PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #2

[**1**] Consider the nonlinear oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 + \tfrac{1}{6}\epsilon b q^6 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.

Solution: In terms of the action variables of the harmonic oscillator, the full Hamiltonian reads:

$$H(\phi_0, J_0) = \nu_0 J_0 + \tfrac{1}{6}\epsilon b\Big(\sqrt{\tfrac{2J_0}{m\nu_0}}\sin\phi_0\Big)^6 \tag{1}$$

where ν_0 is the intrinsic frequency given by $\sqrt{k/m}$. The first order perturbation of the energy is:

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J)\rangle = \tfrac{4}{3}\tfrac{bJ^3}{m^3\nu_0^3}\int\limits_0^{2\pi}\sin^6\phi_0 \tfrac{d\phi_0}{2\pi} = \tfrac{5}{12}\tfrac{bJ^3}{m^3\nu_0^3} \tag{2}$$

Then the first order perturbed frequency is:

$$\nu_1 = \tfrac{5}{4}\tfrac{bJ^2}{m^3\nu_0^3} = \tfrac{5}{16}\tfrac{bA^4}{m\nu_0} \tag{3}$$

The first order of the action is determined by the following first-order differentiate equation

$$\nu_0\tfrac{\partial S_1}{\partial\phi_0} = \langle\tilde{H}_1\rangle - H_1 = \tfrac{bJ^3}{m^3\nu_0^3}\big(\tfrac{5}{12} - \tfrac{4}{3}\sin^6\phi_0\big) \tag{4}$$

Integrating over the above the equation, we obtain

$$S_1 = \tfrac{1}{144}\tfrac{bJ^3}{m^3\nu_0^4}(45\sin 2\phi_0 - 9\sin 4\phi_0 + \sin 6\phi_0) \tag{5}$$

Thus, we have

$$\begin{aligned}\phi &= \phi_0 + \tfrac{\partial S_1}{\partial J} = \phi_0 + \tfrac{\epsilon}{48}\tfrac{bJ^2}{m^3\nu_0^4}(45\sin 2\phi_0 - 9\sin 4\phi_0 + \sin 6\phi_0)\\ J_0 &= J + \tfrac{\partial S_1}{\partial\phi_0} = J + \tfrac{\epsilon}{24}\tfrac{bJ^3}{m^3\nu_0^4}(15\cos 2\phi_0 - 6\cos 4\phi_0 + \cos 6\phi_0)\end{aligned} \tag{6}$$

Inverting the second equation up to $O(\epsilon^2)$, we reach the final answer:

$$J = J_0 - \tfrac{\epsilon}{24}\tfrac{bJ_0^3}{m^3\nu_0^4}(15\cos 2\phi_0 - 6\cos 4\phi_0 + \cos 6\phi_0) \tag{7}$$

[2] Consider the Hamiltonian

$$H(q,p) = \left(1 + \epsilon\,\frac{q^2}{a^2}\right)\frac{p^2}{2m} + \tfrac{1}{2}m\omega_0^2 q^2 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.

Solution: In terms of the action variables of the harmonic oscillator, the full Hamiltonian reads:

$$H(\phi_0, J_0) = \omega_0 J_0 + 2\epsilon \tfrac{J_0^2}{ma^2} \sin^2\phi_0 \cos^2\phi_0 \tag{8}$$

Therefore, the first order perturbation of the energy is:

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J)\rangle = \tfrac{2J^2}{ma^2} \int\limits_0^{2\pi} \sin^2\phi_0 \cos^2\phi_0 \tfrac{d\phi_0}{2\pi} = \tfrac{J^2}{4ma^2} \tag{9}$$

Then the first order perturbed frequency is:

$$\nu_1 = \tfrac{J}{2ma^2} = \tfrac{\omega_0 A^2}{4a^2} \tag{10}$$

The first order of the action is determined by the following first-order differentiate equation

$$\omega_0 \tfrac{\partial S_1}{\partial \phi_0} = \langle \tilde{H}_1 \rangle - H_1 = \tfrac{J^2}{ma^2}(\tfrac{1}{4} - \sin^2\phi_0 \cos^2\phi_0) \tag{11}$$

Integrating over the above the equation, we obtain

$$S_1 = \tfrac{1}{16} \tfrac{J^2}{ma^2\omega_0} \sin 4\phi_0 \tag{12}$$

Thus, we have

$$\begin{aligned} \phi &= \phi_0 + \tfrac{\partial S_1}{\partial J} = \phi_0 + \tfrac{\epsilon}{8} \tfrac{J}{ma^2\omega_0} \sin 4\phi_0 \\ J_0 &= J + \tfrac{\partial S_1}{\partial \phi_0} = J + \tfrac{\epsilon}{2} \tfrac{J^2}{ma^2\omega_0} \cos 4\phi_0 \end{aligned} \tag{13}$$

Inverting the second equation up to $O(\epsilon^2)$, we reach the final answer:

$$J = J_0 - \tfrac{\epsilon}{2} \tfrac{J_0^2}{ma^2\omega_0} \cos 4\phi_0 \tag{14}$$

[**3**] Consider the $n = 2$ Hamiltonian $H(\boldsymbol{J}, \boldsymbol{\phi}) = H_0(\boldsymbol{J}) + \epsilon\, H_1(\boldsymbol{\phi})$, where

$$\begin{aligned} H_0(\boldsymbol{J}) &= \Lambda J_1^{3/2} + \Omega J_2 \\ H_1(\boldsymbol{\phi}) &= \cos\phi_1 \sum_{-\infty}^{\infty} V_n\, e^{in\phi_2} \quad . \end{aligned}$$

(a) Obtain an expression for $J_1(t)$ valid to first order in ϵ.

(b) Which tori are destroyed by the perturbation?

Solution: from the unperturbed part, we obtain the zeroth order of the two frequencies:

$$\begin{aligned} \nu_{1,0} &= \frac{\partial H_0}{\partial J_1} = \frac{3}{2}\Lambda J_1^{1/2} \\ \nu_{2,0} &= \frac{\partial H_0}{\partial J_2} = \Omega \end{aligned} \tag{15}$$

We proceed formally as before, and reach the differential equation that determines S:

$$\nu_{1,0}\frac{\partial S_1}{\partial \phi_{1,0}} + \nu_{2,0}\frac{\partial S_1}{\partial \phi_{2,0}} = \langle H_1 \rangle - H_1 = -\cos\phi_1 \sum_{-\infty}^{\infty} V_n\, e^{in\phi_2} \tag{16}$$

The solution is given by:

$$S_1 = \frac{i}{2}\sum_{-\infty}^{\infty}\left(\frac{V_n}{n\nu_{2,0}+\nu_{1,0}} e^{in\phi_{2,0}+i\phi_{1,0}} + \frac{V_n}{n\nu_{2,0}-\nu_{1,0}} e^{in\phi_{2,0}-i\phi_{1,0}}\right) \tag{17}$$

Therefore,

$$J_{1,0} = J_1 + \epsilon\frac{\partial S}{\partial \phi_{1,0}} = J_1 + \epsilon\sum_{-\infty}^{\infty}\left(\frac{V_n}{n\nu_{2,0}+\nu_{1,0}} e^{in\phi_{2,0}+i\phi_{1,0}} - \frac{V_n}{n\nu_{2,0}-\nu_{1,0}} e^{in\phi_{2,0}-i\phi_{1,0}}\right) \tag{18}$$

where

$$\begin{aligned} \phi_{1,0}(t) &= \phi_{1,0}(0) + \nu_{1,0}t \\ \phi_{2,0}(t) &= \phi_{2,0}(0) + \nu_{2,0}t \end{aligned} \tag{19}$$

When the ratio between $\nu_{0,1}$ and $\nu_{0,2}$ is a integer, one of the terms in the series diverges, implying the breaking down of the perturbation theory. As a consequence, the tori specified by the following condition:

$$\frac{\nu_{0,1}}{\nu_{0,2}} = \frac{3}{2}\frac{\Lambda J_{1,0}^{1/2}}{\Omega} = n \tag{20}$$

are destroyed by arbitrarily small pertubation.

[**4**] Is the following four-dimensional map canonical?

$$\begin{aligned} x_{n+1} &= 2\alpha x_n - \gamma x_n^2 - p_n + X_n^2 \\ p_{n+1} &= x_n \\ X_{n+1} &= 2\beta X_n - P_n + 2x_n X_n \\ P_{n+1} &= X_n \quad . \end{aligned}$$

Solution: The strategy here is to check whether this map preserves the symplectic structure of the Hamiltonian equation, namely whether the Jacobian of the transformation M satisfies $M\mathbb{J}M^T = \mathbb{J}$. Define the original vector $\xi = (x_n, X_n, p_n, P_n)$ and the transformed vector $\Xi = (x_{n+1}, X_{n+1}, p_{n+1}, P_{n+1})$. Explicitly, the Jacobian is:

$$M = \frac{\partial \Xi}{\partial \xi} = \begin{pmatrix} 2\alpha - 2\gamma X & 2X & -1 & 0 \\ 2X & 2x + 2\beta & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{21}$$

Then it is straightforward to show that, indeed,

$$M\mathbb{J}M^T = \mathbb{J} \tag{22}$$

Therefore M is symplectic and the transformation is canonical.