PHYSICS 200B : CLASSICAL MECHANICS
FINAL EXAMINATION

(1) Consider the nonlinear oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 + \tfrac{1}{4}\epsilon a q^4 + \tfrac{1}{4}\epsilon b p^4 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.

Solution:

With $k \equiv m\nu_0^2$, recall the AA variables

$$\phi_0 = \tan^{-1}\!\left(\frac{m\nu_0 q}{p}\right) \qquad , \qquad J_0 = \frac{p^2}{2m\nu_0} + \tfrac{1}{2}m\nu_0 q^2 \quad .$$

Thus, $q = (2J_0/m\nu_0)^{1/2}\sin\phi_0$ and $p = (2m\nu_0 J_0)^{1/2}\cos\phi_0$, so the Hamiltonian is

$$\widetilde{H}(\phi_0, J_0) = \nu_0 J_0 + \epsilon \widetilde{H}_1(\phi_0, J_0) \quad ,$$

where

$$\widetilde{H}_1(\phi_0, J_0) = \frac{aJ_0^2}{m^2\nu_0^2}\sin^4\!\phi_0 + b\, m^2\nu_0^2 J_0^2 \cos^4\!\phi_0 \quad .$$

(a) Averaging over ϕ_0, we have $\langle \sin^4\!\phi_0 \rangle = \langle \cos^4\!\phi_0 \rangle = \frac{3}{8}$, so

$$E_1(J) = \big\langle \widetilde{H}_1(\phi_0, J) \big\rangle = \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{8}J^2 \quad .$$

The perturbed frequencies are $\nu(J) = \nu_0 + \epsilon\nu_1$ where $\nu_1 = \frac{\partial E_1}{\partial J}$. Thus,

$$\nu(J) = \sqrt{\frac{k}{m}} + \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{4}\epsilon J \quad .$$

(b) We only need J to zeroth order in ϵ. Setting $p = 0$ and $q = A$ gives $J = \frac{1}{2}m\nu_0 A^2 + \mathcal{O}(\epsilon)$, in which case

$$\nu(A) = \sqrt{\frac{k}{m}} + \left(\frac{a}{mk} + bmk\right) \times \tfrac{3}{8}\epsilon m\nu_0 A^2 \quad .$$

(c) Recall the desired type-II CT is generated by $S(\phi_0, J) = \phi_0 J + \epsilon S_1(\phi_0, J) + \ldots$, with

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \widetilde{H}_1 \rangle - \widetilde{H}_1}{\nu_0(J)} \quad .$$

Thus,

$$\frac{\partial S_1}{\partial \phi_0} = \frac{aJ^2}{m^2\nu_0^3}\left(\tfrac{3}{8} - \sin^4\!\phi_0\right) + bm^2\nu_0 J\left(\tfrac{3}{8} - \cos^4\!\phi_0\right) \quad .$$

Integrating, we have

$$S_1(\phi_0, J) = \frac{aJ^2}{m^2\nu_0^3}\Big(\tfrac{1}{4}\sin(2\phi_0) - \tfrac{1}{32}\sin(4\phi_0)\Big) - bm^2\nu_0 J^2\Big(\tfrac{1}{4}\sin(2\phi_0) + \tfrac{1}{32}\sin(4\phi_0)\Big) \quad .$$

The constant may be set to zero as it leads to a constant shift of the angle variable ϕ. Thus, we have

$$\begin{aligned} J_0 &= J + \epsilon\,\frac{\partial S_1}{\partial \phi_0} + \mathcal{O}(\epsilon^2) \\ &= J + \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J^2\cos(2\phi_0) - \left(\frac{a + bm^2\nu_0^4}{8m^2\nu_0^3}\right)\epsilon J^2\cos(4\phi_0) + \mathcal{O}(\epsilon^2) \quad . \end{aligned}$$

Thus,

$$J = J_0 - \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J_0^2\cos(2\phi_0) + \left(\frac{a + bm^2\nu_0^4}{8m^2\nu_0^3}\right)\epsilon J_0^2\cos(4\phi_0) + \mathcal{O}(\epsilon^2) \quad .$$

We then have

$$\begin{aligned} \phi &= \phi_0 + \epsilon\,\frac{\partial S_1}{\partial J} + \mathcal{O}(\epsilon^2) \\ &= \phi_0 + \left(\frac{a - bm^4\nu_0^4}{2m^2\nu_0^3}\right)\epsilon J_0\sin(2\phi_0) - \left(\frac{a + bm^2\nu_0^4}{16m^2\nu_0^3}\right)\epsilon J_0\sin(4\phi_0) + \mathcal{O}(\epsilon^2) \quad . \end{aligned}$$

(2) Consider the forced modified van der Pol equation,

$$\ddot{x} + \epsilon(x^4 - 1)\,\dot{x} + x = \epsilon f_0\cos(t + \epsilon\nu t) \quad ,$$

where ϵ is small. Carry out the multiple scale analysis to order ϵ. Following §3.3.2 in the Lecture Notes, find and analyze the equation which relates the amplitude A, detuning ν, and force amplitude f_0 for entrained oscillations. Perform the requisite linear stability analysis and make a plot similar to that in Fig. 3.4 of the Lecture Notes. Is there a region of entrained oscillations which exhibits hysteresis as the detuning parameter is varied? If so, find the corresponding range of f_0 over which this occurs.

Bonus: Use `Mathematica` *or* `Matlab` *to integrate the equation, showing examples of entrained and heterodyne behavior, as in Fig. 3.6 (1000 Quatloos extra credit).*

Solution:

In the multiple scale analysis (MSA), we define a hierarchy of time scales $T_n = \epsilon^n t$, and we expand $x(t) = \sum_{n=0}^{\infty} \epsilon^n \, x_n(T_0, T_1, \ldots)$. The general forced nonlinear oscillator equation is written

$$\ddot{x} + x = \epsilon \, h(x, \dot{x}) + \epsilon \, f_0 \cos(t + \epsilon \nu t) \quad ,$$

where $\epsilon\nu$ is the detuning. We write $\frac{d}{dt} = \sum_{k=0}^{\infty} \epsilon^k \, \frac{\partial}{\partial T_k}$ and derive a hierarchy order by order in ϵ. As shown in §3.3 of the Lecture Notes, to lowest order we have

$$\bigg(\frac{\partial^2}{\partial T_0^2} + 1\bigg) x_0 = 0 \qquad \Rightarrow \qquad x_0 = A \cos(T_0 + \phi) \quad ,$$

where the amplitude $A = A(T_1, T_2, \ldots)$ and phase $\phi = \phi(T_1, T_2, \ldots)$ are independent of T_0. At the next level of the hierarchy, we define $\theta = T_0 + \phi(T_1)$ and $\psi(T_1) \equiv \phi(T_1) - \nu T_1$, where dependences on the scales $\{T_1, T_3, \ldots\}$ are implicit. At order ϵ^1, we have

$$\bigg(\frac{\partial^2}{\partial \theta^2} + 1\bigg) x_1 = 2 \, \frac{\partial A}{\partial T_1} \sin\theta + 2A \, \frac{\partial \phi}{\partial T_1} \cos\theta + h(A\cos\theta, -A\sin\theta) + f_0 \cos(\theta - \psi) \quad .$$

We Fourier transform the function $h(A\cos\theta, -A\sin\theta)$, writing

$$h(A\cos\theta, -A\sin\theta) = \sum_{k=0}^{\infty} \Big[\alpha_k(A) \sin(k\theta) + \beta_k(A) \cos(k\theta)\Big] \quad .$$

We then have

$$\bigg(\frac{\partial^2}{\partial \theta^2} + 1\bigg) x_1 = \sum_{k \neq 1} \Big[\alpha_k(A) \sin(k\theta) + \beta_k(A) \cos(k\theta)\Big] \quad ,$$

where the secular forcing $k = 1$ terms are eliminated by the requirements

$$\begin{aligned} \frac{dA}{dT_1} &= -\tfrac{1}{2} \alpha_1(A) - \tfrac{1}{2} f_0 \sin\psi \\ \frac{d\psi}{dT_1} &= -\nu - \frac{\beta_1(A)}{2A} - \frac{f_0}{2A} \cos\psi \quad , \end{aligned}$$

which may be written as coupled ODEs since the time scales $\{T_2, T_3, \ldots\}$ do not appear. At any fixed point, then, one must have

$$\Big[\alpha_1(A)\Big]^2 + \Big[2\nu A + \beta_1(A)\Big]^2 = f_0^2 \quad .$$

The linearized map in the vicinity of the fixed point (A^*, ψ^*) is given by

$$\frac{d}{dT_1} \begin{pmatrix} \delta A \\ \\ \delta\psi \end{pmatrix} = \overbrace{\begin{pmatrix} -\frac{1}{2}\alpha_1'(A) & \nu A + \frac{1}{2}\beta_1(A) \\ \\ -\frac{\beta_1'(A)}{2A} - \frac{\nu}{A} & -\frac{\alpha_1(A)}{2A} \end{pmatrix}}^{M} \begin{pmatrix} \delta A \\ \\ \delta\psi \end{pmatrix} \quad .$$

In our case, $h(x,\dot{x}) = (1-x^4)\,\dot{x}$, and therefore

$$\begin{aligned} h(A\cos\theta, -A\sin\theta) &= \big(1 - A^4\cos^4\!\theta\big)\big(-A\sin\theta\big) \\ &= \left(\frac{A^5}{8} - A\right)\sin\theta + \tfrac{3}{16}A^5\sin(3\theta) + \tfrac{1}{16}A^5\sin(5\theta) \quad . \end{aligned}$$

Thus,

$$\alpha_1(A) = \tfrac{1}{8}A^5 - A \quad , \quad \alpha_3(A) = \tfrac{3}{16}A^5 \quad , \quad \alpha_5(A) = \tfrac{1}{16}A^5 \quad ,$$

where all other $\alpha_k(A) = 0$ and all $\beta_k(A) = 0$. In particular, $\beta_1(A) = 0$, hence

$$G(y) \equiv \tfrac{1}{64}y^5 - \tfrac{1}{4}y^3 + \big(1 + 4\nu^2\big)y = f_0^2 \quad ,$$

where $y = A^2$; note that $G(0) = 0$. We must analyze the behavior of $G(y)$ for $y \ge 0$. Taking the derivative,

$$G'(y) = \tfrac{5}{64}y^4 - \tfrac{3}{4}y^2 + \big(1 + 4\nu^2\big) \quad .$$

The roots $G'(y) = 0$ lie at $y = y_\pm$, where

$$y_\pm^2 = \tfrac{8}{5}\Big(3 \pm 2\sqrt{1 - 5\nu^2}\,\Big) \quad .$$

Thus, when the argument of the square root is negative, there are no real solutions, which means $G(y)$ is monotonically increasing and $G(y) = f_0^2$ has a unique solution. This occurs for $\nu^2 > \frac{1}{5}$.

For $\nu^2 < \frac{1}{5}$, there are two solutions $G'(y_\pm) = 0$ with $y_\pm > 0$ and another two solutions at $y = -y_\pm$, since $G(y)$ is an odd function of y. Note that $G(y_-) > G(y_+)$. Thus, $G(y) = f_0^2$ has three solutions provided $f_0^2 \in \big[G(y_+)\,,\,G(y_-)\big] \cap \big[0,\infty\big)$. One then finds this is equivalent to the condition

$$(3+2u)^{3/2}\,(1-u) < \sqrt{\tfrac{3125}{512}}\,f_0^2 < (3-2u)^{3/2}\,(1+u) \quad ,$$

where $u = \sqrt{1 - 5\nu^2} \in [0,1]$. Note that for $\nu^2 = \frac{1}{5}$ ther root at $f_0^2 = \big(2^9{\cdot}3^3/5^5\big)^{1/2} = 2.10325$ is a double root. However, we still must check whether these solutions are stable. To do this, we compute the eigenvalues of the matrix M, with

$$M = \frac{1}{16}\begin{pmatrix} 8 - 5A^4 & 16\nu A \\ -16\,\nu A^{-1} & 8 - A^4 \end{pmatrix} \quad .$$

The eigenvalues are $\lambda_\pm = \frac{1}{2}T \pm \sqrt{\frac{1}{4}T^2 - D}$, where

$$T = \mathsf{Tr}(M) = 1 - \tfrac{3}{8}\,y^2 \qquad , \qquad D = \mathsf{det}\,M = \tfrac{5}{256}\,y^4 - \tfrac{3}{16}\,y^2 + \nu^2 + \tfrac{1}{4} = \tfrac{1}{4}\,G'(y) \quad .$$

The fixed point will be unstable if either of the eigenvalues has a positive real part. One possibility is a saddle point, which occurs for $D < 0$. This means $G'(y) < 0$, which means $y \in \big[y_-\,,\,y_+\big]$. Thus, when we have three solutions, the middle one is always unstable.

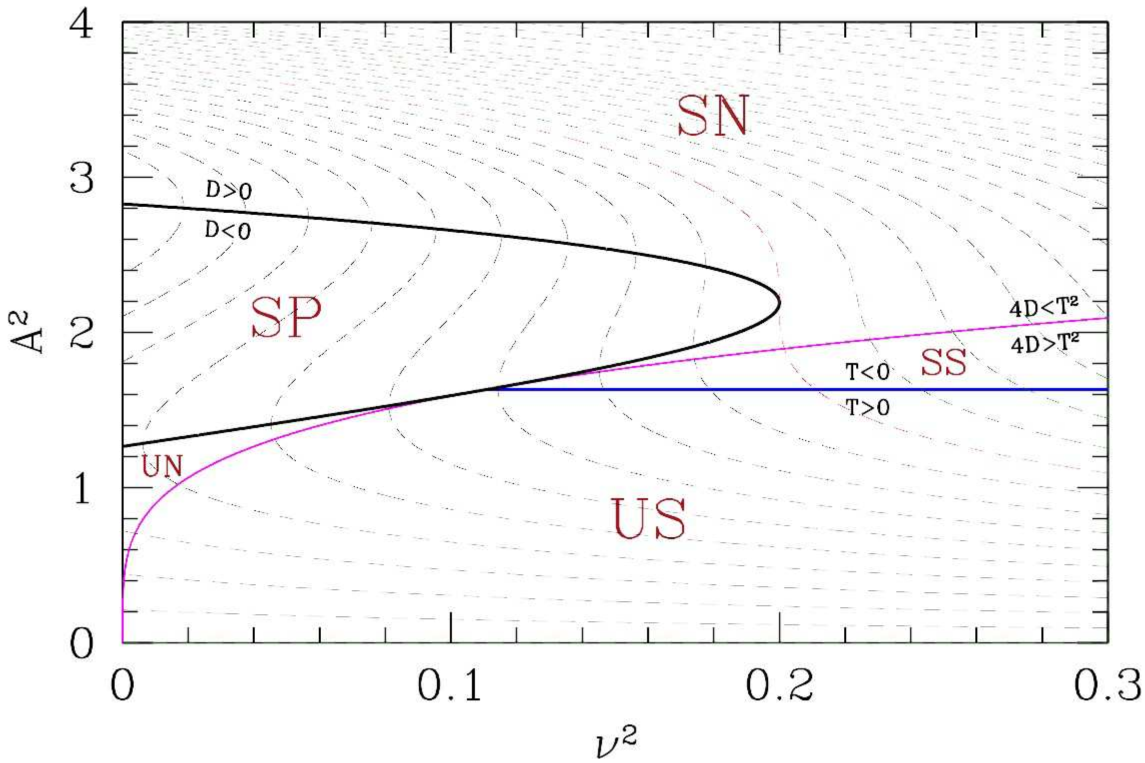

Figure 1: Fixed point solutions corresponding to entrained phases of the forced modified van der Pol oscillator. Thin dashed curves correspond to different values of f_0.

The other possibility is $T > 0$, leading to an unstable spiral or unstable node. This is equivalent to $y^2 < \frac{8}{3}$. Since a global analysis for large A shows the flow is inward, we conclude that the coupled ODEs for A and ψ must have a stable limit cycle in the portion of the (ν, y) plane corresponding to an unstable node or unstable spiral, *i.e.* where $y < \sqrt{\frac{8}{3}}$ and $G'(y) > 0$. The line $y = \sqrt{\frac{8}{3}}$ intersects the curve $D = 0$ at $\nu = \frac{1}{3}$. Thus, the phase diagram resembles that of Fig. 3.4 in the Lecture Notes. To find the range of f_0^2 over which there is hysteretic jumping between stable branches over some interval $\nu \in \big[\nu_- \, , \, \nu_+\big]$, we set $G(y) = G'(y) = 0$ and eliminate y to obtain $f_0^2 = \sqrt{\frac{256}{3125}}\,(3+2u)^{3/2}(1-u)$. We then evaluate $G(y) = f_0^2$, for the same value of ν, at the point where $\mathsf{Tr}(M) = 0$, *i.e.* $y = \sqrt{\frac{8}{3}}$, which yields $f_0^2 = \frac{4}{45}\sqrt{\frac{8}{3}}\,(14 - 9u^2)$. Eliminating f_0^2, we arrive at the quintic equation

$$125\,(14 - 9u^2)^2 = 972\,(3+2u)^3\,(1-u)^2 \quad .$$

The solution over the interval $u \in [0,1]$ is $u = 0.350851$, which gives us $f^2_{0,\text{min}} = 1.87136$. Thus, hysteresis occurs for $f_0^2 \in \big[1.87136 \, , \, 2.10325\big]$, *i.e.* $f_0 \in \big[1.3680 \, , \, 1.4503\big]$. Note one can also have hysteresis between a stable entrained solution and a stable limit cycle as parameters are varied.

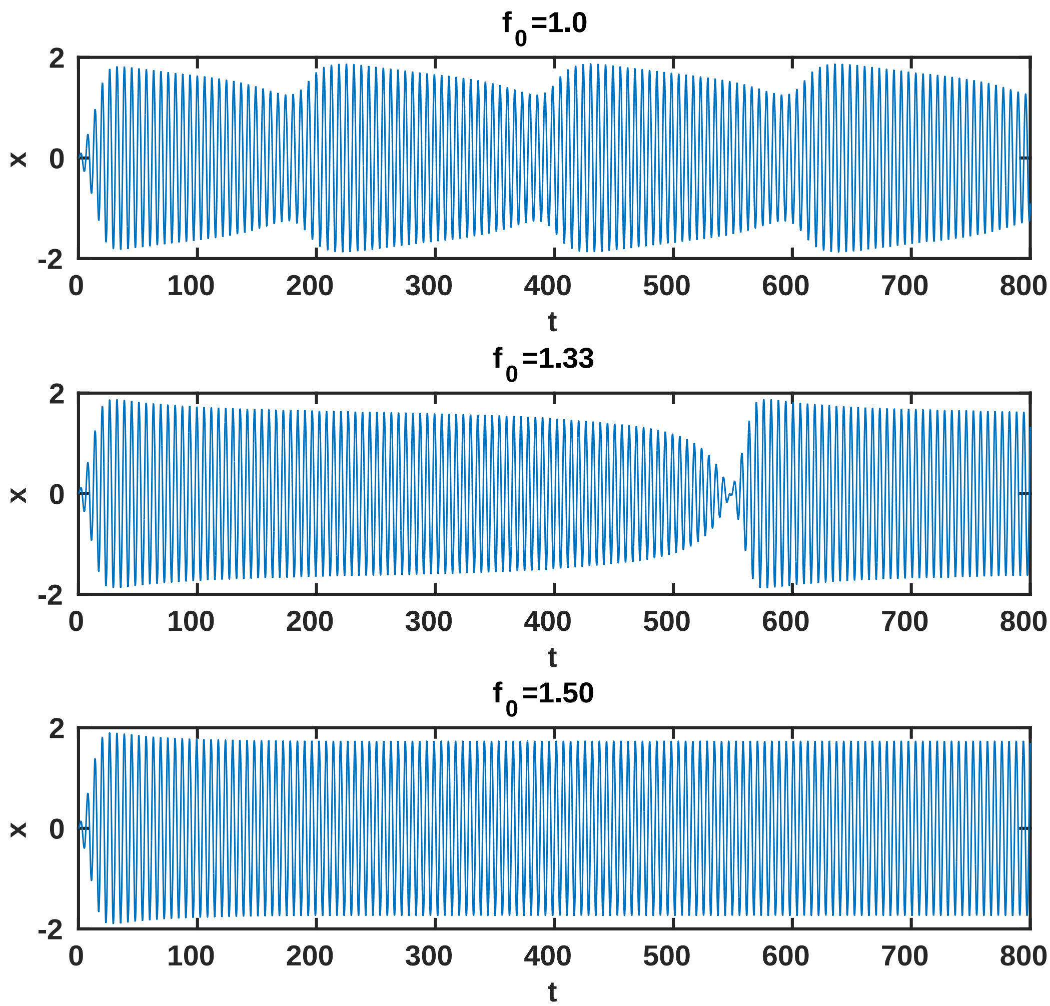

Figure 2: Entrained and heterodyne behavior of the forced modified van der Pol oscillator, with $\epsilon = 0.1$ and $\nu = 0.4$.

(3) Consider shock formation in the inviscid Burgers' equation, $c_t + c\,c_x = 0$. Let the function $c(\zeta) = c(x = \zeta, t = 0)$ be given by the triangular profile,

$$c(\zeta) = c_0 \left(\frac{a - |\zeta|}{a} \right) \Theta\big(a - |\zeta|\big) \quad .$$

(a) Find the break time $t_{\rm B}$.

(b) Implement the shock fitting protocol and find $\zeta_-(t)$, $\zeta_+(t)$, and $x_{\rm s}(t)$.

(c) Find the shock discontinuity $\Delta c(t)$ for $t > t_{\rm B}$.

(d) Sketch $c(x,t)$ *vs.* x for $t/t_{\rm B} = 0$, $\frac{1}{2}$, 1, 2, and 4. Show that for $t \geq t_{\rm B}$, $c(x,t)$ *vs.* x has the form of a right triangle whose area is given by $\int_{-a}^{a} d\zeta\, c(\zeta)$.

(e) Without shock fitting, sketch the characteristics in the (x,t) plane and highlight the region where they cross. Then sketch the characteristics after shock fitting. *Hint: Your sketches should roughly resemble those in Fig. 4.13 of the Lecture Notes.*

Solution:

(a) The break time is

$$t_{\mathrm{B}} = \min_{c'(\zeta)<0} \left(-\frac{1}{c'(\zeta)} \right) \equiv -\frac{1}{c'(\zeta_{\mathrm{B}})} \quad .$$

Thus, $t_{\mathrm{B}} = a/c_0$.

(b) We have two shock fitting equations:

$$x_{\mathrm{s}} = \zeta_- + c_- t = \zeta_+ + c_+ t \quad ,$$

where $c_\pm \equiv c(\zeta_\pm)$, and

$$\tfrac{1}{2}(\zeta_+ - \zeta_-)(c_+ + c_-) = \int_{\zeta_-}^{\zeta_+} d\zeta\, c(\zeta) \quad .$$

Clearly $\zeta_+ > a$ and therefore $c_+ = 0$. We also have $\zeta_- < 0$. The second of our shock fitting equations then gives

$$\zeta_+ = \zeta_- + \frac{a}{a+\zeta_-} \left(a - 2\zeta_- - \frac{\zeta_-^2}{a} \right) \quad .$$

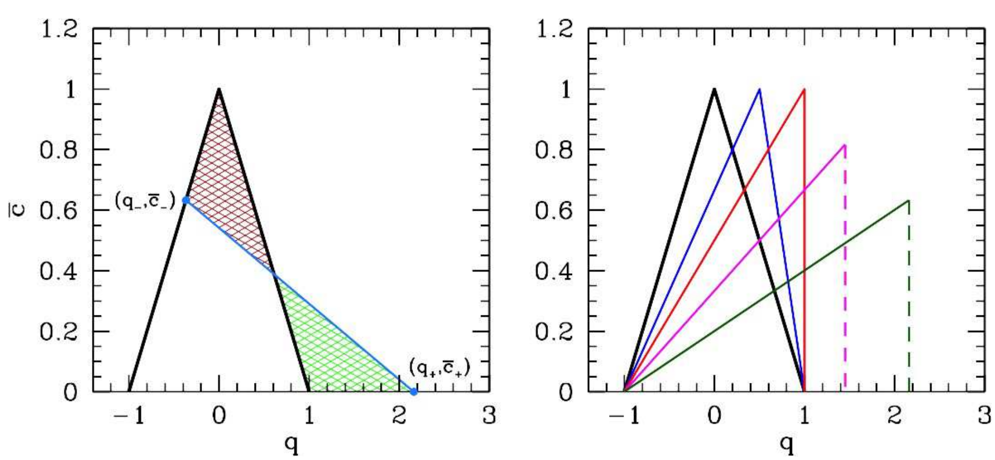

Figure 3: Left: Shock fitting requires the burgundy and green hatched areas to be equal. Right: Evolution of the initial profile at times $\tau = t/t_{\mathrm{B}} = 0$ (black), $\tau = \frac{1}{2}$ (blue), $\tau = 1$ (red), $\tau = 2$ (magenta), and $\tau = 4$ (green). The dashed line shows the shock discontinuity.

The first shock fitting equation gives $\zeta_+ - \zeta_- = c\_t$, and eliminating ζ_+ yields

$$c_0 t = \left(\frac{a}{a+\zeta_-}\right)^2 \left(a - 2\zeta_- - \frac{\zeta_-^2}{a}\right) \quad .$$

At this point it is convenient to define the dimensionless time $\tau \equiv c_0 t/a = t/t_{\rm B}$ as well as $q_\pm \equiv \zeta_\pm/a$. Note $q_{\rm s} = x_{\rm s}/a = q_+$ because $c_+ = 0$. Solving, we obtain

$$q_-(\tau) = -1 + \sqrt{\frac{2}{1+\tau}} \qquad , \qquad q_+(\tau) = -1 + \sqrt{2(1+\tau)} \quad .$$

(c) The dimensionless velocity is $\bar{c} = c/c_0$. Note $\bar{c}_\pm = 1 - |q_\pm|$. The shock discontinuity is then

$$\Delta\bar{c}(\tau) = \sqrt{\frac{2}{1+\tau}} \quad ,$$

where $\tau \geq 1$. Note $\Delta\bar{c}(\tau = 1) = 1$, which is nongeneric, since the discontinuity usually grows from zero starting at the break time. The nongeneric nature here is due to the piecewise linear initial profile. Note also $\Delta\bar{c}(\tau) \propto \tau^{-1/2}$ as $\tau \to \infty$. See Fig. 3.

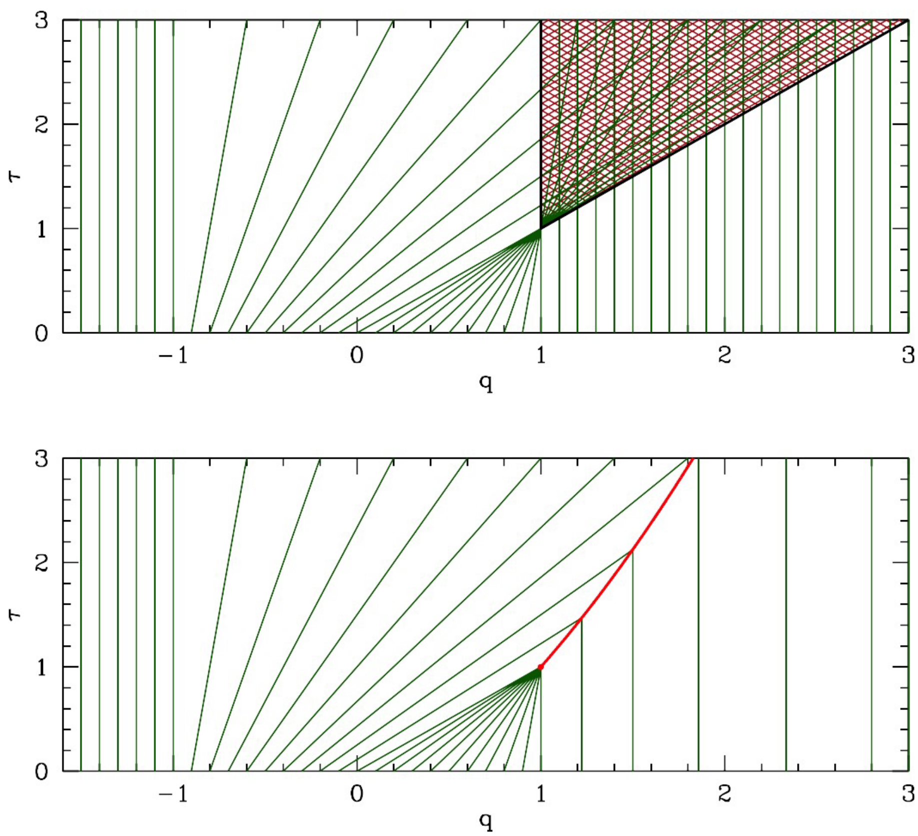

Figure 4: Top: Characteristics prior to shock fitting, showing intersection in the hatched region. Bottom: Characteristics with shock fitting. The shock trajectory is shown in red.

(d) For $t > t_{\rm B}$, the curve $\bar{c}(q,\tau)$ is a right triangle whose base is $1 + q_+(\tau)$ and height is $\Delta\bar{c}(\tau)$. Thus, the dimensionless area is

$$A(\tau > 1) = \tfrac{1}{2}\big(1 + q_+(\tau)\big)\,\Delta\bar{c}(\tau) = 1 = \int\limits_{-1}^{1} dq\,\big(1 - |q|\big) \quad ,$$

and so the area is preserved. For $\tau < 1$, we have $\bar{c}(q,\tau)$ is a triangle connecting the points $(-1,0)$ to $(\tau,1)$ to $(1,0)$, since the peak value moves with $\bar{c}_{\rm max} = 1$. The area is again

$$A(\tau < 1) = \tfrac{1}{2}\,(1+\tau) + \tfrac{1}{2}(1-\tau) = 1 \quad .$$

(e) See Fig. 4.