SELF-SIMILAR SOLUTIONS OF THE SECOND KIND IN NONLINEAR FILTRATION

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Several original self-similar solutions are known in gasdynamics (*), and were classified by Zel'dovich and Raizer [1] as self-similar solutions of the second kind (see also [2]). The formal definition of self-similar solutions of the second kind given in [1] is as follows: the exponent appearing in the expression of self-similar variables cannot, in the case of such solutions, be directly derived from dimensional considerations on the basis of given defining parameters, as is possible with self-similar solutions of the first kind (see [3]). This exponent is derived by solving a system of ordinary differential equations to which the construction of self-similar solution is reduced (from the condition of existence of the entire solution of the boundary value problem corresponding to that system). The reason for the occurrence in various problems of self-similar solutions of the second kind has not been, so far, fully explained. The asymptotic behavior of the nonstationary filtration theory applied to the solution of the Cauchy problem of an elastic fluid in a deformable medium at extended times is considered here. In the case of an elastic medium the asymptotics is represented by the known self-similar solution of the first kind, while the problem of filtration through an elastic-plastic medium reduces to the derivation of a new self-similar solution of the second kind. The obtained results lead to certain general conclusions.

1. A simple example and the statement of problem. (1) Let us first consider the elementary example, viz. the Cauchy problem of the classic heat conductivity equation, which also defines pressure distribution for the case of elastic fluid filtration in an elastic medium

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geqslant 0 \tag{1.1}$$

with the initial condition (see Fig. 1) which is conveniently written

$$U_0 - u(x, 0) = \frac{Q}{l} u_0 \left(\frac{x}{l}\right) \tag{1.2}$$

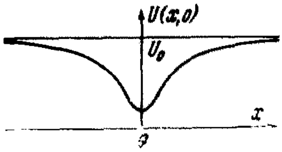

Fig. 1

Here U_0 is an additive constant, l is a certain length scale, Q is the "total extracted heat", i.e. the quantity used in the theory of filtration, proportional to the amount of fluid extracted up to the initial instant, and $u_0(\zeta)$ an even dimensionless function of its dimensionless argument. It is assumed that function $u_0(\zeta)$ is sufficiently smooth and decreases fairly rapidly with increasing ζ, and that

*) The Guderley, Landau and Staniukovich solutions of the convergent shock wave problem, those of Zel'dovich and Weizsäcker of the short shock, and others.

$$\int_{-\infty}^{\infty} u_0\ \zeta) d\zeta = 1$$

Solution of the Cauchy problem (1.1), (1.2) may, by means of dimensional analysis (the Π-theorem, see [3]) be presented in the form

$$U_0 - u = \frac{Q}{\sqrt{a^2t}} F(\xi, \eta), \qquad \xi = \frac{x}{\sqrt{a^2t}}, \quad \eta = \frac{l}{\sqrt{a^2t}} \tag{1.3}$$

where F is a dimensionless function of its dimensionless arguments, easily written in explicit form. We shall now consider the asymptotics of solution (1.3) for $t \to \infty$, assuming that parameter $\xi = x / \sqrt{a^2t}$ remains finite. Argument $\eta = l / \sqrt{a^2t}$ will now tend to zero. Assuming that function $F(\xi, \eta)$ for $\eta \to 0$ and finite ξ, tends to a finite value (this is readily proved in this very simple case), we find that the predominant term of the asymptotics of solution (1.3) for $t \to \infty$ is expressed by the well known formula

$$U_0 - u = \frac{Q}{\sqrt{a^2t}} f(\xi), \qquad f(\xi) = F(\xi, 0) \tag{1.4}$$

Obviously, the transition to limit with $\eta \to 0$ and finite ξ may be considered as corresponding to $l \to 0$ and constant x and t, hence, (1.4) is the solution of Eq. (1.1), corresponding to $l = 0$, i.e. to the initial distribution in the form of δ-function. From Eq. (1.1), the conditions of symmetry and integrability we readily derive

$$f(\xi) = \frac{1}{2\sqrt{\pi}} \exp \frac{-\xi^2}{4}$$

The above example is typical of self-similar solutions of the first kind. This is so, because, when the "not self-similar" argument $\eta = l / \sqrt{a^2t}$ tends to zero, function $F(\xi, \eta)$, which represents the solution, tends to a finite limit for finite ξ, while parameter l vanishes from the limit problem formulation "without trace". Hence, it is possible to derive the self-similar solution (1.4) in the usual manner by determining the exponents on the basis of dimensional considerations, with the assumption that $U_0 - u(x, t)$ depends on parameters x, t, a^2 and Q, and only on these.

2) It was shown in the theory of filtration [4] of a low compressibility fluid through porous plastic media that the fluid pressure u satisfies equation

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial u}{\partial t}\right) \frac{\partial^2 u}{\partial x^2}, \qquad a^2(z) = \begin{cases} a_1^2 & (z > 0) \\ a_2^2 & (z \leqslant 0) \end{cases} \tag{1.5}$$

We shall assume $a_2^2 > a_1^2$ (we consider the very simple case where a single process of consecutive pressure drop and increase occurs).

Let us consider the Cauchy problem for Eq. (1.5) with the same initial condition (1.2). As was shown in [5], the solution of the problem thus formulated exists, is unique and its second derivative with respect to x is continuous. Similarity considerations lead to the following form of solution of the Cauchy problem :

$$U_0 - u = \frac{Q}{\sqrt{a_1^2t}} F(\xi, \eta, \varepsilon) \qquad \left(\xi = \frac{x}{\sqrt{a_1^2t}}, \ \eta = \frac{l}{\sqrt{a_1^2t}}, \ \varepsilon = \frac{a_2}{a_1}\right) \tag{1.6}$$

On similar considerations, and again assuming the finiteness at limit of function $F(\xi, \eta, \varepsilon)$ for finite ξ and $\eta \to 0$, we conclude that the limit solution (1.4) should, as in the previous case, be of the form

$$U_0 - u = \frac{Q}{\sqrt{a_1^2t}} f(\xi, \varepsilon) \tag{1.7}$$

However, there is no solution of Eq. (1.5) in the form (1.7) which would be continuous, would have a continuous derivative with respect to x and would satisfy the natural conditions of symmetry at infinity. In fact, if we substitute (1.7) into (1.5) we obtain for f the following expression:

$$\varepsilon \frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}\xi f = 0 \quad (0 \leqslant \xi \leqslant \xi_0), \qquad \frac{d^2 f}{d\xi^2} + \frac{1}{2}\frac{d}{d\xi}\xi f = 0 \quad (\xi_0 \leqslant \xi < \infty) \tag{1.8}$$

Here ξ_0 is a coordinate at which $(\xi f)'$ proportional to $\partial u / \partial t$ vanishes. After integration we obtain

$$\varepsilon \frac{df}{d\xi} + \frac{1}{2}\xi f = C_1 \quad (0 \leqslant \xi \leqslant \xi_0), \qquad \frac{df}{d\xi} + \frac{1}{2}\xi f = C_2 \quad (\xi_0 \leqslant \xi < \infty) \tag{1.9}$$

Due to symmetry $df / d\xi = 0$ for $\xi = 0$; when $\xi \to \infty$, ξf (function f is assumed to be integrable) and $df / d\xi$ tend to zero. Hence, $C_1 = C_2 = 0$, and integration of the preceding equation yields

$$f = C_3 \exp\frac{-\xi^2}{4\varepsilon} \quad (0 \leqslant \xi \leqslant \xi_0), \qquad f = C_4 \exp\frac{-\xi^2}{4} \quad (\xi_0 \leqslant \xi < \infty) \tag{1.10}$$

where C_3 and C_4 are new constants. The pressure and flow continuity conditions stipulate the continuity of f and $df / d\xi$ when $\xi = \xi_0$, from which and from preceding equations we obtain the system

$$C_3 \exp - \frac{\xi_0^2}{4\varepsilon} = C_4 \exp - \frac{\xi_0^2}{4}, \qquad C_3 \xi_0 \frac{1}{\varepsilon} \exp - \frac{\xi_0^2}{4\varepsilon} = C_4 \xi_0 \exp - \frac{\xi_0^2}{4}$$

When $\varepsilon \neq 1$ this system does not have a nontrivial solution for any finite ξ_0, which proves that solution of the form (1.7) does not exist. Furthermore, Kamenomostskaia, who had investigated this problem, has proved that for a particular selection of function $u_0(x / l)$, with l tending to zero and constant Q, the solution of the Cauchy problem for any $t > 0$ tends at any point x either to the constant U_0, or to infinity (depending on which of the inequalities $\varepsilon \gtrless 1$ holds).

2. Self-similar solution of the second kind. The possible reason of the paradoxical conclusion, – the nonexistence of a limit solution, reached above in the case of $\varepsilon \neq 1$ could be the incorrectness of the assumption of existence of a finite limit of $F(\xi, \eta, \varepsilon)$ at finite ξ and $\eta \to 0$.

If this limit is not finite, the following alternatives arise: either there exists a number α such that $\lim \eta^{-\alpha} F(\xi, \eta, \varepsilon)$ will be finite when $\eta \to 0$, or that such number does not exist.

Let us consider the first alternative in which the asymptotic representation

$$F(\xi, \eta, \varepsilon) = \eta^\alpha f^*(\xi, \varepsilon) \tag{2.1}$$

holds for $F(\xi, \eta, \varepsilon)$ when $\eta \to 0$. Then with $t \to \infty$ the Cauchy problem limit solution will no longer be of the form (1.4), but of the form

$$U_0 - u = \frac{Q l^\alpha}{(a_1^2 t)^{1/2(1+\alpha)}} f^*(\xi, \varepsilon) \tag{2.2}$$

We remind that the tendency of η to vanish at finite ξ may also be realized by the transition to limit for $l \to 0$ and constant x and t. Expression (2.2) shows that, when this transition is made at constant Q, the solution will either tend to zero, or to infinity. This relationship also shows that, if the transition is effected at $l \to 0$, and Q tending either to zero, or infinity (so that $Q l^\alpha$ remains finite), the self-similar solution of Eq. (1.5) obtained at limit, which defines the asymptotics of the Cauchy problem solution, must not be of the form (1.4), but of

$$U_0 - u = \frac{A}{(a_1^2 t)^{1/2(1+\alpha)}} f(\xi, \varepsilon) \qquad (A = \beta Q l^\alpha) \tag{2.3}$$

Here β is a dimensionless constant, while parameter α remains as a kind of trace of the vanished parameters l and Q; α could be calculated, if it were possible to carry out the transition from the solution of the not self-similar problem; in a direct derivation of a self-similar solution (2.3) parameter α is unknown and has to be determined. What is essential, is that parameter α, the trace of the vanished not self-similar parameter, appears explicitly in the formulation of an asymptotic self-similar problem.

Solution (2.3) fulfils the initial singular conditions (this singularity is, however, not the classic δ-function, as in the case of $\varepsilon = 1$), namely:

$$U_0 - u(x, 0) = A\delta_\alpha(x) \tag{2.4}$$

The generalized function $\delta_\alpha(x)$ fulfils here the condition

$$\delta_\alpha(x) = 0 \quad (x \neq 0), \qquad \int_{-\infty}^{\infty} |x|^\alpha \delta_\alpha(x)\, dx = 1$$

so that for function $f(\xi, \varepsilon)$ we have the relationship

$$\int_{-\infty}^{\infty} |\xi|^\alpha f(\xi, \varepsilon)\, d\xi = 1 \tag{2.5}$$

Substituting (2.3) into (1.5) we obtain for f the equation

$$\varepsilon \frac{d^2 f}{d\xi^2} + \frac{1}{2}\xi \frac{df}{d\xi} + \frac{1+\alpha}{2} f = 0 \qquad (0 \leqslant \xi \leqslant \xi_0),$$
$$\frac{d^2 f}{d\xi^2} + \frac{1}{2}\xi \frac{df}{d\xi} + \frac{1+\alpha}{2} f = 0 \qquad (\xi_0 \leqslant \xi < \infty) \tag{2.6}$$

Here ξ_0 is the point at which $f''(\xi, \varepsilon)$ or its equivalent (by virtue of (2.6)) $f'\xi + (1+\alpha) f$ proportional to the derivative $\partial u / \partial t$, vanish.

By virtue of the natural symmetry of function $f(\xi, \varepsilon)$ the boundary condition is

$$f'(0, \varepsilon) = 0 \tag{2.7}$$

Function $f(\xi, \varepsilon)$ and its first derivative must be, moreover, continuous for $\xi = \xi_0$ (pressure and flow of the fluid are continuous at the "unloading wave" $x_0(t) = \xi_0 \sqrt{a_1^2 t}$, where derivative $\partial u / \partial t$ changes its sign).

The solutions of Eq. (2.6) are simply expressed by confluent hyper-geometric, or their contiguous parabolic cylinder functions [6].

When $0 \leqslant \xi \leqslant \xi_0$ the solution of Eq. (2.6) satisfying the first condition (2.7) is of the form

$$f = C \exp\left(-\frac{\xi^2}{8\varepsilon}\right)\left[D_\alpha\left(\frac{\xi}{\sqrt{2\varepsilon}}\right) + D_\alpha\left(-\frac{\xi}{\sqrt{2\varepsilon}}\right)\right] \tag{2.8}$$

where C is a constant, and D_α is the symbol of the parabolic cylinder function [6]. If $\xi_0 < \xi$, the solution of (2.6), for which integral (2.5) is convergent, is

$$f = F \exp(-{}^1/_8 \xi^2) D_\alpha ({}^1/_2 \sqrt{2}\xi) \tag{2.9}$$

where F is a constant. (When $\xi \to \infty$, the second linearly independent solution behaves as $\xi^{-\alpha-1}$, and integral (2.5) is divergent).

By specifying condition

$$(\partial u / \partial t) = 0 \quad \text{for } x = x_0(t) \pm 0$$

or its equivalent

$$\xi f' + (1 + \alpha) f = 0 \quad \text{for} \quad \xi = \xi_0 \pm 0 \tag{2.10}$$

and using the recurrent relationships for the derivatives of the parabolic cylinder functions and the expressions of the latter in terms of confluent hypergeometric functions [6], we obtain

$$D_{\alpha+2}\left(\frac{\xi_0}{\sqrt{2}}\right) = 0, \qquad \Phi\left(-\frac{\alpha}{2} - 1, \frac{1}{2}, \frac{\xi_0^2}{4\varepsilon}\right) = 0 \tag{2.11}$$

Equations (2.11) define the unknown parameter α and the value of ξ_0. This and the condition of continuity of f for $\xi = \xi_0$ yield the relation between constants C and F

$$F = C\left[D_\alpha\left(\frac{\xi_0}{\sqrt{\varepsilon}}\right) + D_\alpha\left(-\frac{\xi_0}{\sqrt{2\varepsilon}}\right)\right] \exp\left[\frac{\xi_0^2}{8}\left(1 - \frac{1}{\varepsilon}\right)\right]\left[D_\alpha\left(\frac{\xi_0}{\sqrt{2}}\right)\right]^{-1} \tag{2.12}$$

(by virtue of (2.10) the condition of continuity of $f'(\xi, \varepsilon)$ is automatically fulfilled), constant C is defined from the normalization condition (2.5). There remains, thus, to investigate the system of transcendental Eqs. (2.11) in order to complete the solution analysis.

Solving the first equation for $\xi_0 / \sqrt{2}$, we obtain the monotonically increasing function α represented in Fig. 2 by Curve 1.

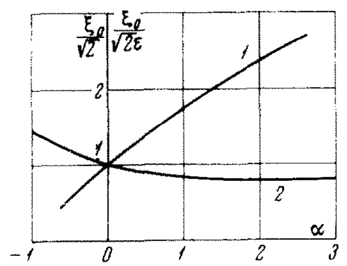

Fig. 2

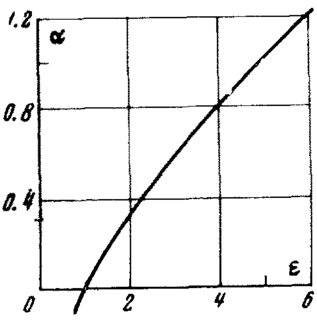

Fig. 3

Solving the second equation for $\xi_0 / \sqrt{2\varepsilon}$ yields a monotonically decreasing function α, Curve 2 of Fig. 2. The dependence of $\xi_0 / \sqrt{2}$ on α, for any given ε is obtained by simply stretching Curve 2 along the ordinate axis. When $\varepsilon = 1$, i.e. $a_1 = a_2$, Curves 1 and 2 intersect at point $\alpha = 0$, as they should according to the results of the classic case presented in Sect. 1. When $\varepsilon > 1$, the intersection point abscissa α is positive. It will be readily seen that due to the character of Curves 1 and 2 their intersection point is unique. Function $\alpha(\varepsilon)$ is shown in Fig. 3 in the form of a curve.

It should be noted that for large α Curves 1 and 2 have also other branches. The second branch of Curve 1 begins at point $\xi_0 = 0$, $\alpha = 1$, and runs monotonically increasing below the first branch, while the second branch of Curve 2 is two-valued and lies above its first branch. Points corresponding to intersections with secondary branches and existing for sufficiently great ε have no physical meaning, as they correspond to nonmonotonic pressure distribution.

3. The case of a "dipole". Discussion of the filtration problem.

1) The preceding Sections dealt with the Cauchy problem for an unbounded space. It is interesting to consider the mixed problem for the half-space $x \geqslant 0$ at the boundary of which $x = 0$, the pressure $u(0, t)$ is constant

$$u(0,t)=U_0 \tag{3.1}$$

The asymptotics of this mixed problem in the classic case of $\varepsilon = 1$ (Eq. (1.1)) is represented by a solution of the dipole type

$$U_0-u(x,t)=\frac{Mx}{2a^3\sqrt{\pi t^3}}\exp\frac{-x^2}{4a^2t} \tag{3.2}$$

which may be easily derived by dimensional analysis, using the readily proved, in this case, law of conservation of the value (the "dipole moment")

$$\int_0^\infty [U_0-u(x,t)]\,x\,dx = M \tag{3.3}$$

When $\varepsilon \neq 1$ (Eq. 1.15), all of the reasoning of the preceding Section holds also without any change in the problem under consideration; however, it must be borne in mind that Eq. (1.5) and the initial condition (1.2) are considered here for $x \geqslant 0$. Furthermore, for function $f(\xi,\varepsilon)$, by virtue of the boundary condition (3.1), condition

$$f(0,\varepsilon)=0 \tag{3.4}$$

is to be substituted for boundary condition (2.7), so that function $f(\xi,\varepsilon)$ satisfying conditions (3.4) and condition

$$\int_0^\infty \xi^\alpha f(\xi,\varepsilon)\,d\xi = 1 \tag{3.5}$$

becomes of the form

$$f=C\exp\frac{-\xi^2}{8\varepsilon}\Big/\left[D_\alpha\left(\frac{\xi}{\sqrt{2\varepsilon}}\right)-D_\alpha\left(-\frac{\xi}{\sqrt{2\varepsilon}}\right)\right] \qquad (0\leqslant\xi\leqslant 0)$$

$$f=F\exp-\frac{\xi^2}{8}\Big/D_\alpha\left(\frac{\xi}{\sqrt{2}}\right) \qquad (\xi\geqslant\xi_0) \tag{3.6}$$

In this case conditions (2.11) are of the form

$$D_{\alpha+2}\left(\frac{\xi_0}{\sqrt{2}}\right)=0, \qquad \Phi\left(-\frac{\alpha+1}{2},\frac{3}{2},\frac{\xi_0^2}{4\varepsilon}\right)=0 \tag{3.7}$$

(it is seen that the first of the above conditions remains unchanged).

Solving numerically Eq. (3.7) we obtain the functional dependence $\alpha(\varepsilon)$ shown in Fig. 4; for $\varepsilon = 1$, obviously $\alpha = 1$ which is in agreement with the classic case. Constants C and F are determined in a manner exactly similar to that of the previous case.

2) In the two cases considered here (initial pressure perturbation in an unbounded space, and in a half-space) the asymptotic behavior of pressure distribution of an elastic fluid in an elastic-plastic medium is defined by the self-similar solution of the second kind

$$U_0-u(x,t)=\frac{A}{(a_1^2t)^{1/2(1+\alpha)}}f(\xi,\varepsilon) \tag{3.8}$$

It follows from the preceding that in the definitions of function f all constants α, ξ_0, C and F are uniquely defined.

Thus, the asymptotic representation of pressure distribution is defined to within constant $A=\beta Q l^\alpha$.

In the classic case of $\varepsilon = 1$ this constant is determined by the conservation law

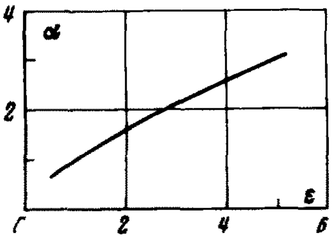

Fig. 4

$$\int_{-\infty}^{\infty} [U_0 - u(x, t)]\, dx = \int_{-\infty}^{\infty} [U_0 - u(x, 0)]\, dx = A \tag{3.9}$$

in the case of unbounded space, and by

$$\int_{0}^{\infty} [U_0 - u(x, t)]\, x dx = \int_{0}^{\infty} [U_0 - u(x, 0)]\, x dx = A \tag{3.10}$$

in the case of half-space. In a general case these two conditions are not fulfilled. For example, when $a_2 \neq a_1$ $(\varepsilon \neq 1)$ we have in the case of unbounded space

$$\frac{d}{dt}\int_{-\infty}^{\infty} [U_0 - u(x, t)]\, dx = -2(a_2^2 - a_1^2)\left(\frac{\partial u}{\partial x}\right)_{x_0(t)} \neq 0 \tag{3.11}$$

and in the case of half-space with constant pressure at the boundary,

$$\frac{d}{dt}\int_{0}^{\infty} [U_0 - u(x, t)]\, x dx = -(a_2^2 - a_1^2)\, x_0(t)\left(\frac{\partial u}{\partial x}\right)_{x_0(t)} -$$

$$-(a_2^2 - a_1^2)[U_0 - u(x_0(t), t)] \neq 0$$

The law of conservation with respect to time applicable to the integral

$$I_1 = \int_{-\infty}^{\infty} [U_0 - u(x, t)]\, |x|^{\alpha}\, dx \tag{3.12}$$

in the case of unbounded space and to the integral

$$I_2 = \int_{0}^{\infty} [U_0 - u(x, t)]\, x^{\alpha} dx \tag{3.13}$$

in the case of a half-space, is not known.

If these conservation laws were known, it would have been possible to determine constant A uniquely from initial conditions. As this is not the case, the only way of finding constant A is by numerical computation of the problem, proceeding from the not self-similar initial conditions. The outcome of such computations should bring the solution to self-similar asymptotics (2.3), and make the determination of constant A possible. A comparison of computation results of the two alternative initial conditions corresponding to the same values of integrals (3.12) and (3.13) is particularly interesting.

4. Certain conclusions. Certain general conclusions relative to self-similar solutions of the second kind may be drawn from the example considered above.

The dimensional analysis is based on the Π-theorem [3] which states that dependence between $(n+1)$-dimensional quantities $a, a_1, \ldots, a_n$

$$a = f(a_1, \ldots, a_k, a_{k+1}, \ldots, a_n) \tag{4.1}$$

may be presented in the form

$$\Pi = F(\Pi_1, \ldots, \Pi_{n-k}) \tag{4.2}$$

$$\Pi = \frac{a}{a_1^{\alpha} \ldots \alpha_k^{\varkappa}}, \quad \Pi_1 = \frac{a_{k+1}}{a_1^{\alpha_1} \ldots a_k^{\varkappa_1}}, \ldots, \Pi_{n-k} = \frac{a_n}{a_1^{\alpha_{n-k}} \ldots a_k^{\varkappa_{n-k}}} \tag{4.3}$$

on the assumption that $a_1, \ldots, a_k$ have independent dimensionalities, and that among $a_1, \ldots, a_n$ there are no $k+1$ quantities of independent dimensionality.

The conclusion as to the shedding of one, or another of the defining parameters a_{k+i}

is usually reached on the basis of an estimate of the corresponding dimensionless parameter Π_i ; this is based on reasoning that when Π_i is either "small" or "large" (as compared with unity) then its limit value, either zero, or infinity, is substituted for Π_i in the expression of function $F(\Pi_i, \ldots, \Pi_{i-1}, \Pi_i, \Pi_{i+1}, \ldots, \Pi_{n-k})$, and a function of $n-k-1$ arguments is considered:

$$\Phi(\Pi_1, \Pi_2, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) = F(\Pi_1, \ldots, \Pi_{i-1}, \overset{0}{\infty}, \Pi_{i+1}, \ldots, \Pi_{n-k})$$

and in this case argument a_{k+1} is called nonessential.

The validity of this reasoning is, however, based on the usually tacit assumption that the limit of function F, for Π_i tending to zero, or infinity, and fixed remaining arguments, exists and is different from either zero, or infinity. There are obviously cases in which this assumption does not hold. In such cases the dependence on parameter Π_i remains essential, no matter how small or large the dimensionless parameter Π_i. The following particular case is significant.

1) The limit of function $F(\Pi_1, \ldots, \Pi_{i-1}, \Pi_i, \Pi_{i+1}, \ldots, \Pi_{n-k})$ for $\Pi_i \to 0$, or to ∞ , is either zero, or infinity, there exists, however, a number α such that for $\Pi_i \to 0$, or to ∞

$$\lim \Pi_i^{-\alpha} F = \Phi(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) \qquad (\Pi_i \to 0, \infty) \qquad (4.4)$$

where Φ is finite when its arguments are finite, and such that for small (or large) values of argument Π_i the following asymptotics hold for function F :

$$F \approx \Pi_i^{\alpha} \Phi(\Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{n-k}) \qquad (4.5)$$

If the limit of function F for Π_i and Π_j, separately tending to zero, exists and is finite, the order and character of transition to limit are obviously immaterial. If, however, this limit does not exist, then in the analysis of asymptotic behavior the character of transition to limit, i.e. the relative rate of tendency to vanish of Π_i and Π_j, becomes essential, no matter how small these parameters may be. A particular case is important in this context.

2) the limit of function $F(\Pi_1, \ldots \Pi_i, \ldots, \Pi_j \ldots, \Pi_{n-k})$ does not exist, but there exist numbers α and β, such that for Π_i and $\Pi_j \to 0$:

$$F \approx \Pi_i^{\alpha} \Phi(\Pi_j / \Pi_i^{\beta}, \Pi_1, \ldots, \Pi_{i-1}, \Pi_{i+1}, \ldots, \Pi_{j-1}, \Pi_{j+1}, \ldots, \Pi_{n-k}) \qquad (4.6)$$

Similar sub-groups may obviously be separated when three, or more parameters tend to zero.

We shall now consider in general terms a certain not self-similar boundary value problem of a system of equations in partial derivatives, describing a certain phenomenon. Its solution may be expressed on the form (4.1), (4.2) (the number a of functions equals the number of unknowns, and variables and parameters of the problem will be represented by $a_1 \ldots a_n$). In the derivation of self-similar solutions, the transition to limit at certain values of parameters Π_i tending to zero, or infinity is applied. The resulting self-similar limit solutions represent at the same time exact particular solutions with singular boundary conditions and a smaller number of defining parameters, as well as the asymptotic representation of not self-similar solutions.

If the limit of F exists and is finite, self-similar solutions of the first kind may be obtained as the result of such transition to limit, in which the exponents of self-similar parameters are derived from the dimensional analysis of the problem input data (e.g. the Sedov problem of strong explosion [3], the problem of heat source (see Sect. 1 and others). The not self-similar parameters vanish "without trace" in such problems, and may be excluded from the formulation of the limit problem.

If function F has no limit, then, generally speaking, a self-similar limit solution does not exist, and the asymptotic representation of the solution of a not self-similar problem will not be self-similar for any arbitrarily small (large) values of parameters tending to zero (infinity).

The existence of self-similar solutions of the second kind is due to the particular Cases (1) and (2).

As regards Case (1) this is illustrated by examples adduced to Sects. 2 and 3. The particular Case (2) is illustrated by the convergent shock wave problem and that of short shock. We shall demonstrate this in somewhat greater detail on the example of the latter problem.

Let a piston, moving at constant velocity U during time intarval τ commencing at instant $t=0$, thrust onto a half-space filled with a perfect gas, of which the density ρ_0 at the initial instant is constant and the pressure equal to zero. At instant $t = \tau$ the piston is removed. The resulting gas motion is obviously not self-similar.

Considerations of similarity applied to the solutions of problems of gas dynamics yield the following expression:

$$p = \rho_0 U^2 F_p(\Pi_1, \Pi_2), \quad \rho = \rho_0 F_\rho(\Pi_1, \Pi), \; u = UF_u(\Pi_1, \Pi_2)$$
$$\Pi_1 = \tau / t, \quad \Pi_2 = x / Ut \tag{4.7}$$

where p, ρ_0 and u are the pressure, density and velocity, respectively; ρ_0, piston velocity U and time τ are taken as the parameters of independent dimensionalities.

It is interesting to consider the limit motion for $t \to \infty$. Function F has no finite limit for $\Pi_1 \to 0$ ($\Pi_1 \to 0$ corresponds to $\tau \to 0$ with finite x, U and t, and all functions tend to zero). However, when $t \to \infty$, Π_1 and Π_2 tend simultaneously to zero. In this problem the singular Case (2) occurs, in which for Π_1 and $\Pi_2 \to 0$, the predominant asymptotic terms of functions F_p, F_ρ and F_u are of the form (4.6):

$$F_p = \Pi_1^{2(1-\alpha)} g_p\left(\frac{\Pi_2}{\Pi_1^{1-\alpha}}\right), \quad F_\rho = g_\rho\left(\frac{\Pi_2}{\Pi_1^{1-\alpha}}\right), \quad F_u = \Pi_1^{1-\alpha} g_u\left(\frac{\Pi_2}{\Pi_1^{1-\alpha}}\right) \tag{4.8}$$

where α is a certain positive number. As far as Π_1 and Π_2 can be made to tend to zero by making U tend to infinity, and τ to zero at constant x and t, the asymptotic solution of the form (4.8) is obtained for such transition to limit, but, in order to retain the pressure and velocity finite, this must be done so, as to have the product $U_0\tau^{1-\alpha}$ finite, as shown by relationships (4.8).

The limit solution form was derived in [1]; parameter α appearing explicitly in the formulation of this problem remains as a kind of trace in the irregular transition to limit.

The presence of a certain constant A appearing amongst the self-similar variables is characteristic of self-similar solutions of the second kind. Its dimension is determined by parameter α derived from the condition of existence of a self-similar solution as a whole. The value of this constant, corresponding to the asymptotics of a particular not self-similar solution, cannot, generally speaking, be derived from the integral conservation laws. It may be determined by tracing (e. g. by numerical computation) the complete process of evolution of the not self-similar solution to the self-similar asymptotics.

(Were it possible to derive constant A from the integral laws of conservation, it would mean that by a proper selection of defining parameters the problem can be restated and reduced to a problem of the first kind. For example, solutions of the problem of strong explosion, and that of the heat source may be obtained as self-similar solutions of the second kind if defining parameters of the pre-self-similar statement of the problem are "unluckily" selected. The possibility of obtaining these solutions in the form of self-similar solutions of the first kind is related to the selection of energy E and total heat Q as the determining parameters which, by virtue of the corresponding integral conservation laws, do not change in time).

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ON THE PERTURBATION OF A FILTRATION FLOW BY A SINGLE CRACK

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1. We consider the problem of steady filtration in a thin layer in the presence of a single crack.

The flow in a crack which can be regarded as a piecewise-smooth line Γ in studying the external filtration field is describable by means of the equations of a lubricant layer, i.e. the pressure can be assumed constant within each cross section but different at each one of them; $p^-(s) = p^+(s) = p(s)$; the fluid velocity u_0 inside the crack can be assumed to have a parabolic profile,

$$u_0 = \frac{n^2 - 2k - h^2}{2\mu}\frac{\partial p}{\partial s} \tag{1.1}$$

Here n is the normal to the crack axis, $2h\,(s)$ is the width of the crack at the cross section $M\,(s)$, μ is the viscosity of the fluid, and k is the permeability of the porous medium.

The volume rate of the fluid flow through the cross section $M\,(s)$ is given by

$$Q\,(s) = \int_{-h}^{h} u_0 dn = \frac{2h\,(h^2 + 3k)}{3\mu}\frac{\partial p}{\partial s} \tag{1.2}$$