Random Matrix Theory and Quantum Transport

S.M. Kravec, E.V. Calman, A.P. Kuczala[1]

[1]*Department of Physics, University of California at San Diego, La Jolla, CA 92093*

We review the basics of random matrix theory and its applications to transport in mesoscale systems such as quantum wires and dots. The goal of this approach is to uncover properties of correlators which are universal, despite impurities or noise, viewed as the the correlations of eigenvalues from random S-matrices.

INTRODUCTION

Random matrix theory begins with the observation that, given some distribution of matrix elements, the correlations of eigenvalues and vectors of these matrices are independent of many of the details of that distribution. The existence of these 'universal features' is very appealing when studying problems involving such complicated matrices whose coefficients we can take to be approximately random. [6]

Several areas of physics lend themselves to such analysis.The methods of random matrix theory have been applied successfully to the study of chaotic systems[3], the spectra of large nuclei[21], and constraining low-energy QCD[19]. In this paper, we summarize how random matrix theory can be applied to study transport properties of disordered and open systems, particularly those for which quantum mechanical effects such as tunnelling and phase coherence are essential to the physics.

The two examples focused on will be transport through a chaotic cavity, a model of a simple quantum dot, and also a series of such cavities as a model of a disordered quantum wire. In these geometries we can understand precisely what the ensemble of S-matrices is and how we can use it to obtain statistics of transport properties, such as conductance fluctuation and shot noise.

We begin with a review of random matrix theory for a Gaussian distribution and its application to closed quantum systems.

THE WIGNER-DYSON ENSEMBLE

Consider an ensemble of $N \times N$ [1] Hermitian matrices, $\mathcal{H}$ obeying the following probability distribution:

$$P(\mathcal{H}) = e^{-\beta \mathrm{Tr}\ V(\mathcal{H})} \tag{1}$$

Where $V(\mathcal{H}) \propto \mathcal{H}^2$ so that this distribution is Gaussian. Consequentially, each matrix element is guaranteed to be independently distributed.[2]

We define β, referred to as the 'symmetry factor', in the above distribution as a number which tells us what sort of elements H_{ij} are. The three cases we will be considering are:

$$\begin{array}{l|l||l} \beta = 1 & H_{ij} \in \mathbb{R} & \text{'Orthogonal'} \\ \beta = 2 & H_{ij} \in \mathbb{C} & \text{'Unitary'} \\ \beta = 4 & H_{ij} \in \mathbb{H} & \text{'Symplectic'} \end{array}$$

The names are chosen for the type of matrices, U, which leave any $\mathcal{H}$ invariant: we can decompose $\mathcal{H} = U\mathrm{diag}(\lambda_1, ..., \lambda_N)U^\dagger$. In applications to a physical system one chooses the smallest β, and thus matrix elements, that manifestly preserves symmetries of the system.

For example, one may use $\beta = 1$ for a system which respects anti-unitary symmetries such as time-reversal but if spin-rotation symmetry is broken one is forced to choose $\beta = 4$.

In this way we can say that β counts the number of degrees of freedom for each matrix element. This makes it a suitable object to play the role of the 'inverse temperature'.

---

[1] The $N \to \infty$ limit will usually be taken.

[2] $\mathrm{Tr}\ \mathcal{H}^2 = \mathrm{Tr}\ \mathcal{H}\mathcal{H}^\dagger = \sum_{i,j} |H_{ij}|^2$

To determine the distribution of eigenvalues, $P(\{\lambda_i\})$, from our matrix distribution 1, we simply preform the basis transformation, $\mathcal{H} = U\text{diag}(\lambda_1, ..., \lambda_N)U^\dagger$, and integrate over the space of U. The trick is in how the volume element transforms.

Consider the measure in the space of $\mathcal{H}$ matrices:

$$d\mu(\mathcal{H}) = \prod_{i<j} d\mathcal{H}_{ij} = J(\prod_i d\lambda_i)d\mu(U) \tag{2}$$

To determine the Jacobian factor, J, we consider the line element: $ds^2 = \text{Tr}\, d\mu(\mathcal{H})d\mu(\mathcal{H}^\dagger) = g_{\mu\nu}dx^\mu dx^\nu$ [3] From which we can derive $J = |\det g|^{1/2}$.[18]

Using $\mathcal{H} = U\mathcal{H}U^\dagger$ above we can express $ds^2 = \sum_i d\lambda_i^2 + \sum_{i,j} dU_i dU_j(\lambda_i - \lambda_j)^{2\beta}$ from which it is clear:

$$J = \prod_{i<j} |\lambda_i - \lambda_j|^\beta \tag{3}$$

Thus we can express $P(\{\lambda_i\})$ as:

$$P(\{\lambda_i\}) \propto \prod_{i<j} |\lambda_i - \lambda_j|^\beta \prod_k e^{-\beta V(\lambda_k)} \tag{4}$$

By inspection we can see 4 can be brought into a form familiar from statistical mechanics: the Gibbs distribution.

$$P(\{\lambda_i\}) \propto e^{\beta[\sum_{i<j} \ln|\lambda_i - \lambda_j| - \sum_k V(\lambda_k)]} \tag{5}$$

The Jacobian factor generates an effective logarithmic repulsion between eigenvalues! If one imagines that these $\{\lambda_i\}$ are classical particles distributed along one dimension then such an interaction could arise from the Coulomb repulsion of infinite parallel lines of charge associated to each λ_i.[8]

As we've seen, the only correlations of eigenvalues enter from the Jacobian factor 3. This led to a rather seductive conjecture that all such correlations are geometric in origin. Is this valid? Generally speaking, it depends heavily on the characteristic length and energy scales of the problem.

The application of the Wigner-Dyson ensemble to the electronic properties of disordered metal grains[13] predicts the same energy-level correlation function as assuming diffusive motion of the electrons in the grain. However both models break down for $|E_i - E_j| > E_c = \frac{\hbar D}{L^2}$, differences above the Thouless energy.[1]

Note that $E_c \propto \frac{1}{L^2}$ indicates that the theory works for length scales where there are many grain boundaries over the relevant length, but breaks down if one is effectively looking inside of single grains. This is unsurprising: at long length scales things look chaotic, but at short length scales they can begin to look ordered, which the individual grains are.

Another interesting application of this formalism is when $\mathcal{H}$ has some dynamical dependance: the distribution remains fixed but one tries to predict the fluctuations of the eigenvalues.

Consider a Hamiltonian:

$$\mathcal{H} = e^{-\tau}\mathcal{H}_0 + (1 - e^{-2\tau})^{1/2}\mathcal{H}_{GU} \tag{6}$$

Where $\tau \in (0, \infty)$ is a time parameter which interprets between a fixed matrix $\mathcal{H}_0$ and a matrix $\mathcal{H}_{GU}$ which is distributed according to 1 for $\beta = 2$: the Gaussian Unitary Ensemble denoted by $P_{GU}(\mathcal{H})$.

This leads to a probability distribution:

$$P(\mathcal{H}, \tau) = \frac{1}{(1 - e^{-2\tau})^{N^2/2}} P_{GU}\left(\frac{\mathcal{H} - e^{-\tau}\mathcal{H}_0}{(1 - e^{-2\tau})^{1/2}}\right) \tag{7}$$

The above distribution is actually the Green's function for an Ornstein Uhlenbeck process: 7 obeys a Fokker-Planck equation![14]

---

[3] The dx^μ are over the spaces of eigenvalues and eigenvectors, $d\lambda^\mu$ and dU^μ respectively.

By performing a series of N^2 Gaussian integrals one obtains the corresponding Fokker-Planck equation for the distribution of eigenvalues $P(\{\lambda_i\},\tau)$:

$$\partial_\tau P = \sum_i \partial_i \left[\lambda_i + \sum_{j\neq i} \frac{1}{\lambda_j - \lambda_i} + \frac{1}{2}\partial_i \right] P \tag{8}$$

One can apply this analysis to the case of disordered metal grains in magnetic field[5], relating τ to changes in field strength. While this correctly predicts the two point functions of densities, the level dynamics are not a Markov process and thus the higher point correlation functions will deviate from the RMT result above.

TRANSMISSION EIGENVALUES AND SCATTERING

We now consider the same formalism of random matrices applied to scattering matrices S, as opposed to Hamiltonians themselves.

Consider the 1-D case of two baths of electrons sandwiching a disordered region where scattering happens. The incoming wavefunctions can be written $\psi_n = e^{\pm k_n x}$, for which we can write $S(n)$ to relate incoming to outgoing states in the usual manner. S then is a $2N \times 2N$ matrix relating the vector of incoming to outgoing channel amplitudes and can be parameterized as:

$$S = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} -\sqrt{1-T} & \sqrt{T} \\ \sqrt{T} & \sqrt{T} \end{pmatrix} \begin{pmatrix} U" & 0 \\ 0 & V" \end{pmatrix} \tag{9}$$

Where T is the diagonal matrix of transmission eigenvalues and the set of U, V, U', V' are $N \times N$ unitary matrices.

We can construct an distribution similar to 1 for the scattering matrices which depends solely on the transmission probabilities $\{T_i\}$:

$$P(S) \propto e^{-\beta \mathrm{Tr}\ f(tt^\dagger)} \tag{10}$$

Where $tt^\dagger$ is the Hermitian matrix with eigenvalues T_i and $f(tt^\dagger) \propto (tt^\dagger)^2$. [4]

Our analogous version of equation 4 :

$$P(\{T_n\}) \propto \prod_{i<j} |T_i - T_j|^\beta \prod_k T_k^{-1+\beta/2} \exp[-\beta f(T_k)] \tag{11}$$

Which has a Gibbs form, ala 5, by parameterizing the above with respect to the ratio of reflection and transmission coefficients: $\lambda_i = \frac{1-T_i}{T_i}$.[4] This then yields

$$P(\{\lambda_n\}) \propto \exp\left[-\beta \left(\sum_{i<j} u(\lambda_i, \lambda_j) + \sum_i V(\lambda_i) \right) \right] \tag{12}$$

with $u(\lambda, \lambda') = -\ln|\lambda - \lambda'|$ the familiar logarithmic potential, and

$$V(\lambda) = [N - \frac{1}{2}(1 - 2/\beta)] \ln(1+\lambda) + f\left((1+\lambda)^{-1}\right)$$

which now contains all the microscopic data of the scattering process.

In practice, the eigenvalue repulsion u is not logarithmic. The form of V above arises from its role as a Lagrange multiplier in the derivation of **??** from maximization of entropy given a fixed mean density of the λ_i. The assumption of geometrical correlations is then valid only for transmission probabilities T_n close to 1 and overestimates repulsion for smaller T_n.

There is some inherent ambiguity in defining the entropy of a random matrix ensemble, for example under variable redefinitions, and it is an open problem whether there is some scheme which can produce the correct distribution for a disordered wire.

---

[4] Note that λ_i is restricted to be positive as opposed to generic eigenvalues.

FUNCTIONAL APPROACH TO CORRELATIONS

We now introduce a method to calculate correlation functions of the ensemble (**??**) with a generic potential $u(\lambda, \lambda')$. The two point correlation function

$$K(\lambda, \lambda') = \langle \sum_{i,j} \delta(\lambda - \lambda_i)\delta(\lambda' - \lambda_j)\rangle - \rho(\lambda)\rho(\lambda')$$

is given by the functional derivative of the mean spectral density $\rho(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i)\rangle$ with respect to the confining potential $V(\lambda)$. Note that

$$\frac{\delta}{\delta V(\lambda')} e^{\beta W} = -\beta \sum_j \delta(\lambda' - \lambda_j)$$

so

$$\rho(\lambda') = \frac{\delta}{\delta V(\lambda')} \ln \int \prod_j^N d\lambda_j \exp(-\beta W),$$

which is strongly reminiscent of the expectation value for an operator in field theory. Taking another functional derivative with respect to V then yields the two point correlation function

$$K(\lambda, \lambda') = \frac{\delta}{\delta V(\lambda)} \frac{\delta}{\delta V(\lambda')} \ln \int \prod_j^N d\lambda_j \exp(-\beta W)$$

$$= \frac{-1}{\beta} \frac{\delta \rho(\lambda)}{\delta V(\lambda')}$$

In the limit of large N, the functional derivative can be explicitly evaluated [20]. The potential V and mean density ρ are related by the integral equation

$$V(\lambda) + \int_{\lambda_-}^{\lambda_+} d\lambda' u(\lambda, \lambda')\rho(\lambda') = \text{constant}$$

where (λ_-, λ_+) contains the support of ρ. Finite N corrections are of order $1/N$ for $\beta = 1, 4$ and $1/N^2$ for $\beta = 2$ [9]. This expression has the natural interpretation that the eigenvalues distribute themselves to obtain equilibrium with the potential V.

Varying this expression,

$$\delta V(\lambda) + \int_{\lambda_-}^{\lambda^+} d\lambda' u(\lambda, \lambda')\delta\rho(\lambda') = \text{constant}$$

Note that $\int_{\lambda_-}^{\lambda^+} d\lambda \rho(\lambda) \propto N$, so varying ρ while holding N fixed yields the constraint

$$\int_{\lambda_-}^{\lambda_+} d\lambda \delta\rho(\lambda) = 0$$

This can be solved for $K_2 = \delta\rho/\delta V$:

$$K_2(\lambda, \lambda') = \frac{1}{\beta} u^{-1}(\lambda, \lambda') \tag{13}$$

where $u^{-1}(\lambda, \lambda')$ is the inverse of the integral kernel u.

UNIVERSAL CONDUCTANCE FLUCTUATIONS

The universal behavior of the random matrix formulation becomes manifest in the fluctuations of the conductance G. The Landauer formula states that the conductance is proportional to the sum of transmission probabilities [11]

$$G = G_0 \sum_n T_n \tag{14}$$

where $G_0 = \frac{2e}{h}$. The fluctuations in current, called shot noise, also depend in a simple way on the transmission coefficients. The power of the shot noise is given by

$$P = P_0 \sum_n T_n(1 - T_n),$$

where $P_0 = 2\text{eV}G_0$. More generally, we will show that any "linear statistic" $A = \sum_n a(T_n)$ exhibits the universal behavior of random matrix theory.

First, we change variables to $\lambda_n = (1 - T_n)/T_n$ so that $A = \sum_n a(\lambda_n)$. For the conductance G, $a(\lambda) = 1/(1 + \lambda)$. The mean of A is

$$\langle A \rangle = \langle \int_{\lambda_-}^{\lambda^+} d\lambda \; a(\lambda) \sum_n \delta(\lambda - \lambda_n) \rangle = \int_{\lambda_-}^{\lambda^+} d\lambda a(\lambda)\rho(\lambda)$$

and the its variance $\langle A^2 \rangle - \langle A \rangle^2$ is

$$\text{var } A = \int_{\lambda_-}^{\lambda^+} d\lambda \int_{\lambda_-}^{\lambda^+} d\lambda' \; a(\lambda)a(\lambda')K_2(\lambda, \lambda')$$

Using the expression for K_2 that we obtained eariler (13),

$$\text{var } A = \frac{1}{\beta} \int_{\lambda_-}^{\lambda^+} d\lambda \int_{\lambda_-}^{\lambda^+} d\lambda' \; a(\lambda)a(\lambda')u^{-1}(\lambda, \lambda')$$

This equation highlights the universality of fluctuations in linear statistics. The expression is proportional to $1/\beta$, which means that the variance depends in a universal way on the symmetries of the system. Furthermore, the expression is independent of the potential V which depends on the microscopic details of the system.

The universality of the conductance fluctuations can be argued as follows: Each of the N channels in the conductor is either nonlocalized, in which case it will contribute to the conductance (with a transmission coefficient close to unity), or localized, in which case it will not contribute to the conductance (transmission coefficent exponentially small). These are called open and closed channels, respectively.[15]

The conductance is therefore proportional to the number of open channels only, $N_{eff} \equiv G/G_0$. The fraction of open channels is approximately l/L, where l is the mean free path and L is the length of the conductor. Therefore the conductance $G \approx G_0 N \frac{l}{L}$.

QUANTUM DOTS

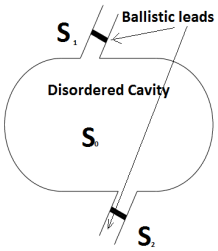
Random matrix theory is well suited to the problem of scattering off of a quantum dot. A defect in an otherwise ordered system can be treated as a scattering site with some unknown properties. Consider the circular ensemble developed in random matrix scattering theory.

$$P(\{\phi_n\}) \propto \prod_{n<m} |e^{i\phi_n} - e^{i\phi_m}|^\beta \tag{15}$$

Remember that the S matrices are constrained by probability conservation, so are uniformly distributed in the unitary group. We work in $T_n = \frac{1}{\cosh x_n{}^2}$, so that

$$P(\{x_n\}) \prod_i dx_i = \frac{1}{\int d\mu(S)} d\mu(S) \tag{16}$$

The usual trick of looking at the transformation and taking Jacobian of said transformation applies, but now we can also say that $d\mu(S) = d\mu(S \cdot U)$ where U is a matrix of the type determined by β. We say that this measure is invariant, and averages in S are equivalent to averages in the appropriate group of matrices.
Now let's apply this to a one dimensional version of the quantum dot problem. Imagine a small chaotic cavity with ideal point contracts on either side.



There is some impedence on the inputs, leading to three scattering sites, S_0 for the actual cavity and S_1 and S_2 of the contacts 1 and 2 respectively. Transport in the contacts is otherwise ballistic. To see why the circular ensemble is a reasonable choice, consider the "most random" possible distribution of scattering matrices, by which we mean those with maximal entropy $\mathcal{S} = -\int d\mu P(S) \log P(S)$. If subjected to the constraint $\int d\mu P(S) S^q = \bar{S}^q$, Gopar [12] showed that

$$P(S) \propto |Det(1 - S^\dagger S)|^{-\beta(N_1+N_2-1+2/\beta} \tag{17}$$

Which is the Poisson kernel used to solve $\vec{\nabla}_r^2 P = 0$ on the disk. It turns out that this is also the solution for our quantum dot. Gopar [12] also showed that if S_0 is distributed on the circular ensemble, then the total matrix S is distributed on the Poisson kernel. Where S_{total} is is obtained by multiplying the transfer matrices $M_{total} = M_1 M_0 M_2$ and then converting back to S.

Before applying the circular ensemble to the quantum dot let's try the Wigner-Dyson ensemble for this system, so we consider the Hamiltonian

$$H = \sum |\alpha\rangle E_f \langle\alpha| + \sum_{\mu,\nu} |\mu\rangle \mathcal{H}_{\mu\nu} \langle\nu| + \sum_{\mu,\alpha} |\mu\rangle W_{\mu\alpha} \langle\alpha| + |\alpha\rangle W^*_{\mu\alpha} \langle\mu| \tag{18}$$

Where the states $|\alpha\rangle$ run over the states at the Fermi surface of the leads, as those are the ones which can scatter. Recall that the leads are ordered material with ballistic transport, so it makes sense to talk about a well defined set of states on the Fermi surface and their energy E_f. The states $|\mu\rangle$ and $|\nu\rangle$ are the bound states for the disordered cavity. That Hamiltonian $\mathcal{H}$ is distributed as 1. However, E_f and W should be well defined. So if

$$S = 1 - 2\pi i W^\dagger (E_f - \mathcal{H} + i\pi W W^\dagger)^{-1} W \tag{19}$$

by polar decomposition. Therefore the average of S is just an average over $\mathcal{H}$ for the cavity, since the other elements are fully determined. This seem like something of an over-simplification having developed the machinery for RMT for the circular ensemble. However, we will see that the Wigner-Dyson ensemble can make accurate predictions about the conductance of the disordered cavity, and it is in fact the gaussian distribution of $\mathcal{H}$ is in fact a valid approximation of the circular ensemble in the limit $M \to \infty$ The average

$$\bar{S} = \frac{M\delta - \pi^2 W^\dagger W}{M\delta + \pi^2 W^\dagger W} \tag{20}$$

Where M is the number of states at the cavity (S_1) and δ is the mean energy spacing in the contacts energy levels.

If we consider the case of zero impedance on the inputs, so the only scattering events are inside the quantum dot itself, the probability of transmission from state n on one lead to state to the state to state m is given by $\langle |S_{mn}|^2 \rangle$ We will see that for the case of no time reversal symmetry with S uniformly distributed on the unitary group,

$$\langle |S_{mn}|^2 \rangle = \int d\mu(S) S_{nm} S^*_{nm} = \frac{1}{N_1 + N_2} \tag{21}$$

since $\beta = 2$ and for $\beta = 1$, this becomes

$$\langle |S_{mn}|^2 \rangle = \int d\mu(S) \sum_{k,k"=1}^{N_1+N_2} S_{nk} S_{km} S^*_{nk"} S^*_{mk"} = \frac{1 + \delta_{nm}}{N_1 + N_2} \tag{22}$$

because S is now uniformly distributed on the orthogonal symmetric matrices. So now scattering back into the same state n is twice as likely as any other scattering event. This is known as weak localization, and it results in a measurable increase in the conductance of a material when time reversal symmetry is broken by a magnetic field (ie when weak localization disappears). Let's compute the conductance as the probability of all possible scattering events which go from one contact to the other.

$$G = G_0 \sum_{n=1}^{N_1} \sum_{m=N_1+1}^{N_1+N_2} |S_{mn}|^2 \tag{23}$$

Taking the average of S from above, we get that

$$\bar{G} = \frac{2e^2}{h} \frac{N_1 N_2}{N_1 + N_2 - 1 + \frac{2}{\beta}} \tag{24}$$

so if a magnetic field is introduced to a quantum dot, the symmetry is broken and the conductance will increase! The presence of time reversal symmety makes scattering of an electron back into the state it came from twice as likely as any other, as there are two ways for the events to happen. This makes backscattering overall more likely then forward scattering, so the conductance drops slightly. If we now consider the corrections from the scattering due to impedance at the contacts, we can no longer just evaluate the values of S_{mn} uniformly over the unitary group. Instead the distribution now follows the poisson kernel and if we re-write $S_{barrier} = \begin{pmatrix} \bar{S} & A \\ C & B \end{pmatrix}$ for some unitary matrices A,B& C. In the limit of $N\Gamma >> 1$ for Γ_n =transmission probability, Beenakker [6] showed that through perturbative expansion.

$$\frac{\bar{G}}{G_0} \approx \frac{g_1 g_1^{'}}{g_1 + g_2} + (1 - \frac{2}{\beta}) \frac{g_2 g_1^{'2} + g_1 g_2^{'2}}{g_1 + g_2}^3 + \cdots \tag{25}$$

Where the sum of tranmission probabilities $g_p = \sum_{n=1}^{N_1} \Gamma_n^p$ The firs term in 25 represents the classical sum of conductances in series, and the first term in Beenakker's expansion represents the leading order effect of symmetry breaking. Namely, if $\beta = 2$ and time reversal is absent, the conductance rises to the classical series sum. This has a direct experimental realization. Chang [7] realized this in both stadium and ring shaped quantum dots. The stadium (rectangle with circular caps) geometry is classicaly non-integrable for ballistic motion, and so we expect our conditions for RMT to be satisfied. Indeed, We can see that There is a narrower peak in the center for the case 1.

As to why the peaks have the finite width they do can look at this in terms of the two time scales (and therefore energy scales) $E_{c,open} << E_{c,closed}$. Since we have $E \simeq \frac{h}{\tau}$ the energy of a closed system is related to the ergodic time and the energy scale of an open system is related to the dwelling time $E_{c,open} \simeq \frac{h}{\tau_{dwell}}$. Then if we imagine changing $\mathcal{H}$ in a manner similar to 6 where

$$\mathcal{H} = \mathcal{H}_0 + i\alpha\mathcal{A} \tag{26}$$

Where $\mathcal{H}$ is real and symmetric and $\mathcal{A}$ is real and anti-symmetric to the total hamiltonian goes from being uniformly distributed in the orthogonal matrices to uniformly distributed in the unitary matrices as $\alpha \to 1$. If we simply do perturbation theory to leading order, the result is the energy shift is

$$\delta E_i = \alpha^2 \sum_{i \neq j} \frac{A_{ij}^2}{E_i - E_j} \simeq M\alpha^2\delta \tag{27}$$

for mean energy spacing δ and M states in the system. We will say TR symmetry is broken when δE is of the same order as E. This is better written in terms of the flux through the cavity rather than the field strength, since we can relate the former to the energies of the cavity. The critical flux at which the symmetry breaks and there is a transition from the GOE to the GUE is

$$\Phi_c \simeq \frac{h}{e} (\frac{E_{c,open}}{E_{c,close}})^{1/2} \simeq \frac{h}{e} (\frac{N\Gamma L^2 \delta}{\hbar v_f min(l, L)} \tag{28}$$

Where l is the mean free path and L is the size of the disordered region. Thus in figure 1 the change in normalized conductance is plotted against the flux through the disordered cavity (in units of flux quanta). If we use the machinery of the circular ensemble and rather than inserting a particular value of β we use the distribution for $\mathcal{H}$ and predict the Lorentzian shape of the conductance peak for the stadium geometry.

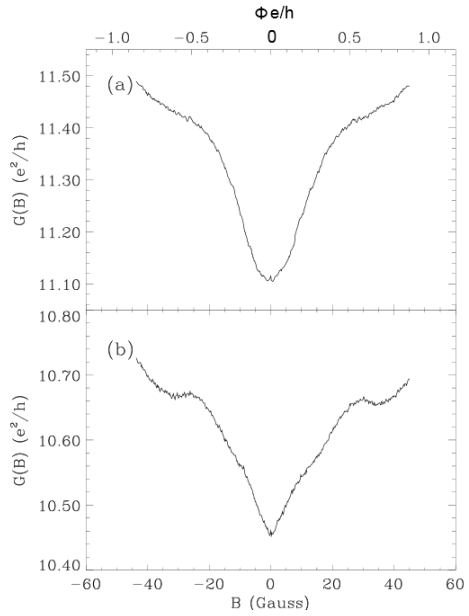


FIG. 1: Figure from [7] showing normalized concudtances for the circular and stadium geometries. The strength of the magnetic field is in terms of the flux through a single dot

QUANTUM WIRE

One dimensional quantum chains don't obey Ohm's law. Rather the conductance scales, at any strength of disorder, as $G \propto e^{-L/\xi}$ where L is the chain length and ξ the coherence length which depends on the mean free path l.

This was demonstrated [2] precisely by considering a $1D$ chain with weak scattering ($l \gg \lambda_F$) and computing how the transmission probability scales with L and attained the exponential scaling for $L > l$.

A more realistic model of a wire is not one dimensional however: when the width $W > \lambda_F$ we should describe the transmission of the N transverse modes. This leads to considering the eigenvalues T_i of the trasmission matrix product $tt^\dagger$ and their joint probability distribution $P(T_1, ..., T_N, L)$ where correlations arise from eigenvalue repulsion.

This has the effect of scaling $\xi \propto Nl$ and allows one to distrinquish between a metallic regime ($Nl \gg L \gg l$) and an insulating regime ($L \propto Nl$). In the metallic regime Ohm's law is recovered and the conductance decreases linearly in L.

The aforementioned probability distribution is known to satisfy a Fokker-Planck equation, known as the DMPK equation[16]:

$$\partial_L P = \frac{2}{l(\beta N + 2 - \beta)} \sum_{i=1}^{N} \partial_i [\lambda_i (1 + \lambda_i) J \partial_i] \frac{P}{J} \tag{29}$$

Where $\lambda_i \equiv \frac{1-T_i}{T_i}$ and the Jacobian is given by 3. An interesting equivalent approach to studying localization and transport in the thick wire ($N \gg 1$) limit is the use of a one dimensional supersymmetric non-linear σ model! [10]

Let us verify our expectations in the metallic regime. We can derive from 29 a coupled set of evolution equations for $M_q \equiv \sum_{n=1}^{N} T_n^q$.[17] We can think of this in the context of a large N expansion for which one such equation, to leading order in N, is given by:

$$\partial_s \langle M_1^p \rangle = \frac{-p}{N} \langle M_1^{p+1} \rangle + \mathcal{O}(N^{p-1}) \tag{30}$$

Note that β is not present in this truncation, $s \equiv L/l$. We can solve this precisely with the initial condition of $T_n = 1$ for all n if $s = 0$:

$$\langle M_1^p \rangle = N^p (1 + s)^{-p} \tag{31}$$

Combining this with 14 implies that $\langle G/G_0 \rangle = \langle M_1 \rangle$ which, in the diffusive limit of $s >> 1$ is precisely the linear dependence expected.

Carrying our the expansion of 30 further once can determine the variance of G as well: $\mathrm{Var}G/G_0 = \frac{2}{15}\beta^{-1}$ which agrees with diagrammatic perturbation theory results. That this coefficient is not $\frac{1}{8}$ implies that the repulsion between eigenvalues is not truly logarithmic, as previously discussed.

CONCLUSIONS

Random matrix theory can be both a valuable calculational tool and source of insight into problems of highly chaotic or disorder driven behaviour. While sensitive to the scales within a given problem, it is perhaps the best tool for understanding physics in regimes where other tools fail precisely because of their dependence on structure. It is also for this reason a very broad method and, as we have shown, has promise in determining characteristics of transport in these disordered systems.

Acknowledgements
Authors would like to thank John McGreevy for useful discussions and Dan Arovas for a wonderful class and many bad puns.

[1] BL Altshuler and BI Shklovskii. Repulsion of energy levels and the conductance of small metallic samples. *Zh. Eksp. Teor. Fiz.*, 91(1):220–234, 1986.
[2] PW Anderson, DJ Thouless, E Abrahams, and DS Fisher. New method for a scaling theory of localization. *Physical Review B*, 22(8):3519, 1980.
[3] AV Andreev, O Agam, BD Simons, and BL Altshuler. Quantum chaos, irreversible classical dynamics, and random matrix theory. *Physical review letters*, 76(21):3947, 1996.
[4] Harold U Baranger and Pier A Mello. Mesoscopic transport through chaotic cavities: A random s-matrix theory approach. *Physical review letters*, 73(1):142, 1994.
[5] C. W. J. Beenakker and B. Rajaei. Exact solution for the distribution of transmission eigenvalues in a disordered wire and comparison with random-matrix theory. *Phys. Rev. B*, 49:7499–7510, Mar 1994.
[6] Carlo WJ Beenakker. Random-matrix theory of quantum transport. *Reviews of modern physics*, 69(3):731, 1997.
[7] AM Chang, HU Baranger, LN Pfeiffer, and KW West. Weak localization in chaotic versus nonchaotic cavities: A striking difference in the line shape. *Physical review letters*, 73(15):2111, 1994.
[8] Freeman J Dyson. Statistical theory of the energy levels of complex systems. i. *Journal of Mathematical Physics*, 3(140), 1962.
[9] Freeman J Dyson. A class of matrix ensembles. *Journal of Mathematical Physics*, 13(90), 1972.
[10] KB Efetov. Supersymmetry and theory of disordered metals. *Advances in Physics*, 32(1):53–127, 1983.
[11] Daniel S Fisher and Patrick A Lee. Relation between conductivity and transmission matrix. *Physical Review B*, 23(12):6851–6854, 1981.
[12] Víctor A Gopar, Moisés Martínez, Pier A Mello, and Harold U Baranger. The invariant measure for scattering matrices with block symmetries. *Journal of Physics A: Mathematical and General*, 29(4):881, 1996.
[13] LP GorKov and GM Eliashberg. Minute metallic particles in an electromagnetic field. *Sov. Phys. JETP*, 21(940), 1965.
[14] Fritz Haake. *Quantum signatures of chaos*, volume 54. Springer, 2010.
[15] Y. Imry. Active transmission channels and universal conductance fluctuations. *EPL (Europhysics Letters)*, 1(5):249, 1986.
[16] PA Mello, P Pereyra, and N Kumar. Macroscopic approach to multichannel disordered conductors. *Annals of Physics*, 181(2):290–317, 1988.
[17] Pier A MELLO. Random matrix theory and maximum entropy models for disordered conductors. *Mesoscopic Phenomena in Solids*, page 369, 1991.
[18] Charles E Porter. *Statistical theories of spectra: fluctuations*. Academic Press New York, 1965.
[19] JJM Verbaarschot and T Wettig. Random matrix theory and chiral symmetry in qcd. *arXiv preprint hep-ph/0003017*, 2000.
[20] Eugene P Wigner. On the distribution of the roots of certain symmetric matrices. *The Annals of Mathematics*, 67(2):325–327, 1958.
[21] Eugene P Wigner. Random matrices in physics. *Siam Review*, 9(1):1–23, 1967.