically simplified our considerations is that the noise is assumed to be white and Gaussian. In real experimental systems the situation may not be so simple. For example, rigorous considerations have shown, that in a Josephson junction thermal noise is given by a superposition of two white Gaussian noises [64]. Moreover, the noise characteristics in

Josephson junctions can be tuned by changing the experimental setup. In a beautiful experiment with the use of a delay line it has been demonstrated, that, in agreement with theoretical predictions [44], the resonant interaction between noise and the phase oscillations in an underdamped junction results in an oscillating dependence of the escape rate on the delay time [45]. This experimental result gives the most direct confirmation of the correctness of our ideas on the activated decay processes. We have cited these theoretical and experimental results in order to demonstrate the present state of the art (for detailed information see ref. [43]).

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