PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAMINATION WINTER 2017

The time limit for this exam is six consecutive hours. You may consult the course lecture notes, problem set solutions, and the course text, but no other sources.

(1) Consider the Hamiltonian for one-dimensional particle motion in a gravitational field,

$$H(z,p) = \overbrace{\frac{p^2}{2m} + mgz}^{H_0} + \overbrace{\varepsilon\alpha z^3}^{\varepsilon H_1} \quad ,$$

where ε is small. The particle is constrained such that $z \geq 0$. It should be useful to consult §1.5.5 of the Lecture Notes.

(a) Find the unperturbed Hamiltonian $\widetilde{H}_0(J_0)$ and the unperturbed frequency $\nu_0(J_0)$.

(b) Find the unperturbed frequencies $\nu_0(h)$, where h is the amplitude of the z motion. Your result should look familiar.

(c) Find the energy $E(J)$ to lowest nontrivial order in ε.

(2) Consider relaxation oscillations for the second order ODE

$$\ddot{x} + \mu F'(x)\,\dot{x} + x = 0 \quad ,$$

where $\mu \gg 1$, and where the function $F(x)$ is given by

$$F(x) = \begin{cases} +Ax(x-a) & \text{if } x \geq 0 \\ -Bx(x+b) & \text{if } x < 0 \end{cases} \quad ,$$

where the constants A, B, a, and b are all positive. *Warning : the function $F(x)$ is not necessarily antisymmetric!*

(a) Sketch the function $F(x)$, and identify the location of all local minima and maxima.

(b) Find the stable limit cycle, and identify the slow and fast sections.

(c) Find the duration of each of the slow sections, and find expressions for the period of the limit cycle in the $\mu \gg 1$ limit. Evaluate your results for the case $A = 1$, $B = 2$, $a = \frac{3}{2}$, and $b = \frac{1}{2}$.

(3) Consider the forced nonlinear oscillator

$$\frac{d\Psi}{dt} = (1 + i\alpha)\Psi - \big(1 + i\beta|\Psi|\big)|\Psi|\,\Psi + \varepsilon\,\cos(\omega t) \quad ,$$

where $\Psi(t)$ is a complex function of time, and α and β are real numbers.

(a) For $\varepsilon = 0$, what is the stable limit cycle and what is its frequency ω_0? *Hint: Write* $\Psi = R\exp(i\Theta)$.

(b) Find the equation for the isochrones $\phi(R, \Theta)$ for the unperturbed ($\varepsilon = 0$) system.

(c) What is the resonance condition in terms of ω and ω_0?

(d) Derive the equation $\dot{\psi} = -\nu + \varepsilon\, G(\psi)$, where $\psi = \langle\phi\rangle - \omega t$, where $\langle\phi\rangle$ is the average of the phase ϕ on time scales short compared with $|\nu|^{-1}$ but long compared with the period of the nonresonant terms in $\dot{\phi}$. Find the function $G(\psi)$.

(e) What is the condition on the detuning ν in order for synchronization to occur?

(f) When ν lies outside the regime of synchronization, what is the period $T(\nu)$ over which $\psi(t)$ advances by 2π?