PHYSICS 200B : CLASSICAL MECHANICS
SOLUTION SET #1

[**1**] Evaluate all cases of $\{A_i, A_j\}$, where

$$A_1 = \tfrac{1}{4}\big(x^2 + p_x^2 - y^2 - p_y^2\big) \qquad A_3 = \tfrac{1}{2}\big(x\,p_y - y\,p_x\big)$$

$$A_2 = \tfrac{1}{2}\big(x\,y + p_x\,p_y\big) \qquad A_4 = x^2 + y^2 + p_x^2 + p_y^2 \ .$$

Solution : Recall

$$\{A,B\} = \sum_{\sigma=1}^{n} \left(\frac{\partial A}{\partial q_\sigma}\,\frac{\partial B}{\partial p_\sigma} - \frac{\partial B}{\partial q_\sigma}\,\frac{\partial A}{\partial p_\sigma} \right) .$$

Using

$$\begin{array}{llll}
\dfrac{\partial A_1}{\partial x} = \tfrac{1}{2}x & \dfrac{\partial A_1}{\partial y} = -\tfrac{1}{2}y & \dfrac{\partial A_1}{\partial p_x} = \tfrac{1}{2}p_x & \dfrac{\partial A_1}{\partial p_y} = -\tfrac{1}{2}p_y \\
\dfrac{\partial A_2}{\partial x} = \tfrac{1}{2}y & \dfrac{\partial A_2}{\partial y} = \tfrac{1}{2}x & \dfrac{\partial A_2}{\partial p_x} = \tfrac{1}{2}p_y & \dfrac{\partial A_2}{\partial p_y} = \tfrac{1}{2}p_x \\
\dfrac{\partial A_3}{\partial x} = \tfrac{1}{2}p_y & \dfrac{\partial A_3}{\partial y} = -\tfrac{1}{2}p_x & \dfrac{\partial A_3}{\partial p_x} = -\tfrac{1}{2}y & \dfrac{\partial A_3}{\partial p_y} = \tfrac{1}{2}x \\
\dfrac{\partial A_4}{\partial x} = 2x & \dfrac{\partial A_4}{\partial y} = 2y & \dfrac{\partial A_4}{\partial p_x} = 2p_x & \dfrac{\partial A_4}{\partial p_y} = 2p_y \ ,
\end{array}$$

we obtain

$$\{A_i, A_j\} = \varepsilon_{ijk}\,A_k$$
$$\{A_i, A_4\} = 0 \ ,$$

where i, j, and k are elements of $\{1,2,3\}$, and ε_{ijk} is the completely antisymmetric tensor of rank 3, with $\varepsilon_{123} = +1$.

[**2**] Determine the generating function $F_3(p,Q)$ which produces the same canonical transformation as the generating function $F_2(q,P) = q^2 \exp(P)$.

Solution : We have

$$F_2(q,P) = q^2 \exp(P) \quad \Rightarrow$$

$$p = \frac{\partial F_2}{\partial q} = 2q\exp(P) \quad , \quad Q = \frac{\partial F_2}{\partial P} = q^2 \exp(P) \ .$$

The generator $F_3(p,Q)$ is given by

$$F_3(p,Q) = F_2(q,P) - qp - QP \ .$$

To represent F_3 in terms of its proper arguments p and Q, we must find $q = q(p, Q)$ and $P = P(p, Q)$, which are easily obtained. We first eliminate $\exp(P)$ to obtain $q = 2Q/p$. Then we eliminate q, yielding $p^2 = 4Q\exp(P)$, or $P = \ln(p^2/4Q)$. Thus,

$$\begin{aligned}
F_3(p, Q) &= q^2 \exp(P) - qp - QP \\
&= \frac{4Q^2}{p^2} \cdot \frac{p^2}{4Q} - \frac{2Q}{p} \cdot p - Q \cdot \ln(p^2/4Q) \\
&= -Q - Q\ln(p^2/4Q) \ .
\end{aligned}$$

One can now check explicitly that $F_3(p, Q)$ generates the same transformation:

$$q = -\frac{\partial F_3}{\partial p} = \frac{2Q}{p} \quad , \quad P = -\frac{\partial F_3}{\partial Q} = \ln(p^2/4Q) \ .$$

[**3**] Show explicitly that the canonical transformation generated by an arbitrary $F_1(q, Q, t)$ preserves the symplectic structure of Hamilton's equations. That is, show that

$$M_{aj} \equiv \frac{\partial \Xi_a}{\partial \xi_j}$$

is symplectic. *Hint : Start by writing* $p_\sigma = \frac{\partial F_1}{\partial q_\sigma}$ *and* $P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma}$*, and then evaluate the differentials* dp_σ *and* dP_σ.

Solution :

From

$$p_\sigma = \frac{\partial F_1}{\partial q_\sigma} \qquad , \qquad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \ ,$$

we take the differential of p_σ and P_σ to arrive at

$$\begin{aligned}
dp_\sigma &= \frac{\partial^2 F_1}{\partial q_\sigma \partial q_{\sigma'}} \, dq_{\sigma'} + \frac{\partial^2 F_1}{\partial q_\sigma \partial Q_{\sigma'}} \, dQ_{\sigma'} + \frac{\partial^2 F_1}{\partial q_\sigma \partial t} \, dt \\
&\equiv A_{\sigma\sigma'} \, dq_{\sigma'} + C_{\sigma\sigma'} \, dQ_{\sigma'} + u_\sigma \, dt
\end{aligned}$$

and

$$\begin{aligned}
dP_\sigma &= -\frac{\partial^2 F_1}{\partial q_{\sigma'} \partial Q_{\sigma'}} \, dq_{\sigma'} - \frac{\partial^2 F_1}{\partial Q_\sigma \partial Q_{\sigma'}} \, dQ_{\sigma'} + \frac{\partial^2 F_1}{\partial Q_\sigma \partial t} \, dt \\
&\equiv -C^{\rm t}_{\sigma\sigma'} \, dq_{\sigma'} - B_{\sigma\sigma'} \, dQ_{\sigma'} - v_\sigma \, dt \ ,
\end{aligned}$$

with

$$A_{\sigma\sigma'} = \frac{\partial^2 F_1}{\partial q_\sigma \partial q_{\sigma'}} \quad , \quad B_{\sigma\sigma'} = \frac{\partial^2 F_1}{\partial Q_\sigma \partial Q_{\sigma'}} \quad , \quad C_{\sigma\sigma'} = \frac{\partial^2 F_1}{\partial q_\sigma \partial Q_{\sigma'}}$$

and

$$u_\sigma = \frac{\partial^2 F_1}{\partial q_\sigma \partial t} \quad , \quad v_\sigma = \frac{\partial^2 F_1}{\partial Q_\sigma \partial t} \ .$$

Putting the dQ and dP terms on the LHS of the equations, and suppressing indices, we have

$$\begin{pmatrix} B & 1 \\ C & 0 \end{pmatrix} \begin{pmatrix} dQ \\ dP \end{pmatrix} = \begin{pmatrix} -C^{\mathrm{t}} & 0 \\ -A & 1 \end{pmatrix} \begin{pmatrix} dq \\ dp \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} dt \ .$$

Thus, *assuming C is invertible*,

$$M_{aj} = \frac{\partial \Xi_a}{\partial \xi_j} = \begin{pmatrix} B & 1 \\ C & 0 \end{pmatrix}^{-1} \begin{pmatrix} -C^{\mathrm{t}} & 0 \\ -A & 1 \end{pmatrix} ,$$

from which we obtain

$$\mathsf{det}(M) = \big[(-1)^n \,\mathsf{det}(C)\big]^{-1} \cdot (-1)^n \,\mathsf{det}(C^{\mathrm{t}}) = 1 \ .$$

We must however show more than $\mathsf{det}(M) = 1$. We must show that M is symplectic, *i.e.* $M^{\mathrm{t}} J M = 1$, where $J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$. To this end, we write

$$M = \begin{pmatrix} -C^{-1} A & C^{-1} \\ BC^{-1} A - C^{\mathrm{t}} & -BC^{-1} \end{pmatrix} ,$$

which follows from writing $dQ = -C^{-1} A \, dq + C^{-1} dp$ and then substituting this into $dP = -C^{\mathrm{t}} \, dq - B \, dQ$. We have here set $dt = 0$ since we are interested only in how changes in (q,p) affect (Q,P). Now $A = A^{\mathrm{t}}$ and $B = B^{\mathrm{t}}$ are explicitly symmetric, hence

$$M^{\mathrm{t}} = \begin{pmatrix} -AC^{\mathrm{t}\,-1} & AC^{\mathrm{t}\,-1} B - C \\ C^{\mathrm{t}\,-1} & -C^{\mathrm{t}\,-1} B \end{pmatrix} .$$

Clearly

$$JM = \begin{pmatrix} BC^{-1} A - C^{\mathrm{t}} & -BC^{-1} \\ C^{-1} A & -C^{-1} \end{pmatrix}$$

It is then a simple matter to verify

$$M^{\mathrm{t}} J M = J \ .$$

[4] Consider the small oscillations of an anharmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m \, \omega^2 \, q^2 + \alpha \, q^3 + \beta \, q \, p^2$$

under the assumptions $\alpha \, q \ll m \, \omega^2$ and $\beta \, q \ll \frac{1}{m}$.

(a) Working with the generating function

$$F_2(q,P) = qP + a \, q^2 \, P + b \, P^3 ,$$

find the parameters a and b such that the new Hamiltonian $\tilde{H}(Q,P)$ does not contain any anharmonic terms up to third order (*i.e.* no terms of order Q^3 nor of order QP^2).

(b) Determine $q(t)$.

(a) We have

$$p = \frac{\partial F_2}{\partial q} = P + 2aqP$$
$$Q = \frac{\partial F_2}{\partial P} = q + aq^2 + 3bP^2 \ .$$

We invert the latter equation to obtain $q(Q,P)$, then substitute into the former equation to get $p(Q,P)$:

$$q = Q - aQ^2 - 3bP^2 + \ldots$$
$$p = P + 2aQP + \ldots \ .$$

We now write the Hamiltonian in terms of Q and P:

$$\tilde{H}(Q,P) = \frac{P^2}{2m} + \tfrac{1}{2}m\omega^2 Q^2 + \left(\alpha - m\omega^2 a\right) Q^3 + \left(\beta + \frac{2a}{m} - 3m\omega^2 b\right) QP^2 + \ldots \ .$$

Setting the coefficients of the cubic terms to zero, and solving for a and b,

$$a = \frac{\alpha}{m\omega^2} \quad , \quad b = \frac{2\alpha}{3m^3\omega^4} + \frac{\beta}{3m\omega^2} \ .$$

With these choices for a and b, the transformed Hamiltonian becomes

$$\tilde{H}(Q,P) = \frac{P^2}{2m} + \tfrac{1}{2}m\omega^2 Q^2 + \mathcal{O}(Q^4, Q^3P, Q^2P^2, QP^3, P^4)$$

(b) The solution to Hamilton's equations for Q and P is now

$$Q(t) = A\cos(\omega t + \delta) \quad , \quad P = -m\omega A \sin(\omega t + \delta) \ .$$

We substitute these expressions into the earlier result,

$$q = Q - aQ^2 - 3bP^2 + \ldots \tag{1}$$

to obtain

$$q(t) = A\cos\theta + \left(\frac{3\alpha}{m\omega^2} + m\beta\right) \cdot \tfrac{1}{2}A^2\cos(2\theta) - \left(\frac{\alpha}{m\omega^2} + m\beta\right) \cdot \tfrac{1}{2}A^2 + \ldots \ ,$$

with $\theta = \omega t + \delta$. Note that the center of the oscillation has shifted to the left by an amount proportional to A^2. This is because the original Hamiltonian $H(q,p)$ is no longer symmetric under the parity operation $q \to -q$.

[5] A particle of mass m moves in one dimension subject to the potential

$$U(x) = \frac{k}{\sin^2(x/a)} \ .$$

(a) Obtain an integral expression for Hamilton's characteristic function.

(b) Under what conditions may action-angle variables be used?

(c) Assuming that action-angle variables are permissible, determine the frequency of oscillation by the action-angle method.

(d) Check your result for the oscillation frequency in the limit of small oscillations.

Solution :

(a) We must solve

$$\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + \frac{k}{\sin^2(x/a)} = Q \ .$$

Note that $Q = E$, the total energy, which is conserved. The motion is therefore between the turning points

$$x_-(E) = n\pi a + a\sin^{-1}\sqrt{k/E} \quad , \quad x_+(E) = (n+1)\pi a - a\sin^{-1}\sqrt{k/E} \ ,$$

where n is any integer. We may then write

$$W(x,E) = \sqrt{2m}\int\limits_{x_-(E)}^{x} dx' \sqrt{E - \frac{k}{\sin^2(x'/a)}} \ .$$

The lower limit may be left as unspecified; this only changes the result by a constant.

(b) We need that the motion is bounded. In our case, $x_-(E) \le x \le x_+(E)$.

(c) We have

$$\begin{aligned} J &= \frac{1}{2\pi}\sqrt{2m}\oint dx \sqrt{E - \frac{k}{\sin^2(x'/a)}} \\ &= \frac{a}{2\pi}\sqrt{2mE}\int\limits_0^{2\pi} du \, \frac{1-\cos u}{\frac{E+k}{E-k} - \cos u} \ , \end{aligned}$$

where we have substituted

$$\cos(x/a) = \sqrt{1 - \frac{k}{E}}\,\cos(\tfrac{1}{2}u) \ .$$

Mathematical Interlude : We are interested in evaluating

$$\int_0^{2\pi} du\, \frac{1-\cos u}{b-\cos u} = 2\pi + (1-b)\int_0^{2\pi} \frac{du}{b-\cos u} \ ,$$

where $b > 1$. We do this by the method of contour integration. Consider the integral

$$\begin{aligned} \mathcal{I} &= \int_0^{2\pi} \frac{du}{2\pi}\, \frac{1}{b-\cos u} = \oint_{|z|=1} \frac{dz}{2\pi i z}\, \frac{2}{2b - z - z^{-1}} \\ &= -\oint_{|z|=1} \frac{dz}{2\pi i}\, \frac{2}{z^2 - 2bz + 1} = -\oint_{|z|=1} \frac{dz}{2\pi i}\, \frac{2}{(z-z_+)(z-z_-)} \ , \end{aligned}$$

where

$$z_\pm = b \pm \sqrt{b^2 - 1} \ .$$

Note above we have used $z = e^{iu}$, $du = dz/iz$ in obtaining the contour integral. The root z_- lies within the circle $|z| = 1$; z_+ lies outside; note that $z_+ z_- = 1$. We therefore have

$$\mathcal{I} = -\frac{2}{z_- - z_+} = \frac{1}{\sqrt{b^2 - 1}} \ .$$

Using the results from our pleasant interlude, with $b = (E+k)/(E-k)$, we find

$$J = \sqrt{2m}\, a \left(\sqrt{E} - \sqrt{k}\right) \quad , \quad E = \left(\frac{J}{\sqrt{2m}\, a} + \sqrt{k}\right)^2 \ .$$

Note that the minimum energy is $E_{\text{min}} = k$. The oscillation frequency is given by

$$\nu(J) = \frac{\partial E}{\partial J} = \frac{J}{ma^2} + \sqrt{\frac{2k}{ma^2}} = \sqrt{\frac{2E}{ma^2}} \ .$$

(d) With $U(x) = k/\sin^2(x/a)$ we have

$$\begin{aligned} U'(x) &= -\frac{2k}{a} \cdot \frac{\cos(x/a)}{\sin^3(x/a)} \\ U''(x) &= \frac{2k}{a^2} \cdot \frac{\sin^4(x/a) + 3\sin^2(x/a)\cos^2(x/a)}{\sin^6(x/a)} \ . \end{aligned}$$

Setting $U'(x^*) = 0$ we obtain $x^* = (n + \frac{1}{2})\,\pi a$, where $n \in \mathbb{Z}$. At any of these equilibria, $U''(x^*) = 2k/a^2$. Therefore, the frequency of small oscillations is

$$\omega_{\text{s.o.}} = \sqrt{\frac{U''(x^*)}{m}} = \sqrt{\frac{2k}{ma^2}} \ ,$$

which agrees with the result from part `(c)`.