HW5: Burgers' equation and the 4/5 Kolmogorov's law

To be returned on February 23, 2017

I. TRAVELING SHOCK IN BURGERS' EQUATION

During our class, we found the steady shock for the 1D Burgers' equation

$$\partial_t u(x,t) + \frac{1}{2}\partial_x u^2 = \nu \partial_x^2 u \,, \tag{1}$$

when we maintain the velocities $u \to \pm U$ as $x \to \mp\infty$.

1) Generalize the solution to the case when we maintain the velocities at infinity $u(-\infty) = U_-$ and $u(+\infty) = U_+$ with $U_- > U_+$.

2) Verify that the dissipation remains finite in the limit $\nu \to 0$.

3) The velocity of the shock that you have found above is a special case of the Rankine-Hugoniot condition. By integrating the conservation law

$$\partial_t w(x,t) + \partial_x f(w) = 0 \,, \tag{2}$$

around a shock moving with velocity V_s, derive the Rankine-Hugoniot relation $V_s = \frac{f_+ - f_-}{w_+ - w_-}$ where the subscripts $\pm$ refer to the value on the proximal right and left of the shock, respectively.

II. PRE-SHOCK IN BURGERS' EQUATION

During our class, we have shown that the first shock occurs at a time t^* determined by the minimum value of the gradient field at the initial time. The initial gradient and velocity fields are denoted $g_0(a)$ and $u_0(a)$, with $g_0 = \partial_a u_0$. Denote the absolute minimum as $\min_a g_0(a) = -G$ and expand $g_0(a) \simeq -G + \alpha/2a^2 + \ldots$ around its minimum, which is assumed to be at the origin.

1) Denote by $X(a,t)$ the position at time t of a particle that was initially at a. Write down the expression for X in terms of a and $u_0(a)$.

2) Re-derive the relation between the first shock time t^* and G.

3) Show that the inverse Lagrangian map $a(X,t)$ at $t = t^*$ behaves singularly around the origin $X = 0$, namely it has a $1/3$ power-law behavior.

4) Show that at the time t^* the velocity $u(X,t)$ also develops a $1/3$ singular behavior at the origin.

5) Show that the enstrophy $\Omega(t) = \int (\partial u/\partial x)^2 \, dx$ diverges as $(t^* - t)^{-1/2}$ as t approaches t^*.

III. HOPF-COLE TRANSFORMATION FOR BURGERS' EQUATION

Define the stream function as $u = -\partial_x \psi(x,t)$. By taking the space derivative of the Burgers' equation, write down the equation for ψ_t. An unknown function $g(t)$ appears when a space-derivative is factored out. Suppose first that the unknown time-dependent function $g(t) = 0$.

1) Use the Hopf-Cole transformation $\psi(x,t) \equiv 2\nu \log\theta(x,t)$ to reduce the previous equation for ψ_t to the heat equation.

2) Use the Gaussian propagator for the heat equation to obtain the expression of $\theta(x,t)$.

3) In the limit $\nu \to 0$, show that $\psi(x,t) = \max_a \left[\psi_0(a) - \frac{(x-a)^2}{2t}\right]$. Interpret the max as follows: considers a parabola $\frac{(a-x)^2}{2t} + C$ centered at x, start with a big constant C and reduce it until you contact the curve $\psi_0(a)$ for the first time.

4) Interpret double contacts, i.e. first contact at two different a values for a given x. What is the velocity profile resulting from multiple x's having the same contact point a?

5) Can you adapt the arguments in 1) to the case $g(t) \neq 0$?

IV. KOLMOGOROV ”4/5” LAW FOR BURGERS’ EQUATION

By using the same procedure that we discussed in class for the Navier-Stokes equation, derive the relation

$$\langle (u(x)-u(0))^3\rangle = -12\varepsilon x\,, \tag{3}$$

for the Burgers’ equation. Here, ε is the dissipation rate.

V. MISSING PARTS IN THE DERIVATION OF THE KOLMOGOROV 4/5 LAW

We shall complete the derivation sketched in class of the 4/5 Kolmogorov law.

1) Use incompressibility to show that the two functions $B_{NN}(r)$ and $B_{LL}(r)$ in

$$\langle (v_i(\boldsymbol{r})-v_i(\mathbf{0}))\,(v_j(\boldsymbol{r})-v_j(\mathbf{0}))\rangle \equiv B_{NN}\delta_{ij} + (B_{LL}-B_{NN})\,\frac{r_i\,r_j}{r^2}\,, \tag{4}$$

are related as $B_{NN} = (1 + r/2\partial_r)\,B_{LL}$.

Use this expression to prove that the component proportional to δ_{ij} in the time-dependent term $\partial_t\langle v_i(\mathbf{0})v_j(\boldsymbol{r})\rangle$ and the viscous term $2\nu\Delta\langle v_i(\mathbf{0})v_j(\boldsymbol{r})\rangle$ in the Navier-Stokes equation read:

$$\left(1+\frac{r}{2}\partial_r\right)\left(-\frac{2}{3}\varepsilon - \frac{1}{2}\partial_t S_2(r,t)\right)\,; \qquad -\nu\left(1+\frac{r}{2}\partial_r\right)\left(\frac{1}{r^4}\partial_r\left(r^4\partial_r S_2(r,t)\right)\right)\,, \tag{5}$$

where the second-order longitudinal structure function $S_2 = B_{LL}$.

2) Use incompressibility to show that the three functions $F_1(r)$, $F_2(r)$ and $F_3(r)$ in

$$b_{ij,k} \equiv \langle v_i(\boldsymbol{r})v_j(\boldsymbol{r})v_k(\mathbf{0})\rangle \equiv F_1\delta_{ij}\frac{r_k}{r} + F_2\left(\delta_{ik}\frac{r_j}{r} + \delta_{jk}\frac{r_i}{r}\right) + F_3\frac{r_ir_jr_k}{r^3}\,, \tag{6}$$

are related by

$$F_2 = -\left(1+\frac{r}{2}\partial_r\right)F_1\,; \qquad F_3 = -F_1 + r\partial_r F_1\,. \tag{7}$$

3) Show that the longitudinal third-order structure $S_3(r) \equiv \langle (v_i(\boldsymbol{r})-v_i(\mathbf{0}))\,(v_j(\boldsymbol{r})-v_j(\mathbf{0}))\,(v_k(\boldsymbol{r})-v_k(\mathbf{0}))\rangle\,\frac{r_ir_jr_k}{r^3} = 12F_1(r)$.

4) Show that the component proportional to δ_{ij} for the non-linear terms $\partial_{r_k}\left[b_{kj,i} - b_{j,ki}\right]$ in the Navier-Stokes equation reads

$$-2\left(1+\frac{r}{2}\partial_r\right)\left(\frac{1}{r^4}\partial_r\left(r^4F_1(r,t)\right)\right)\,. \tag{8}$$

5) Gather all bits and pieces to obtain (34.20) in Landau-Lifchitz book

$$-\frac{2}{3}\varepsilon - \frac{1}{2}\partial_t S_2 = \frac{1}{6r^4}\partial_r\left(r^4S_3\right) - \nu\frac{1}{r^4}\partial_r\left(r^4\partial_r S_2\right)\,, \tag{9}$$

which finally yields the 4/5 law.