Wikipedia

Poisson limit theorem

In probability theory, the **law of rare events** or **Poisson limit theorem** states that the Poisson distribution may be used as an approximation to the binomial distribution, under certain conditions. [1] The theorem was named after Siméon Denis Poisson (1781–1840).

Contents

Theorem

Let p_n be a sequence of real numbers in $[0, 1]$ such that the sequence np_n converges to a finite limit λ. Then:

$$\lim_{n\to\infty}\binom{n}{k}p_n^k(1-p_n)^{n-k}=e^{-\lambda}\frac{\lambda^k}{k!}$$

Proofs

$$\begin{aligned}\binom{n}{k}p^k(1-p)^{n-k}&\simeq\lim_{n\to\infty}\frac{n(n-1)(n-2)\dots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}\\&=\lim_{n\to\infty}\frac{n^k+O\left(n^{k-1}\right)}{k!}\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^{n-k}\\&=\lim_{n\to\infty}\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}\end{aligned}.$$

Since

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n=e^{-\lambda}$$

and

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}=1$$

This leaves

$$\binom{n}{k}p^k(1-p)^{n-k}\simeq\frac{\lambda^k e^{-\lambda}}{k!}.$$

Alternative Proof

Using Stirling's approximation, we can write:

$$\begin{aligned}\binom{n}{k}p^k(1-p)^{n-k}&=\frac{n!}{(n-k)!k!}p^k(1-p)^{n-k}\\&\simeq\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi\left(n-k\right)}\left(\frac{n-k}{e}\right)^{n-k}k!}p^k(1-p)^{n-k}\\&=\sqrt{\frac{n}{n-k}}\frac{n^n e^{-k}}{(n-k)^{n-k}k!}p^k(1-p)^{n-k}\end{aligned}$$

Letting $n\to\infty$ and $np=\lambda$:

$$\begin{aligned}\binom{n}{k}p^k(1-p)^{n-k}&\simeq\frac{n^n\,p^k(1-p)^{n-k}e^{-k}}{(n-k)^{n-k}k!}\\&=\frac{n^n\left(\frac{\lambda}{n}\right)^k(1-\frac{\lambda}{n})^{n-k}e^{-k}}{n^{n-k}\left(1-\frac{k}{n}\right)^{n-k}k!}\\&=\frac{\lambda^k\left(1-\frac{\lambda}{n}\right)^{n-k}e^{-k}}{\left(1-\frac{k}{n}\right)^{n-k}k!}\\&\simeq\frac{\lambda^k\left(1-\frac{\lambda}{n}\right)^{n}e^{-k}}{\left(1-\frac{k}{n}\right)^{n}k!}\end{aligned}$$

As $n\to\infty$, $\left(1-\frac{x}{n}\right)^n\to e^{-x}$ so:

$$\binom{n}{k}p^k(1-p)^{n-k} \simeq \frac{\lambda^k e^{-\lambda} e^{-k}}{e^{-k} k!} = \frac{\lambda^k e^{-\lambda}}{k!}$$

Ordinary Generating Functions

It is also possible to demonstrate the theorem through the use of Ordinary Generating Functions of the binomial distribution:

$$G_{\text{bin}}(x;p,N) \equiv \sum_{k=0}^{N}\left[\binom{N}{k}p^k(1-p)^{N-k}\right]x^k = \Big[1+(x-1)p\Big]^N$$

by virtue of the Binomial Theorem. Taking the limit $N \to \infty$ while keeping the product $pN \equiv \lambda$ constant, we find

$$\lim_{N\to\infty} G_{\text{bin}}(x;p,N) = \lim_{N\to\infty}\left[1+\frac{\lambda(x-1)}{N}\right]^N = \mathrm{e}^{\lambda(x-1)} = \sum_{k=0}^{\infty}\left[\frac{\mathrm{e}^{-\lambda}\lambda^k}{k!}\right]x^k$$

which is the OGF for the Poisson distribution. (The second equality holds due to the definition of the Exponential function.)

See also

- De Moivre–Laplace theorem
- Le Cam's theorem

References

1. Papoulis, Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th Edition