HW7: Self-similarity and dimensional methods

To be returned on March 16, 2017

I. SELF-SIMILAR SOLUTION FOR THE NONLINEAR BARENBLATT EQUATION

Consider the following nonlinear porous-medium equation in d dimensions

$$\partial_t \phi(r,t) = \Delta_d \phi^{1+n}(r,t) = \frac{1}{r^{d-1}} \partial_r \left(r^{d-1} \partial_r \phi^{1+n}(r,t) \right) \,. \tag{1}$$

1) Show that the radially symmetric similarity solution has the form :

$$\phi(r,t) = \frac{Q}{(Q^n t)^{d\theta}} \, f\left(\frac{r}{(Q^n t)^{\theta}} \right) , \tag{2}$$

where $\theta = 1/(2+nd)$ and the conserved mass $Q = \int \phi(r,0)\, d^d r$.

2) Show that

$$f(\xi) = \left[\frac{n\theta}{2\,(n+1)} \left(\xi_0^2 - \xi^2 \right) \right]^{1/n} , \tag{3}$$

for $\xi \leq \xi_0$ and vanishes for $\xi \geq \xi_0$. Here, the scaling variable $\xi \equiv r/\left(Q^n t\right)^{\theta}$ and ξ_0 is determined below.

3) Use the conservation of mass to determine ξ_0.

4) For $n = 0$ the equation reduces to the diffusion equation, which has Gaussian tails extending to infinity, contrary to the spreading front (3). Show how the solution above for general n crosses to a Gaussian in the limit $n \to 0$.

II. ANOMALOUS SELF-SIMILARITY

Consider the problem above with a modified diffusivity:

$$\partial_t \phi(r,t) = \kappa \Delta_d \phi^{1+n}(r,t) \,, \tag{4}$$

where $\kappa = 1$ for $\partial_t \phi > 0$ and $\kappa = 1 + \epsilon$ for $\partial_t \phi < 0$.

1) Show that mass is not conserved any longer.

2) Introduce a self-similar solution of the second type $\phi(r,t) \propto \frac{1}{t^{d\theta+\alpha}} \, f\left(\frac{r}{t^{\theta+\beta}}\right)$ and use arguments developed during our class to show that the two anomalous exponents α and β are related as : $n\theta\alpha + (1 - nd\theta)\,\beta = 0$.

3) Assume ϵ small. Use perturbation theory to calculate the anomalous exponents α and β at the first order in ϵ.