

Poisson limit theorem

In probability theory, the **law of rare events** or **Poisson limit theorem** states that the Poisson distribution may be used as an approximation to the binomial distribution, under certain conditions. ^[1] The theorem was named after Siméon Denis Poisson (1781–1840).

Contents

Theorem

Proofs

Alternative Proof

Ordinary Generating Functions

See also

References

Theorem

Let p_n be a sequence of real numbers in $[0, 1]$ such that the sequence np_n converges to a finite limit λ . Then:

$$\lim_{n \rightarrow \infty} \binom{n}{k} p_n^k (1 - p_n)^{n-k} = e^{-\lambda} \frac{\lambda^k}{k!}$$

Proofs

$$\begin{aligned} \binom{n}{k} p^k (1 - p)^{n-k} &\simeq \lim_{n \rightarrow \infty} \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \lim_{n \rightarrow \infty} \frac{n^k + O(n^{k-1})}{k!} \frac{\lambda^k}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \lim_{n \rightarrow \infty} \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^{n-k} \end{aligned}$$

Since

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

and

$$\lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^{-k} = 1$$

This leaves

$$\binom{n}{k} p^k (1-p)^{n-k} \simeq \frac{\lambda^k e^{-\lambda}}{k!}.$$

Alternative Proof

Using Stirling's approximation, we can write:

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &= \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} \\ &\simeq \frac{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-k)} \left(\frac{n-k}{e}\right)^{n-k} k!} p^k (1-p)^{n-k} \\ &= \sqrt{\frac{n}{n-k}} \frac{n^n e^{-k}}{(n-k)^{n-k} k!} p^k (1-p)^{n-k} \end{aligned}$$

Letting $n \rightarrow \infty$ and $np = \lambda$:

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &\simeq \frac{n^n p^k (1-p)^{n-k} e^{-k}}{(n-k)^{n-k} k!} \\ &= \frac{n^n \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} e^{-k}}{n^{n-k} \left(1 - \frac{k}{n}\right)^{n-k} k!} \\ &= \frac{\lambda^k \left(1 - \frac{\lambda}{n}\right)^{n-k} e^{-k}}{\left(1 - \frac{k}{n}\right)^{n-k} k!} \\ &\simeq \frac{\lambda^k \left(1 - \frac{\lambda}{n}\right)^n e^{-k}}{\left(1 - \frac{k}{n}\right)^n k!} \end{aligned}$$

As $n \rightarrow \infty$, $\left(1 - \frac{x}{n}\right)^n \rightarrow e^{-x}$ so:

$$\begin{aligned} \binom{n}{k} p^k (1-p)^{n-k} &\simeq \frac{\lambda^k e^{-\lambda} e^{-k}}{e^{-k} k!} \\ &= \frac{\lambda^k e^{-\lambda}}{k!} \end{aligned}$$

Ordinary Generating Functions

It is also possible to demonstrate the theorem through the use of Ordinary Generating Functions of the binomial distribution:

$$G_{\text{bin}}(x; p, N) \equiv \sum_{k=0}^N \left[\binom{N}{k} p^k (1-p)^{N-k} \right] x^k = \left[1 + (x-1)p \right]^N$$

by virtue of the Binomial Theorem. Taking the limit $N \rightarrow \infty$ while keeping the product $pN \equiv \lambda$ constant, we find

$$\lim_{N \rightarrow \infty} G_{\text{bin}}(x; p, N) = \lim_{N \rightarrow \infty} \left[1 + \frac{\lambda(x-1)}{N} \right]^N = e^{\lambda(x-1)} = \sum_{k=0}^{\infty} \left[\frac{e^{-\lambda} \lambda^k}{k!} \right] x^k$$

which is the OGF for the Poisson distribution. (The second equality holds due to the definition of the Exponential function.)

See also

- De Moivre–Laplace theorem
- Le Cam's theorem

References

1. Papoulis, Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th Edition

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